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"The Electrician" Series.

# THE ALTERNATE CURRENT TRANSFORMER

IN

THEORY AND PRACTICE.

By J. A. FLEMING,
D.Sc., M.A.

Vol. I.—NEW EDITION.

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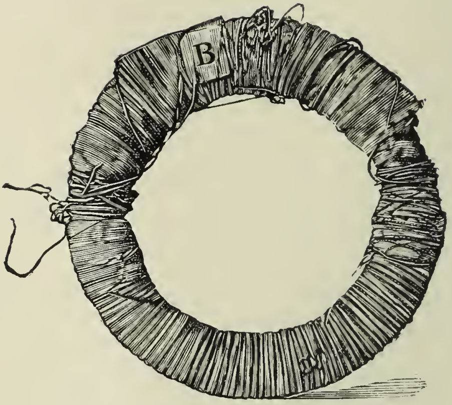

THE FIRST TRANSFORMER.

The Iron Ring, with insulated Primary and Secondary Coils wound on it, with which Faraday made the discovery of Magneto-Electric Induction. Photographed from the original, preserved in the Museum of the Royal Institution.

[*Frontispiece.*

# THE ALTERNATE CURRENT TRANSFORMER

## IN THEORY AND PRACTICE.

BY

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ETC., ETC.

## VOLUME I.

## THE INDUCTION OF ELECTRIC CURRENTS.

**NEW EDITION.**

LONDON:
"THE ELECTRICIAN" PRINTING AND PUBLISHING COMPANY,
LIMITED,
SALISBURY COURT, FLEET STREET.

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Printed and Published by
"THE ELECTRICIAN" PRINTING AND PUBLISHING CO., LIMITED,
1, 2 and 3, Salisbury Court, Fleet Street,
London, E.C.

# PREFACE.

IN the seven years which have elapsed since the first edition of this Treatise was published, the study of the properties and applications of alternating electric currents has made enormous progress. At the outset the aim of the author was to collect, and present in a form suitable for students, a general statement of the facts and principles of electromagnetic induction, and the manner in which these are applied in the design and construction of the Induction Coil and Transformer. At that time most of the practical information on the subject was embedded in technical journals and original papers. Confident that alternating electric currents would play a very important part in the evolution of the electrical industry, the author believed that service would be rendered to engineering students by an attempt, even if an imperfect one, to place a brief systematic treatise on the subject of the Alternating Current Transformer within reach. The result, so far, has justified the belief. At the present time, however,

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much of the subject matter in the first edition has become antiquated, and it became necessary to revise the book thoroughly, to eliminate those parts which had been seen to be imperfect or unnecessary, and to bring the remainder of the information as far as possible into line with recent views and experience.

The author has, accordingly, rewritten the greater part of the chapters, and availed himself of various criticisms, with the desire of removing mistakes and remedying defects of treatment. In the hope that this will be found to render the book still useful to the increasing numbers of those who are practically engaged in alternating-current work, he has sought, as far as possible, to avoid academic methods and keep in touch with the necessities of the student who has to deal with the subject not as a basis for mathematical gymnastics but with the object of acquiring practically useful knowledge.

It is, perhaps, in some ways, a positive disadvantage that alternating-currents lend themselves so easily to mathematical treatment and, by a few assumptions akin to that of the perfectly frictionless machine, offer an attractive field for mathematical ingenuity. Real difficulties are thus often passed over, and an intimate knowledge only gained of a perfectly hypothetical transformer.

The ever-increasing progress of electrical knowledge, and the constant necessity for recasting electrical theories, renders it a most difficult matter to secure in an electrical treatise of any length, uniformity of treatment; whilst views must always differ as to the mode in which any special subject should be approached.

The author ventures to hope, however, that in its revised form, the information here collected may be of use to those who are in any way concerned with alternating-current practice or investigations, and that it may be effective as an introduction to treatises of a more advanced character, which deal with the properties of periodic currents and their utilization in various technical applications.

J. A. F.

*University College, London,*
*April,* 1896.

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63. In the table at the bottom of page 63, for the power wasted in watts per cubic centimetre per 100 cycles per second, in all the numerical values except the first for very soft annealed iron, the decimal point is displaced. The numbers should be divided by ten.

117. Line 6 from top, *for* lies down at $b$ *read* dies down at $b$.

122. Line 7 from top, *for* $10^9$ linkages of current and magnetic force *read* $10^8$ linkages of current and magnetic force; also, in equation (25), *for* $\frac{\text{B S N}}{10^9}$ *read* $\frac{\text{B S N}}{10^8}$, and in line 7 from bottom,

$$\textit{for } \frac{4\pi}{10^{10}} \frac{\text{AS}}{\text{L}} \text{N}^2 \textit{ read } \frac{4\pi}{10^9} \frac{\text{AS}}{\text{L}} \text{N}^2.$$

162. The equation numbered (12) should be numbered (39).

184. Equation (47), *instead of* $v = \text{V}\,(1 + e^{-\overline{\text{RC}}})$

$$\textit{read } v = \text{V}\,(1 + e^{-\frac{t}{\text{RC}}}).$$

195. Last line equation (64), *for* $i = \text{A}\,e^{-\frac{\text{R}}{\text{L}t}} +$, &c.,

$$\textit{read } i = \text{A}\,e^{-\frac{\text{R}}{\text{L}}t} +, \text{ \&c.}$$

203. Line 11 from bottom, *for* of the mean is $\frac{\pi}{2}$ *read* of the mean is $\frac{1}{2}$.

219. In the first line of paragraph § 4, *for* 1888 *read* 1838.

375. Line 7 from bottom, *for* then Sir William Thomson *read* then Prof. William Thomson.

377. In equation (116), *for* $q = \text{A}\,e^{m1t} + \text{B}\,e^{m2t}$ *read* $q = \text{A}\,e^{m_1 t} + \text{B}\,e^{m_2 t}$.

383. At the end of line 7 insert a full stop, and in line 8 *for* when $t = 0$ *read* When $t = 0$.

588. Equation (156), *for* $\text{N}_1\,\text{S}\,\frac{d\,b_2}{d\,t}$ *read* $\text{N}_1\,\text{S}\,\frac{d\,b_1}{d\,t}$; also, in equation (157), *for* $+\text{N}_2\,\text{S}\,\frac{d\,b_2}{d\,t}$ *read* $-\text{N}_2\,\text{S}\,\frac{d\,b_2}{d\,t}$, in line 7 from bottom, *for* $\text{N}_2\,\text{S}\,\frac{d\,b_2}{d\,t}$ *read* $\text{N}_1\,\text{S}\,\frac{d\,b_1}{d\,t}$, and in line 4 from bottom, *for* $-\text{N}_2\,\text{S}\,\frac{d\,b_2}{d\,t}$ *read* $\text{N}_2\,\text{S}\,\frac{d\,b_2}{d\,t}$,

# CHAPTER I.

## HISTORICAL INTRODUCTION.

§ **1. Faraday's Discoveries.**—The autumn days of the year 1831 are ever memorable in the annals of electrical discovery. At that time Faraday, then in the prime of his intellectual powers, began the continuous series of "Electrical Researches," which enriched physical science with discoveries of far-reaching importance, and laid the firm foundations on which much of the modern applications of electricity rests. Looking on the whole of electrical phenomena with an eye eager to see physical analogies, and confident that where these exist they may prove suggestive for further research, he had already asked himself if it were possible there was any effect in the case of electric currents analogous to that known as electrostatic induction. An insulated conductor possessing an electric charge when introduced into a closed chamber having conducting walls calls forth upon them an equal charge of an opposite sign. This *induced electrification* is invariably present, no matter how far off the walls of the enclosing chamber may be, and all surrounding conductors share in the duty of carrying a portion of the induced charge. At a later date, when Faraday viewed this phenomenon of electrostatic induction by the aid of the education he had received in dealing with magnetic lines of force, he was able to picture to himself lines of electrostatic force proceeding in all directions from the surface of a charged body. Wherever they terminated, whether on neighbouring conductors or on the walls of an enclosing chamber, they developed on these "corresponding points" a charge equal and opposite to that of the surface at the point from which they took their rise. Just eleven years previously H. C. Oersted had made

Copenhagen famous as the birthplace of the discovery that an electric current passing through a metallic wire magnetises it circularly and creates round it a magnetic field, the direction of the lines of magnetic force being closed curves surrounding the axis of the wire. Faraday placed these two phenomena side by side before his mental vision, and he asked himself whether it was possible that the magnetic field of force generated round a current-carrying conductor could develop in an adjacent circuit an *induced current* just as the charged body calls forth an *induced electrostatic charge* on the neighbouring conductors.

Some notions on this subject of electric current induction had before 1831 occupied his mind at intervals, but these early experiments did not lead to any satisfactory results.

In 1825, in the month of November, Faraday stretched alongside of a wire connected with a galvanometer another through which an electric current was flowing, but both then and on December 2, 1825, and on April 22, 1828, he had to record of his experiment that it gave "no result." A very little step in experimental research often separates failure from success. A reversal of operations, a change of some dimension, an alteration of some proportion, is often all that is needed to step from the region of failure into the field of discovery and achievement. In this case it may have been the apparently trivial one of starting the electric current in one wire *before* completing the circuit of the galvanometer.

The one thing, it seemed, that this preliminary work did disprove was the notion that a continuous steady current in one conductor could generate a continuous current in another adjacent conductor relatively at rest to the first. It is possible that some conception of the above nature had been dominant in the mind of Faraday before these trials had convinced him that the effect, if existing at all, was not detectable with his apparatus. Three years later he returned to the attack, and we cannot describe the experimental results of the autumn months of 1831 better than they have been given in Faraday's own words in the laboratory note-books of the Royal Institution.* On the 29th day of August, 1831, he thus records the epoch-making discovery by which he will be for ever known. He wrote:—"I have had an iron ring made (soft iron), iron round and $\frac{7}{8}$in. thick, and ring 6in. in external diameter. Wound many coils of copper round one-half of it, the coils being separated by twine and calico; there were three lengths of wire, each about 24ft. long, and they could be connected as one length or used as separate lengths. By trials with a trough, each was insulated from the other. Will call this side of the ring A. On the other side, but separated by an interval, was wound wire in two pieces, together amounting to about 60ft. in length, the direction being as with former coils. This side call B. Charged a battery of ten pairs of plates 4in. square. Made the coil B side one coil, and connected its extremities by a copper wire passing to a distance and just over a magnetic needle (3ft. from wire ring), then connected the ends of one of the pieces on A side with battery; immediately a sensible effect upon needle. It oscillated, and settled at last in original position. On breaking connection of A side with battery, again a disturbance of the needle." (*See* Frontispiece.)

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* *See* Dr. Bence Jones's "Life of Faraday,' Vol. II., p. 2.

On September 24th he resumed his attack. He prepared an iron cylinder and wound on it a helix of insulated wire. The ends of the helix were connected with a galvanometer. The iron was then placed between the poles of bar magnets. Every time the magnet poles were brought in contact with the ends of the iron cylinder the galvanometer needle indicated a current, the effect being, as in former cases, not permanent, but a mere momentary impulse or deflection.

But the full meaning of this hardly appeared clear, and on October 1st he once more laid siege to the fortress. Preparing a battery of 100 pairs of plates, each 4in. square, and charged with a mixture of nitric and sulphuric acids, he arranged to send the current from this through a wire of copper 203ft. long wound round a block of wood. Round the same block, and wound parallel to the first, was a second wire, of equal length to the first, but insulated from it. This second wire he joined up to the terminals of his galvanometer, and then when the battery connection was *made* or *broken* with the first wire he noticed a small but sudden jerk of the needle, one way when the current was made, the other way when it was broken. The clue to the real phenomenon was now in his hand, and guided by it he stepped over a series of confirmatory experiments, and entered as a triumphant conqueror into the stronghold wherein the whole truth lay hid.

Writing on November 29th to his friend, Mr. R. Phillips, he says:—"Now, the pith of all this I must give you very briefly. When the electric current is passed through one of two parallel wires it causes at first a current in the same direction through the other, but this induced current does not last a moment, notwithstanding the inducing current (from the voltaic battery) is continued. All seems unchanged except that the principal current continues its course. But when the current is stopped, then a return current occurs in the wire under induction of about the same intensity and momentary duration, but in the opposite direction to that first formed. Electricity in currents, therefore, exerts an inductive action like ordinary electricity, but subject to peculiar laws. The effects are a current in the same direction when the induction is established, a reverse current when the induction ceases, and a *peculiar state* in the interim."

The path for valuable discovery now lay open. Fully familiar with the work of Ampère and Arago, Faraday knew that a closed circuit conveying an electric current affects all surrounding space with magnetic force, and that, in particular, a small closed circular current can, as far as magnetic action is concerned, be exactly replaced by a very thin disc of steel, whose edge coincides with the line of the closed current, and which is magnetised everywhere in a direction perpendicular to its surface. Such a normally magnetised disc is called a *magnetic shell.* It follows that a helix of wire, which may be regarded as a number of closely approximate circular currents nearly in the same plane, should be magnetically equivalent to a number of magnetic shells piled one above the other, with similar polar faces turned the same way. But such an arrangement of shells would form a cylindrical magnet, and therefore a helix of wire or solenoid in which a current is flowing is for all external space the magnetic equivalent of a cylinder of steel of the same dimensions magnetised uniformly in a longitudinal direction. It remained, therefore, to test this hypothesis. The fifth day of his experiments was October 17th, and on that day he thus notes in the laboratory book the results:—"A cylindrical bar magnet $\frac{3}{4}$in. in diameter and $8\frac{1}{2}$in. in length had one end just inserted into the end of a helix of wire 220ft. long. It was then quickly thrust in the whole length, and the galvanometer needle moved; then pulled out again, and again the needle moved, but in the opposite direction. This effect was repeated every time the magnet was put in or out, and therefore a wave of electricity was so produced from mere approximation of a magnet."

Exactly twenty years afterwards, in the 28th and 29th series of his "Researches," Faraday illuminated, by the exactness and clearness of his experimental method, the whole behaviour of magnets towards closed conducting circuits. It is probable that even at this time he had learned to think of a magnet as carrying with it, as part of itself, a whole system of lines of magnetic force, which emanate from it and surround it. The system of lines of force moves with the magnet wherever it goes. Regarding the production of a current in the helix by a magnet thrust into it, Faraday pictured to himself the advancing magnet as pushing its lines of magnetic force across the coils of wire of the helix, and "cutting" or intersecting them in its progress towards its final position in the coil. The conclusion to which he was led by this reflection seemed to be that the very essence of the effect was the movement across one another of a line of force and a portion of a conducting circuit. If this was so, then the result could be obtained by a more simple and obvious method. The ninth day, October 28th, saw these ideas put to further crucial test. Taking the great permanent horseshoe magnet of the Royal Society, he placed a copper disc so that it was free to revolve on an axis placed in the line of the poles. Soft iron pole pieces were then adjusted to create a powerful magnetic field, the lines of force of which passed through the disc at right angles to its surface. The wires of the galvanometer were made to press against the disc, one near the axis, and the other near the edge. When the disc remained stationary, no current whatever was manifested, but on causing the disc to revolve on its axis a permanent and steady current traversed the galvanometer. This experiment was conclusive. The operation taking place during the revolution of the disc could be viewed as consisting simply in the continual movement of any radial section of the disc across a stream of lines of magnetic force flowing at right angles to its surface. The continuous current resulted from the fact that the motion of that radial section of the disc was always the same relatively to the stream of force. On November 4th Faraday reduced the conception to its utmost simplicity. Taking in his hand the mere closed galvanometer wires, he passed a portion of the loop between the poles of his large permanent magnet in such a way that the direction of that part of the loop between the poles was at right angles to the direction of the magnetic force, and the direction of the movement was at right angles to the direction of the force and that portion of the conductor. The galvanometer deflected, and showed the presence of a momentary current *at* the instant when the intersection took place.

§ 2. **Faraday's Theories.**—In ten days of splendid and conclusive experiment in the autumn of 1831, Faraday had therefore not only discovered the law of induction of currents, but the facts of magneto-electricity as well; and more, for he had not merely accumulated a mass of experimental results, but had reduced the whole valuable store of knowledge to one fundamental principle of exquisite simplicity, namely, that the passage of a line of magnetic force *across* a line of a conducting circuit generates in that portion of the circuit an electromotive force, or a force setting electricity, or tending to set electricity, in motion.

The subsequent work of all experimentalists and mathematicians has been to work out the applications of this principle in countless forms; but no one has since added any essential discovery of fact which is not implicitly contained in the series of discoveries by which, in this short space, Faraday stepped from happy conjectures into possession of facts, which have proved more fertile in far-reaching practical consequences than any of those which even his genius bestowed upon the world. Faraday's theoretical views, however, on the phenomena underwent, in process of time, some modification. He apparently distinguished at first between the induction of currents by a current, which he called *volta-electric induction*, and the production of currents by a conductor moving in a magnetic field, which he called *magneto-electric induction*. That which seemed to impress him most forcibly was, however, the fact that it was only the beginning and ending of the inducing current which had any effect upon the other circuit. He considered that, since the mere cessation of the inducing current was accompanied by a wave of induced current, that could only be because the induced current circuit was, meantime, in a peculiar condition, to which he gave the name of the *electrotonic state*, the annulment of which gave rise to a current in the circuit. The same state he considered to be found in a wire or circuit at rest in a magnetic field. The circuit was in the electrotonic state whilst in the field, but withdrawing the circuit or removing the magnetic field annulled the electrotonic state and gave rise to a current. To use his own words at a later date (Ser. XXVIII., § 3172, "Exp. Researches"), "Mere motion would not generate a relation which had not a foundation in the existence of some previous state;" and (Ser. XXIX., § 3269, *ibid.*) "Again and again the idea of an electrotonic state has been forced upon my mind." The mere motion of an external body, such as a copper wire, in a magnetic field cannot, he considers, be the sole cause of the current, unless there is a previous peculiar state as regards the wire which, when motion is superadded, produces the current. When, however, subsequent thought and diverse experiment had clarified his ideas and adjusted facts in proper relation, he came to see that that which he had denominated the *electrotonic state* is really the amount of electromagnetic momentum which the circuit possesses in virtue of its being in a magnetic field. In modern language, it is the equivalent of that which is now called *the number of lines of magnetic force* passing through the circuit. Every line of magnetic force is a closed loop or continuous line, and if we set out at any point on a line of magnetic force and travel forwards along that line we shall come back to that same point again. If this line of force is originated by a permanent magnet or an electromagnet, then part of our journey will be performed through the iron or steel and part through the air or other diamagnetic surrounding it. If, then, a closed conducting circuit is so situated that the line of force considered passes through it or is linked with it, the line of force and the closed circuit form, as it were, two links of a chain, and cannot be separated except by pulling one through the other (Fig. 1). When they are so pulled through one another the line of force "cuts" and is cut by the circuit. The number of lines of force, therefore, which at any instant are linked with a given circuit represent potentially the greatest amount of "cutting" possible. The existence of lines of magnetic force linked with the circuit is an essential antecedent to the appearance of a current of induction in that circuit when removed from the magnetic field. At a later stage of his investigations Faraday was able to modify his earlier notions of the electrotonic state, and learnt to look on the induced current appearing under these circumstances as due not to a state of things *in* the circuit, but to a condition of things *outside* the circuit, or, more precisely, to the relation in which the circuit stands to the magnetic field of force around it.

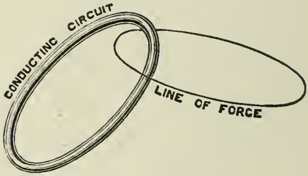

Fig. 1.

In the 28th and 29th series of his "Experimental Researches," Faraday exhausted all possible means of experiment in proving that this conception of the linking or unlinking of loops of force and loops of conducting circuits was an unerring guide to the solution of all problems of electromagnetic induction. The circuit being given, he was able to show by a course of rigid demonstration that the process of linking with it a loop of magnetic force was always accompanied by the passage of a wave of current round the circuit in one direction, and the unlinking was invariably associated with the flow of an opposite pulsation of electricity. Moreover, and most important of all, he built up a quantitative conception around the term "a line of magnetic force," so that it came to him to mean not merely a geometrical line or a direction, but a definite physical magnitude, which represented the product of a certain area of space, and a certain mean intensity of magnetic force over that area.* Armed with this idea, he proceeded to show that the quantity of electricity represented by each current of induction is the numerical equivalent of the "number of lines of force" which are linked or unlinked with the circuit by any operation. He found that this hypothesis never failed to enable him to render a satisfactory and a logical explanation of all his results, and with this clue in hand he could find his way about amidst the entanglements of experimental inquiry, and return always from each fresh excursion after fact with new confirmation of its consistency, and with fresh power to predict the results of other experiments.

So strong became at last his conviction that these lines of force could hardly have such powers if they were mere geometrical conceptions, like lines of latitude and longitude, that he gives expression to it by speaking of them as *physical lines of force.* He intends to imply that he thinks "a line of force" must be taken to be a definite *action* going on in a certain region of space, and that, whatever may be its real nature, we must accord to it a definite physical character in some sort or sense, as much as we do an electric current of unit strength flowing along a prescribed circuit. Faraday was not a professed mathematician, and it was perhaps fortunate that his inability to employ the mechanical aid of symbolic reasoning forced him to make clear to himself each step by experimental demonstration. He was thereby compelled to keep to the main track of discovery, and prevented from deviating into the more abstract lines of thought. The special abilities of Kelvin and Helmholtz, and subsequently those of Clerk Maxwell, were, however, directed to the complete elucidation of these conceptions of Faraday, and the great treatise of Maxwell, as he himself has stated, was undertaken mainly with the hope of making these ideas the basis of a mathematical method. The one cardinal principle which may be said to be at the base of the mode of viewing electrical and magnetic phenomena introduced by these investigators is the denial of action at finite distances, and accounting for the phenomena by the assumption of the existence of a *medium* which is the active agent in the transmission of energy from one place to another, and which is itself capable of storing up energy in a potential and kinetic form.

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* Faraday's notion of "a line of force" was at first merely a geometrical conception, representing a certain line of action, but his ultimate applications of the term showed that he had come to think of it as a *surface integral.*

The mathematical methods and hypotheses of the French school of physicists, represented chiefly by Ampère, Arago, Poisson, and Coulomb, consisted in the assumption that material particles in special states, called electric and magnetic, could act on one another at finite distances without any intervening mechanism according to certain laws of force varying with the distance. Faraday may be said to have raised the standard of revolt against this notion, and indeed he was able to quote in his support the great authority of Newton in rejecting the idea that matter could act on matter across intervening distance without aid from any mechanism. He never considers bodies as existing with nothing between them but their distance, and acting on one another according to some function of that distance. He conceives all space as a field of force, the lines of force being in general curved, and those due to any body extending from it on all sides, their direction being modified by the presence of other bodies. A magnet, an electrified conductor, or a wire conveying an electric current, are thus the focus and source of a system of radiations of force lines or loops which are to be thought of as part and parcel of it. This force system is capable of deformation or change by the presence of other bodies, but it moves with the magnet, electrified body, or current-carrying wire. These force radiations penetrate surrounding bodies, and the apparent actions between bodies at a distance are in reality actions due to immediate action of the field of force of one body upon the other at the place where it is. Then rises for solution the important problem: What *are* these lines of force? Faraday answered the question by saying that they consist in some sort of operation or action going on in a *medium* along certain lines or axes, and Maxwell added to this the suggestion that the electromagnetic medium must be identical with the medium postulated to account for the phenomena of light.

The question which yet remains unanswered is: What is the nature of the action or operation along certain lines in this medium which causes a line of force to exist? The future of electric and magnetic investigation will, perhaps, conduct us step by step to the solution of this supremely important problem.

§ 3. **Henry's Investigations.**—At the same time that Faraday was pursuing in England a career of triumphant discovery in the field of electromagnetic science a young philosopher of hardly less intellectual power, but more limited opportunities for research, was following hard on the same path of investigation in America. The name of Joseph Henry is one which we must link with that of our own great countryman as a co-worker, nay, even an anticipator in some things, in the region of fundamental discovery in electromagnetism.

To Henry clearly belongs the credit of having improved Sturgeon's electromagnet by substituting for the single layer of copper wire wound on the iron horse-shoe a spool or bobbin of insulated copper wire. By this means he made what he then called *intensity magnets*, or electromagnets, suitable for excitation by an intensity battery or battery of many cells. Henry in this manner constructed in 1829 or 1830 a very large electromagnet, capable of supporting a weight of 600lb. or 700lb. Before having any knowledge of Faraday's experiments, and guided apparently by the notion that as electric currents can produce magnetism, so magnetism should be able to generate electric currents, Henry experimented as follows:—A piece of wire about 30ft. long and covered with an elastic varnish was closely coiled round the middle of the soft iron armature of this large electromagnet. The wire was wound upon itself so as only to occupy about 1in. in length of the armature which was 7in. in all its length. The ends of this wire were connected by long copper wires with a distant galvanometer. The armature with its coil was laid upon the poles of the electromagnet, and the galvanic plates connected with the helix of the electromagnet immersed in the trough of acid. At the moment of immersion the needle of the galvanometer was seen to be deflected about 30deg., but it immediately returned to its normal position. On withdrawing the battery plates from the acid it was noticed that the galvanometer needle made a sudden deflection in the opposite direction of about 20deg. A similar effect was produced by pulling off or putting on the armature whilst the magnet remained excited. Henry, in his account of this experiment, says:—"From the foregoing facts it appears that a current of electricity is produced for an instant in a helix of copper wire surrounding a piece of soft iron whenever magnetism is induced in the iron, and a current in the opposite direction when the magnetism ceases; also, that an instantaneous current in one or other direction accompanies every change in the magnetic intensity of the iron."

This very lucid statement of experiments, made probably in August, 1831, shows that Henry was at least an independent discoverer of the induction of electric currents. In April, 1832, an account reached him of Faraday's discovery in the previous year, and Henry then repeated his former experiments, and was able by means of larger helices of wire wound on the armature of his electromagnet to greatly increase the magnitude of the induced current. Henry, therefore, not only discovered independently the facts of electromagnetic induction, but correctly interpreted them as well. He early laid a firm grasp upon the essential principles involved, and he came almost within reach of anticipating that discovery which is, and will remain, the crowning glory of his illustrious rival. Between 1831 and 1840, or later, Henry continued to add fresh knowledge to the original facts, and in a later chapter a description will be given of his important investigations on the self and mutual induction of conducting circuits.

# CHAPTER II.

## ELECTRO-MAGNETIC INDUCTION.

§ **1. Magnetic Force and Magnetic Fields.**—Certain substances, such as iron, nickel, cobalt, steel, and some of their compounds, particularly a native oxide of iron, possess peculiar physical properties, and either exist in, or can be put into, a condition in which they are said to be *magnetised.* When in this condition they exhibit physical qualities which are called magnetic properties, the most obvious of which is the power of producing attraction and repulsion upon other magnetic substances. Some bodies, notably hardened steel, can acquire marked permanent magnetic qualities. The neighbourhood round these bodies when in this state, and within which they exercise these actions, is called a *magnetic field.* If a small magnetised steel needle is suspended freely at its centre of gravity and held in a magnetic field it is found that it takes up a certain direction under the influences of forces acting upon it. If disturbed from this position it returns to it again.

It is found that there is a line in the needle round which it can be revolved without changing the *set* of that line when the needle is left free to obey the forces acting upon it. The direction of this line in the needle is called its *magnetic axis.* Oersted discovered that a magnetic field exists in the neighbourhood of a conductor conveying an electric current, and that it imposes a certain directive influence upon a magnetic needle held near to it. If a small steel magnetised needle is placed in any region containing either conductors carrying electric currents or substances in a permanent magnetic state it is found that at every point of the field the magnetic axis of this small exploring needle takes up a definite position if it is freely suspended so as to be removed from the influence of gravity. The direction so assumed by its magnetic axis is called the *direction of the magnetic force* at that point. The magnetic force has at every point in the magnetic field of these active agents a certain direction. On examining the behaviour towards one another of two magnetised steel needles we find that their magnetic properties are exhibited chiefly at the two extremities, and these are called the *magnetic poles.* The two poles of a magnetic needle are not identical in quality. If a uniformly magnetised steel needle is broken in the middle, the ends where it is broken immediately become new magnetic poles, whereas before rupture that portion of the needle exhibited no apparently active magnetic properties. If the two poles which make their appearance at the broken ends are tested it will be found that they attract one another. If these poles are placed one centimetre apart and the force with which they attract one another measured in absolute units, the square root of the number which expresses this attraction is called the numerical value of the *strength of these poles.* Hence, a unit magnetic pole is a pole which at a unit of distance attracts another unit pole of opposite kind with a unit of force. The earth as a whole is a magnetic body, and if a small magnetic needle is freely suspended at its centre of gravity, its magnetic axis assumes a certain position at each point on the earth's surface which is called the direction of the *terrestrial magnetic force* at that point. The pole of the needle which points in our latitude in any direction north of the true east and west line is called the north pole or north-seeking pole of the needle. If we take a very long thin magnetised needle, called for shortness a magnetic filament, we can employ one pole of it, say the north pole, for exploration in a field, whilst the other pole is so far removed as not to be affected. If such a pole, called for shortness a free north pole, is placed in any magnetic field it is acted upon by the magnetic force and urged to move in the direction of this force. If this free north pole is a pole of unit strength, then the force dynamically measured in absolute units which acts upon it is called the numerical measure of the magnetic force at that point.

The direction in which a free north pole tends to move is called the *positive direction of the magnetic force* at that point. The magnetic force at any point in the magnetic field of magnetic bodies, whether magnetised substances or conductors conveying electric currents, is thus a quantity which has *direction* as well as *magnitude*, and we have defined above how both of these can be measured. By means of a free north magnetic pole of unit strength we may thus explore and define a magnetic field at every point.

A magnetic field in which the magnetic force is the same in magnitude and direction at every point is called a uniform magnetic field. The magnitude of the magnetic force at any point is a measure of the *strength of the magnetic field* at that point.

There are several simple and yet important cases in which it is possible to calculate the strength of the magnetic field or the magnetic force at certain assigned points in the neighbourhood of conductors conveying electric currents. The pre-determination of the field strength at points near to magnets and conductors conveying electric currents is, generally speaking, except in these simple cases, a very difficult matter.

### *The Magnetic Force near to a very long Straight Wire conveying an Electric Current.*

If a current flows in a thin circular wire we may call a very short length of this conductor, denoted by $d\,s$, an element of the circuit or of the current. Ampère showed by a classical series of experiments that the magnetic force due to an element of a current at any point near it was numerically equal to the product of the strength of the current, the length of the element, and the sine of the angle between the direction of the element of the circuit and the line joining the centre of that element with the point, and inversely as the square of this distance. Thus, if $d\,s$ (Fig. 2) represents the element P of a circuit in which is flowing a current of strength I in absolute electromagnetic measure, and if $\alpha$ is the angle which any line O P makes with the direction of the element, and $r$ is the length of the line O P, then the magnetic force at the point O due to that element of the current is numerically equal to

$$\frac{\mathrm{I}\,d\,s\sin\alpha}{r^2},$$

and this force is in a direction at right angles to the plane containing the element of the circuit and the line joining it to the given point. Starting with this fundamental law, we can deduce expressions for the strength of the magnetic field due to currents flowing in conductors of certain forms at certain assigned points. Consider, for instance, a very long, practically infinite straight wire in which a current is flowing, the return wire being at a very great distance. Take any point P in the neighbourhood of this conductor (Fig. 3). It is required to find the magnetic force at the point P. Draw P M perpendicular to the wire from P. Let N N′ be any element of the conductor. Then the magnetic force at P due to the element $ds = \mathrm{N\,N'}$ of the conductor is in a direction at right angles to the plane of the paper, and if the length N P is called $r$ and the angle P N M is called $\alpha$, the magnetic force at P due to the element N N′ is equal numerically to

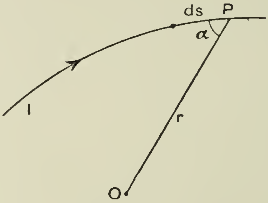

Fig. 2.

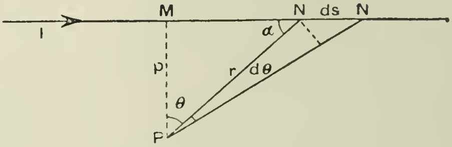

Fig. 3.

$$\mathrm{I}\,\frac{ds \sin \alpha}{r^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where I is the current flowing in the element. Let P M be denoted by $p$. In order to find the magnetic force due to the whole wire at P we have to integrate the above expression throughout the whole length of the wire. To do this we transform it as follows:—Let the angle $M P N = \theta$, then $N P N'$ is the increment of this angle, call it $d\theta$. From the geometry of the figure it is easily seen that $\sin \alpha = \cos \theta$, and that when $ds$ is very small $\frac{ds}{r\,d\theta} = \frac{r}{p}$, or $\frac{ds}{r^2} = \frac{d\theta}{p}$.

Hence substituting these values for $\sin \alpha$ and $ds/r^2$ in equation (1), we have as the expression for the value of the magnetic force at P, due to the element N N′ of the current, the formula

$$dF = I \frac{\cos \theta\, d\theta}{p}.$$

The magnetic force due to the whole infinitely long straight current is obtained by integrating this expression between the limits $\theta = 0$ and $\theta = \frac{\pi}{2}$ and then doubling this value. Hence the magnetic force of the whole wire at P is equal to

$$\frac{2I}{p}\int_0^{\frac{\pi}{2}} \cos \theta\, d\theta = \frac{2I}{p}. \quad . \quad . \quad . \quad . \quad (2)$$

In other words, the magnetic force at any point due to the current I flowing in an infinitely long straight conductor is in magnitude inversely proportional to the perpendicular distance of the point from the wire; and, as regards direction, it is everywhere perpendicular to the plane containing the wire, and the perpendicular let fall on it from the given point. This conclusion was experimentally verified by Biot and Savart by vibrating a small magnetic needle at different distances from a long straight current, and counting the square of the number of oscillations in a given time made by the said small needle in these different positions. If the current in the wire is measured in amperes, then, since ten amperes equal one unit current in absolute electromagnetic measure, and if A is the current so measured in amperes, the magnetic force at any point $p$ centimetres from the wire is equal to

$$\frac{2}{10}\frac{A}{p} = \frac{1}{5}\frac{A}{p}.$$

Thus the magnetic force due to a current of one ampere flowing in a long straight wire at a point one centimetre from the wire is equal to one-fifth of a unit of magnetic force. This is nearly equal to the value of the earth's horizontal magnetic force in England. It will be seen that the magnetic force due to powerful currents in long straight cables may be sensible at points very far removed from the cable. This magnetic force is at every point perpendicular to the conductor, and hence the direction of the force of such a straight conductor must be everywhere a tangent to a circle drawn round the wire with its plane perpendicular to the axis of the wire and its centre in that axis. Hence a freely suspended magnetic needle tends to stand perpendicular to a straight conductor when this last is traversed by a current. If a magnetic pole of strength $m$ is placed at any point in the field of such a straight conductor the magnetic force tends to drive the pole in a circle round the wire with a force equal to $\frac{2mI}{p}$ dynamical units or dynes, where $p$ is the distance of the pole from the axis of the wire and I is the absolute value of the current flowing in it. It follows that the lines of magnetic force of such a linear current are circles described round the wire with planes perpendicular to it and centres in the axis of the wire. Oersted was aware of this fact, and he expressly says,* "The electric conflict" (that is, magnetic field) "performs circles round the wire."

We may next proceed to determine

### *The Magnetic Force at the Centre of a Circular Current.*

If a thin wire is bent into a circle, and a current of strength I is sent round it, the magnetic force, estimated at the centre of the circle, due to each element of the length of the current, is in a direction at right angles to the plane of the circle. Let $ds$ be an element of length and let $r$ be the radius of the circular wire, then the magnetic force due to $ds$ at the centre is equal to $\frac{I\,ds}{r^2}$. But, since the force due to each element is the same, the magnetic force due to the whole length of the circular wire is equal to

$$\frac{2\pi I r}{r^2} = \frac{2\pi I}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If the current is measured in amperes and denoted by A, then the magnetic force at the centre of the circular current is

$$\frac{\pi}{5}\frac{A}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

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* *Annals of Philosophy*, Oct., 1820 Vol. XVI., p. 274.

This magnetic force at the centre of a circular current is in a direction perpendicular to the plane of the circle. If the circular current consists of a current of A amperes flowing in a very thin wire wound $n$ times round a circular groove of mean radius $r$, then the magnetic force at the centre is equal to

$$\frac{\pi}{5}\frac{n\mathrm{A}}{r} \quad \ldots\ldots\ldots \quad (5)$$

The expression for the magnetic force at a point in the plane of the circle not in the centre is less simple (*see* Appendix, Note A).

### *The Magnetic Force due to a Circular Current of* n *turns at a point on its axis out of its own plane.*

The third case of importance is to find the value of the magnetic force due to a circular current at a point on a line drawn through its centre and perpendicular to its plane. Let the circular current be X Y Z (Fig. 4), and let P be any point on a line O P drawn through the centre O and perpendicular to the plane of X Y Z. The magnetic force at P, due to an element $ds$ of the circuit at X, acts along a line perpendicular to X P, and is in the plane of X O P. If $r$ stands for O X, and $x$ for O P, the magnetic force due to the element $ds$ at P resolved in the direction O P is equal to

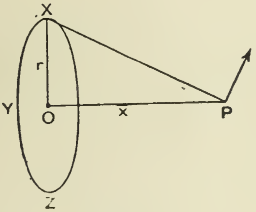

Fig. 4.

$$\frac{\mathrm{I}\,ds}{(r^2 + x^2)} \sin \mathrm{XPO},$$

where I is the strength of the current in the element. The above is equal to

$$\frac{\mathrm{I}\,r\,ds}{(r^2 + x^2)^{\frac{3}{2}}},$$

and hence the magnetic force due to one whole turn of the conductor is equal to

$$\frac{\mathrm{I}\, 2\pi r^2}{(r^2+x^2)^{\frac{3}{2}}}.$$

If the circuit makes $n$ turns, the magnetic force at P, due to the current I flowing $n$ times round the circular conductor X Y Z estimated in the direction O P, is equal to

$$2\pi n \mathrm{I} \frac{r^2}{(r^2+x^2)^{\frac{3}{2}}}. \quad \ldots \ldots \quad (6)$$

This, then, is the expression for the magnetic force, due to a circular current of $n$ turns at a point on its axis but outside of its own plane. The calculation of the magnetic force due to the circular currrent at points other than those on the axis O P, is a much more difficult matter. In the above formula I is measured in the electromagnetic units. If the current is measured in amperes and denoted by A, then (6) becomes

$$\frac{\pi}{5} n \mathrm{A} \frac{r^2}{(r^2+x^2)^{\frac{3}{2}}}. \quad \ldots \ldots \quad (7)$$

The above expression may be put into another useful form.

Since

$$\frac{r^2}{(r^2+x^2)^{\frac{3}{2}}}$$

is the differential with respect to $x$ of

$$\frac{x}{\sqrt{r^2+x^2}},$$

which last, as can be seen from Fig. 4, is equal to cos X P O, we may write (7) in the form

$$\frac{\pi}{5} n \mathrm{A} \frac{d}{dx} (\cos\theta), \quad \ldots \ldots \quad (8)$$

where $\theta$ stands for the angle X P O.

### *The Magnetic Force due to a long closely-coiled Helical Current at points on the axis near the centre.*

Another useful case in which it is possible to calculate the magnetic force due to a current is in the case of points in the interior of a very long closely-coiled helical current called a solenoid. Such a case is practically realised by coiling insulated wire round a tube. Let the length of the helix be $l$, and let there be N turns of wire per unit of length. Then if a slice of this helix is considered of thickness $dx$, the number of turns of wire in this slice is $N\,dx$. Let a current I flow through the wire. Take a point on the axis of the helix somewhere near the centre (*see* Fig. 5), and take any element of length of the helix at a distance $x$ from this point. Then by (8) the magnetic force due to this element of length of the helix at the point P is

$$2\pi N I \frac{d\cos\theta}{dx}\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

where $\theta$ is the angle O P X.

Let the slice of the helix be taken at successive distances from the point P, beginning with $x=0$ and ending with the end of the helix. The sum of all the magnetic forces due to each element of the helix to the left of the point P is then equal to the integral of (9) taken between the limits $\theta=90$deg. or $\cos\theta=0$ and $\theta=\theta_1$, where $\theta_1$ is the angle O′ P X′, or the angle subtended by half the mean diameter of the end of the helix at the point P. Similarly, to obtain the whole force at P due to the elements of the helix lying to the right of P we have to integrate (9) from $\theta=0$ to $\theta=\theta_2$, where $\theta_2$ is the angle subtended by half the aperture of the other end of the helix at P. Adding these forces together we have as value of the whole magnetic force of the whole helix at P the expression

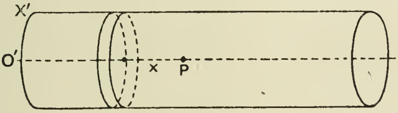

FIG. 5.

$$F = 2\pi N I\,(\cos\theta_1 + \cos\theta_2).$$

If the helix is so long that the half diameter of the aperture of the ends of the helix, as seen from the point P, is practically zero, then $\theta_1$ and $\theta_2$ are both practically zero, and therefore $\cos\theta_1 + \cos\theta_2 = 2$, nearly, or

$$F = 4\pi N I. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

The magnetic field at P is then equal to $4\pi$ times the absolute current-turns per unit of length. If the current is measured in amperes, the force is equal to $\frac{2}{5}\pi N A$, or to $\frac{2}{5}\pi$ times the ampere-turns per unit of length of the coil. Since $\frac{2}{5}\pi$ is nearly 1·25, the approximate practical rule for the magnetic force in the neighbourhood of the centre of a long helix of this kind is that the magnetic force is numerically equal to $1\frac{1}{4}$ times the ampere-turns per unit of length of the helix. The above formula is only strictly true for points on the axis of the helix and for helices very long compared with their diameters. It is very nearly true for all points in the interior of a fairly long helix. Thus, for instance, if the helix is twelve diameters long, the magnetic force in the interior throughout one quarter of its length on either side of the central point does not differ by much more than one per cent. from the value it has at the central point. Hence this fact presents us with an easy and practical method of procuring a magnetic field of known strength. On a long pasteboard or metal tube provided with cheeks wind covered copper wire carefully and evenly in any number of layers. Count the turns and layers of wire, and measure the length between the cheeks; this gives us the turns per unit of length. Then pass a known current through the wire, and calculate by formula (10) the field at the centre. The coil should be at least twelve diameters long. We may approximately apply (10) to calculate the magnetic force in the interior of such long bobbins as are used in winding the field-magnets of dynamos. The magnetic force in the interior of a long bobbin is strongest in the centre of the bobbin, and falls off towards either end, and it is slightly stronger at points nearer the wire than on the central axis even at the centre. The complete calculation of the field at any point in the interior or exterior of a not very long helix is a rather difficult matter, but the above formula (10) will be sufficient for most practical purposes.

A final, practical, and useful case is that of the predetermination of *the magnetic force in the interior of a circular closed solenoid or endless helical current.* Let a wooden ring of circular cross-section be wound over closely with insulated wire so that the turns of the wire are contiguous and one or more layers are put on. This is called a circular solenoid, and we can calculate the magnetic force for points in the interior when the circular solenoid is traversed by a current. Let R be the mean radius of the solenoid, and $a$ that of the mean circular section. If the wire is wound in one layer on a wooden ring, then $a$ will be the mean between the half diameter of the section of the ring and the half diameter measured over all after the wire is wound on it.

The magnetic force is not the same at all points over the circular cross-section of the solenoid. To find out what it is at any point we may proceed as follows:—A solenoid of any size, meaning by that a spiral current with turns closely adjacent, is electrically equivalent to a bundle of elementary solenoids or spiral currents of exceedingly small cross-section. Consider such a very small-sectioned solenoid, which may be called a spiral filament. It may be obtained in practice by winding insulated wire of small size on a very fine knitting needle as a core, and then withdrawing the needle. The section of this solenoid being very small, the magnetic force in its interior is everywhere nearly the same over the cross-section, and if the spiral is long the force in the centre in the interior is equal to $4\pi n \mathrm{I}$, where I is the absolute current flowing in the wire, and $n$ is the number of turns per unit of length. Let this long elementary solenoid be bent round into a circle so as to form a closed or endless solenoid, let $x$ be the mean radius of the circle which it forms and let $n_1$ be the number of turns of wire of the spiral in an arc of the solenoid equal to one unit angle in circular measure. Then, the number of turns per unit of length of the spiral being $n$, we have $n x = n_1$, and we may write the expression for the magnetic force in the interior of the solenoid as

$$\frac{4\pi n_1 \mathrm{I}}{x}. \quad \ldots \ldots \quad (11)$$

A little consideration will then show that the magnetic effect of any circular solenoid must be the same as that of a bundle of elementary solenoids, so wound and arranged as that the number of turns of wire of each spiral per length of arc subtending one unit angle in circular measure is the same. The magnetic force in the interior of the circular solenoid at any point in the cross-section is then equal to the product of $4\pi n_1$ I, and the reciprocal of the perpendicular distance of this point from the axis of the circular solenoid. The force over the cross-section is not uniform, but has a particular value for every point, but the same value at all points at an equal radial distance from the axis of the circular solenoid or ring coil.

**§ 2. Magnetomotive Force and Magnetic Induction.**—We have in the foregoing section defined magnetic force, and shown how it can be determined in a few simple cases from a fundamental principle. If any line is drawn in a magnetic field of force, and we sub-divide this line into very small elements of length, and estimate the magnitude of the magnetic force at the centre of each element resolved in the direction of this element, and then sum up all the products obtained by multiplying the length of each element by the strength of the magnetic force along its direction, we obtain the *line integral of magnetic force* along that line. This is also called the *magnetomotive force* along that line.

In mathematical language, if $\mathsf{H}$ is the magnetic force at any point on the line, and $\theta$ the angle this force makes with the line and $d\,s$, an element of length of that line at that point, then $\int \mathsf{H} \cos\theta\, d\,s$ is the line integral of magnetic force along that line. From the definition of magnetic force, it is clear that this line integral is the *work done* in carrying a free unit magnetic pole along that line. This magnetomotive force along a line is likewise called the *difference of magnetic potential* between the two ends of the line. In those cases in which the magnetic force has a uniform value and is in the direction of the path chosen, it becomes a simple matter to calculate the magnetomotive force along that line, for it is the simple product of the numerical values of the magnitude of the magnetic force and the length of the line.

Let us consider two simple cases. First, when the line integral is taken along a closed line or loop in a magnetic field drawn in air or other non-magnetic medium, but not linked with or encircling a circuit conveying a current. In this case the value of the line integral is zero, because no work is done in carrying a free pole around a closed path in an air field. Second, when the line integral is taken along a path which is a closed loop, and which surrounds or is linked with a circuit conveying an electric current. Consider the simplest case. Let a straight wire convey an electric current C, the return being at a great distance. Describe a circular line round the wire at a distance $r$ from the axis of the wire. The length of this line is $2\pi r$; the magnetic force at a distance $r$ from a straight wire is $\frac{2C}{r}$ units; and the line integral along this line is $2\pi r \times \frac{2C}{r} = 4\pi C$. Hence the line integral of the magnetic force taken once round the circuit is $4\pi$ times the total current through the line of force. This can be shown to be generally true, and is the general relation between magnetic force and current.*

If a looped line is taken through a helical current which links itself round the line $n$ times, then, if A amperes traverse the conductor, the total quantity of current flowing through the loop is $n$ A (equal to the ampere-turns), or in absolute C.G.S. measurement is $\frac{nA}{10}$; hence the line integral of the magnetic force taken along any closed line threading $n$ times through the circuit of a current A is $\frac{4\pi}{10} nA$, or $1\frac{1}{4}$ times the ampere-turns of the current which are linked with the closed line. It is useful to remember that the value of $\frac{4\pi}{10}$ is very nearly 1·25.

We have here introduced the student to the notion of a *line integral.* Another similar mathematical idea which has to be grasped is that of a *surface integral.* If, in any field of magnetic force, we describe a surface of any form bounded by a closed line, the magnetic force at all points of this surface will have a certain value, call it **H**. Let the surface be supposed to be divided in a number of very small elements of surface each equal to $dS$. At the centre of each element estimate the value of the magnetic force perpendicularly or normally to that element, take the product of the value of this normal value and that of the element of area. If $\theta$ is the angle between the direction of the force and that of the normal to the surface, then the product of the normal force, or $\mathsf{H} \cos\theta$ and $dS$ is to be taken for all elements of the surface. The sum of all such products is called the *surface integral of the force*, and is expressed in mathematical language by the integral $\int \mathsf{H} \cos\theta \, dS$.

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* See *Electrician*, Vol. X., p. 7: Mr. Oliver Heaviside on "The Relation between Magnetic Force and Electric Current."

This is also called the *flux of the force* through the area, for if we suppose that, instead of dealing with magnetic force, we were considering the velocity of a moving fluid, the surface integral of the velocity over any area would represent the whole quantity of liquid which flows in one second through the line bounding the area or the flux of the fluid.

Returning to the measurement of magnetomotive force, the reader will notice that, as a consequence of the above general theorem, in those cases in which we are dealing with the magnetic force due to a spiral current or solenoid making a number of turns round, or *linkages* with, the line of the magnetomotive force, the measurements of this magnetomotive force is practically made in ampere-turns, or by the products of the number of turns of the wire and the ampere current conveyed by it.

Owing to the fact that the circumference of a circle is $2\pi$ times its radius in length we get a numeric $\frac{4\pi}{10}$ introduced which makes the magnetomotive force, measured along a line linked with a line of current making $n$ turns round it, numerically equal to $1\frac{1}{4}$ times the ampere-turns, but it is not difficult to remember or to use this simple factor.

When magnetomotive force acts on any body, whether magnetic like iron or non-magnetic like wood or air, it produces in it an effect called *magnetic induction.* The student must think of magnetic induction as something which is produced by magnetomotive force, just as electric current is produced by electromotive force. Magnetic induction is a quantity which is called a flux, and the magnetic induction in magnetic bodies results from magnetomotive force or difference of magnetic potential, just as the flow of water results from difference of pressure or head of water, and the flow of electricity in conductors from difference of electric potential, and the flow of heat in thermal conductors from difference of temperature.

The quality in virtue of which magnetomotive force can produce magnetic induction in a magnetic body is called its *magnetic inductivity*, or *magnetic permeability*. A body having large permeability is one in which a given magnetic force produces a relatively large magnetic induction. Similarly, we might say that the quality of bodies in virtue of which a hydrostatic force or pressure produces a flow of fluid through them is called their *porosity*, and a very porous body is one in which a given hydrostatic pressure produces a relatively great flow of liquid. Quite similarly we define *electric* and *thermal conductivity* as those qualities of bodies in virtue of which electromotive force and difference of temperature produces in them electric current or flow of heat. Hence magnetic permeability, electric conductivity, thermal conductivity, porosity, and, we may add, specific inductive capacity, in electrostatics are all analogous qualities of bodies numerically capable of being measured which are of importance in that they determine the amount of the *flux* produced by a unit of *force* of the corresponding kind.

The magnetic induction, like magnetic force, has a definite direction as well as magnitude at every point where it exists. Both magnetic induction and force belong to that class of quantities which in mathematics are called *vector quantities*, and possess both magnitude and direction. In order to define the direction and amount of magnetic induction we fall back upon the fundamental discovery of Faraday. The ground fact of all his investigations is that if a conducting circuit is placed in a field of magnetic induction any *change* in the magnitude or strength of the magnetic induction will create an electromotive force in that circuit urging, or tending to urge, an electric current round it, provided that the plane of this circuit has any but one particular direction. We may, therefore, use a small conducting circuit to explore a field of magnetic induction, just as we employed a free magnetic pole to explore a field of magnetic force. Let a small circular conducting circuit, formed, say, of one turn of very thin wire, and enclosing one unit of area, be placed in a field of magnetic induction. Let this small loop or unit circuit be held in various positions, and let changes be made in the induction by varying the magnetomotive force. It will be found that there is a particular position or positions of the circuit in which no change in the strength of the induction produces any electromotive force in this small circuit. The direction of the induction at the centre of the circuit is then parallel to the plane of the circuit. The axis round which the circuit can be revolved without affecting this inactive condition of the circuit is the direction of the induction at that place. Hence we can map out at all points of the field of induction the *direction of the magnetic induction.* If the circuit is turned into any other position such that a change in the induction does produce an electromotive force acting in the circuit, we may find by trial another position of the circuit at any point in the field in which the total suppression of the induction, or its instantaneous reversal in direction, produces the maximum electromotive force in the circuit. The direction of the induction is then normal to the plane of that exploring circuit. At every point in the field of induction the induction has a certain magnitude as well as direction. If we suppose any plane surface placed normally to the direction of the induction in a field of uniform induction, the product of the strength of the induction and the area of the surface is called the *total induction through that area.* If we take any surface placed in any position, and suppose it divided up into small elements of surface $d\,S$, and at the centre of each element of the surface estimate the magnitude of the induction in a direction normal to the surface at the centre of the element, and sum up all the products obtained by multiplying the normal value of the induction and the area of the element, the sum so obtained is called the *surface integral of induction.* If we draw in a field of induction a line such that the direction of its tangent at any point is the direction of the induction at that point, such a line is called a *line of induction.* Suppose a surface of any kind, for simplicity a plane surface, placed in a field of induction, and let it be divided up into small areas such that over each of them the surface integral of induction is equal to unity, and if through the centre of each of these small areas we draw a line of induction, then we may make the following statements:—The bounding line of the surface is said to be perforated by induction, or to have a flux of induction taking place through it, or to have lines of induction passing through it. Since the total surface integral of induction through this surface is, by definition, equal numerically to the number of lines of induction passing through the boundary of the surface, we may also speak of the *number of lines of induction* passing through the surface. Faraday used the phrase *number of lines of magnetic force* instead of induction. Hence the student should notice that the following expressions all denote the same thing:

1. The surface integral of induction over a surface.
2. The total induction through the surface (Maxwell).
3. The flux of induction through the surface.
4. The number of lines of induction passing through the surface.
5. The number of lines of force (Faraday) passing through the surface.

If a circuit be placed normally in a field of uniform induction, then the numerical product of the area of the circuit and the strength of the induction gives us the total induction through that surface, or total number of lines of induction (force) passing through that circuit.

In the twenty-eighth series of his "Experimental Researches on Electricity," Faraday examined afresh with elaborate care the notion of *lines of magnetic force* which had guided him at all stages of his electromagnetic discoveries. He there gathers together his ideas on this subject, and by a series of researches, inimitable for physical insight and exquisite experimental skill, he has shown how the definition of a line of force or induction can be raised from a merely qualitative or directive one into a quantitative conception by which not only the direction but the magnitude of the induction can be denoted. Having placed clearly before his mind the idea of the surface integral of induction or the total induction through any surface or circuit as the important one to hold in view, he proceeds in the latter part of his investigations (§ 3,152 and § 3,199 "Exp. Res.") to show experimentally that when any closed circuit, such as a loop of wire, is placed in a field of magnetic induction, and if in any way the total induction through that circuit is changed, there is a flow of electricity round the circuit, and the total quantity so set flowing is proportional to the conducting power of the circuit, and to the change in the total magnetic induction passing through it. Employing for this purpose a ballistic galvanometer, or a galvanometer with a needle having a long periodic time of vibration inserted in the circuit, he placed circuits of various forms in fields of induction, and exhausted every possible method of experimental proof that in every case any change which altered the total induction through the circuit was accompanied by the production of an electromotive force in the circuit, and that the product of the total quantity of electricity so set in motion, and the number representing the resistance of the whole circuit, was in every case proportional to the change in the total induction through the circuit. This provides us with the means of defining the strength or magnitude of induction and stating what is meant by a unit of total induction.

*A unit of magnetic induction is a flux or amount of induction such that, when passing through or linked once with a circuit of unit resistance, it gives rise, if suppressed, to a flow of a unit quantity of electricity round that circuit.*

In other words, the above is the definition of what is meant by *one line of induction* (or force), linked once with a single turn of a circuit of unit resistance. We can then define the meaning of the term *density of magnetic induction* or *induction density*. It is the total induction through a unit of area taken normally to the lines of induction at that place. Instead of the term density of induction, it is usual to speak simply of *the induction* at any point in a field of induction, or the number of lines of induction (force) passing normally through a unit of area.

We shall employ the letter $\mathsf{B}$ to stand for *the induction* at any point in a field of magnetic induction. Hence in the notation of the differential calculus $d\,\mathsf{B}$ will stand for any small change in the induction. Let a circuit formed of one single turn of wire whose resistance is R be placed in a uniform field of induction of strength $\mathsf{B}$, and let S be the area of that circuit; the total induction through the circuit will be $\mathsf{B}\,S$. Let the induction be changed by an amount $d\,\mathsf{B}$, then the total induction is changed by an amount $d\,(\mathsf{B}\,S)$. According to Faraday's experiments the change will set a small quantity of electricity, which may be denoted by $d\,q$, flowing round the circuit, and we have

$$d\,(\mathbf{B}\,\mathrm{S}) = \mathrm{R}\,d\,q.$$

If the whole of this change of induction takes place in a very short interval of time, which may be denoted by $d\,t$, then during that time the total flow of electricity is equivalent to an average current of strength $i$, and we have

$$i\,d\,t = d\,q.$$

Hence also

$$d\,(\mathbf{B}\,\mathrm{S}) = \mathrm{R}\,i\,d\,t.$$

But $\mathrm{R}\,i$ is the instantaneous value of the induced electromotive force in the circuit; let this be denoted by $e$, and we have

$$d\,(\mathbf{B}\,\mathrm{S}) = e\,d\,t.$$

or

$$\mathrm{S}\frac{d\,\mathbf{B}}{d\,t} = e. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

Therefore the magnitude of the induced electromotive force in the circuit is at any instant expresssed by the time-rate of change of the total induction passing through the circuit.

If we bear in mind clearly the meaning which Faraday attached to the phrase "a line of force" or "the number of lines of force" as expressing the total induction through any area or conducting circuit, we can express in his language the above fact in a statement which may be called Faraday's Law of Induction; it is as follows:—

If there be in any field of magnetic induction a circuit which is traversed by induction, then any change either of the size or position of the circuit or of the direction or strength of the induction which changes the number of lines of force (induction) passing through the circuit creates an induced electromotive force in this circuit which is numerically equal at any instant to the rate of change in the number of lines of force (induction) so passing through it. The above defines the magnitude of the electromotive force; we have next to define its direction in the circuit. To do this we must make certain conventions. If a watch-face is held in front of the observer, then the *positive direction through* that watch-face is away from the observer, and the *positive direction round* that disc is in the direction in which the hands rotate. The connection between positive direction round and positive direction through can be also fixed by thinking of the direction of the thrust and twist of a corkscrew or other right-handed screw. The negative direction of rotation is therefore the counter-clockwise direction.

In regard to the direction of the induced electromotive force, the following is the law :—The insertion of lines of induction into a circuit in the positive direction, or the increase of positively-directed lines of induction, gives rise to negatively-directed electromotive force. Let the reader bear in mind the following rule: Imagine a magnetic north pole and a magnetic south pole placed opposite to each other, as in the case of a horse-shoe magnet or dynamo field-magnet. The positive direction of the magnetic force in the interspace is by convention *from* the north pole *to* the south pole. This field of force creates magnetic induction in the air-space, and the positive direction of the lines of induction is in the same direction *from* the north pole *to* the south pole in the air-space. Place a watch (non-magnetic) in this space with its watch-face facing the north pole; the watch-face is traversed by a certain flux of induction. Let the rim of this face be thought of as a conducting circuit. Then demagnetise the magnets or withdraw them so as to diminish the induction perforating through the watch-face; it will generate an induced electromotive force in the rim which will tend to set an induced current flowing round the rim in the direction in which the hands of the watch usually rotate, or in the positive direction. Hence as time increases induction diminishes, and we get positively-directed electromotive force. If, then, N represents the number of lines of induction passing positively at any instant through the watch-face and $-\frac{dN}{dt}$ the rate of decrease of induction at any instant, since this creates positively-directed electromotive force, we must write

$$-\frac{dN}{dt}=e.$$

The differential coefficient must, therefore, have the negative sign.

The above rules accordingly settle the magnitude and the direction of the induction and of the rate of change of induction, and hence of the induced electromotive force.

**§ 3. The Magnetic Circuit.—Magnetic Resistance.**—We have in the foregoing section defined magnetic force, magnetomotive force, and magnetic induction, and explained how they are measured. We have in the next place to examine the conditions under which magnetic induction is produced by magnetomotive force. The same magnetomotive force will not always produce the same magnetic induction. What it will produce depends upon the nature ot the *magnetic circuit.* The path of a line of induction is always a closed loop—that is, it is an endless line. In its path it passes either through a magnet or is linked with an electric circuit, and the path of a line of induction is called a magnetic circuit. This circuit may pass wholly through air, or it may pass partly or wholly through iron masses, or of masses partly of iron, air, brass, wood, &c. There are three principal kinds of magnetic circuits. First, those in which the path of a line of induction lies wholly in iron or magnetic metals. Second, those in which it travels wholly in air or in non-magnetic metals. Third, those in which it passes partly through iron masses and partly through air or non-magnetic materials. This distinction is founded upon the fact that there is a very great difference between circuits in which the lines of induction lie wholly in iron, wholly in air or non-magnetic materials, or partly in magnetic and partly in non-magnetic materials.

A given magnetomotive force produces an enormously greater induction when the path of the lines of induction is wholly in iron masses than it does when they are wholly in air. This is analogous to the fact that a given electromotive force produces a much greater current when the electric circuit of given dimensions is composed wholly, say, of copper, than it does when it is composed wholly, say, of carbon. The number expressing the ratio between the numerical value of the electromotive force and that of the total current produced by it in any electric circuit is a measure of the value of the *electrical resistance* of that circuit, and briefly we may say that for unvarying or continuous currents the electrical resistance of a circuit

$$=\frac{\text{The electromotive force acting in the circuit}}{\text{The total current in circuit}}.$$

Similarly the term *magnetic resistance* is applied to that quality of the magnetic circuit in virtue of which a given magnetomotive force produces a certain definite total magnetic induction, and the numerical measure of this magnetic resistance of a circuit is obtained by taking the quotient of the numbers representing the magnetomotive force and the total induction, or the magnetic resistance of a circuit

$$=\frac{\text{The magnetomotive force acting in the circuit}}{\text{The total induction in circuit}}.$$

There is, however, a great difference between electric and magnetic circuits. In the case of electric circuits, though the electric resistance is affected by temperature changes and other physical alterations, yet, when all necessary corrections are made, it is found practically that the electric resistance of a circuit does not depend upon the current flowing through it, whereas in the case of magnetic circuits formed wholly or partly of magnetic metals the magnetic resistance of the circuit does depend essentially upon the value of the induction.

It has not yet been found necessary to coin words analogous to volt, ampere, and ohm, as names for the practical units of magnetomotive force, magnetic induction, and magnetic resistance.

All magnetic circuits which are wholly composed of non-magnetic substances, such as air, wood, brass, &c., have the same magnetic resistance for the same dimensions, and the *specific magnetic resistance* of these substances, or the magnetic resistance per unit length and unit cross-section, is taken as unity. As an instance of the great difference in magnetic resistance between an air circuit and an iron circuit let us consider the following simple case.

Let two rings be made, one of wood and the other of the best soft iron. Let these rings have a circular cross-section of radius $a$, and let the mean radius of the circular axis of the ring be denoted by R, the value of R being large compared with that of $a$. Let each of these rings be wound uniformly and closely over with insulated wire, and let there be N turns on each ring. We have then two circular solenoids, and if a current is passed through each coil of which the strength is A amperes, we have equal magnetomotive forces acting round the axis of

$4\pi n I$

each circular solenoid. This magnetomotive force along the circular axis of the ring is equal to the product of the length $2\pi R$ of the path of the line of induction and the strength of the field along this line, and it is therefore approximately equal to

$$\frac{4\pi}{10}\,\frac{N\,A}{2\pi R}\times 2\pi R = 1\tfrac{1}{4}\,N\,A.$$

The sectional area of the magnetic circuit is $\pi a^2$, since all the lines of induction are confined to the inside of the circular solenoid and from endless circular lines. The mean length of the magnetic current is $2\pi R$. The total magnetic resistance is numerically obtained by multiplying the specific magnetic resistance (which in the case of the wooden ring is equal to unity) by the length and dividing by the section of the path. Hence, the magnetic resistance is equal to $1\times\frac{2\pi R}{\pi a^2}=2\frac{R}{a^2}$, and this is the total magnetic resistance of the circuit. The total induction is numerically equal to the quotient of the magnetomotive force, viz., $\frac{4\pi}{10}N\,A$, by the magnetic resistance, viz., $\frac{2\,R}{a^2}$, and is therefore equal to $\frac{2\pi N\,A\,a^2}{10\,.\,R}$, and *the induction* or *induction density* is equal to $\frac{1}{5}\cdot\frac{N\,A}{R}$, since the section is $\pi a^2$. The above is very nearly true if $a$ is very small compared with R, but if this is not the case, we have to take account of the fact that in this last instance the magnetic force has different values at different points over the cross-section of the solenoid (*see* Appendix, Note B).

As an actual example we may take the case of a wooden ring in which $a = 1{\cdot}27015$ centimetres, $R = 12{\cdot}8291$ centimetres, and $N = 623$. If a current of 1 ampere is sent round this wire, the magnetomotive force on the axis is $\frac{4\pi}{10}\times N\,A = 1\tfrac{1}{4}\times 623 = 779$, and the induction density along the central axis is $\frac{1}{5}\,\frac{N\,A}{R}=\frac{1}{5}\,\frac{623}{12{\cdot}829}=9{\cdot}7$ C.G.S. units nearly, or about 10 lines of induction per square centimetre.

If, however, a soft iron ring of the same dimensions is employed and wound over with the same number of turns and the same current employed, it would have been found that the induction density on the axis would have been about 10,000 C.G.S. units or lines of induction (force) per square centimetre. The same magnetomotive force in both cases, viz., 779, would in one case give rise only to an induction density of 10 C.G.S. units, and in the other case to an induction density of many thousands of C.G.S. units. This is because soft iron has a far less specific magnetic resistance than wood, and a given magnetic force produces far more induction through it than it does in wood.

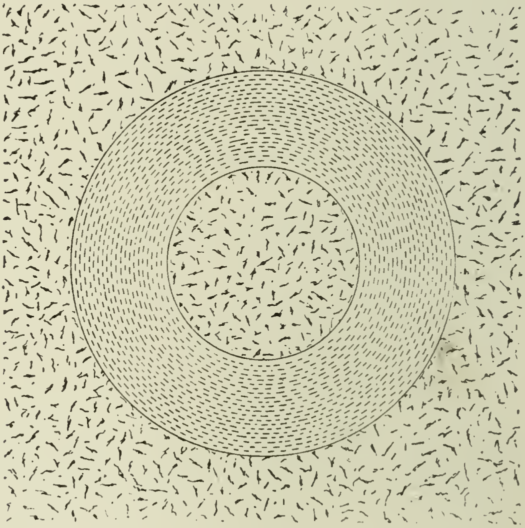

FIG. 6.

Diagram showing the path of the Lines of Induction in a closed Iron Ring when magnetised by a closed endless solenoid or coil wound on it. The dotted lines inside the ring represent the Lines of Induction. There is no sensible field outside the ring.

There is, very roughly speaking, about the same numerical difference between the minimum specific magnetic resistance of soft wrought iron and that of air as there is between the specific electrical resistance of copper and gas retort carbon.

If, instead of forming the core of the solenoid above mentioned wholly of iron or wholly of wood, we make it partly of iron and partly of wood, or employ an iron ring with a cut or air-gap in it, and apply the same magnetomotive force, the result will be that the induction in different parts of the core will be found to be different. There will be a certain number of lines of induction which will be continuous right round the core, and which are determined by the resultant magnetic resistance of the path which is in part of iron and in part of wood or air, and which resultant magnetic resistance will be something intermediate between that of a complete iron core and a complete wooden or air core. There will be, however, an additional induction in the iron, which will be represented by an extra number of lines of induction which pass through the iron but turn back and complete themselves through the space outside. In Figs. 6, 7, and 8 are shown three diagrams illustrating the form of the lines of induction for three cases of complete and incomplete iron circuits. Where the lines of induction enter and leave the iron parts of the core they give rise to *magnetic poles* in the iron, and

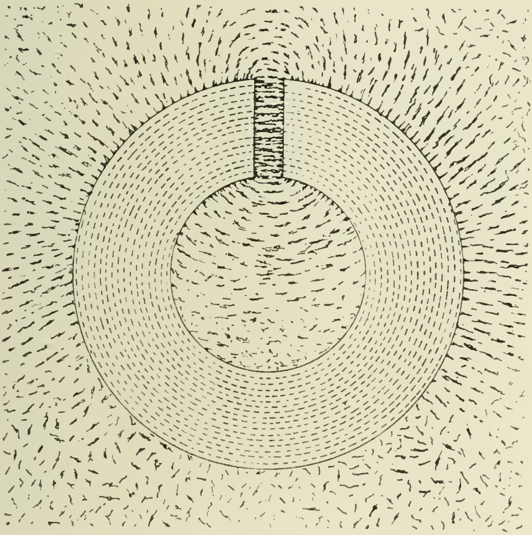

FIG. 7.

Diagram showing the paths of the Lines of Induction in and outside an Iron Ring with an air-gap in it when magnetised by a solenoid wound on it. The lines exterior to the ring are supposed to be delineated by iron filings. The lines inside the ring are represented by the dotted lines.

this additionally complicates the case. These magnetic poles produce a reverse magnetising force in the interior of the iron which opposes the magnetic force due to the magnetising solenoid. Such an imperfect iron circuit is often called an *open magnetic circuit*, whilst the complete iron ring core would be called a *closed magnetic circuit*. The wood, or air, or other body of unit specific magnetic resistance is said to form *a gap* in the iron magnetic circuit.

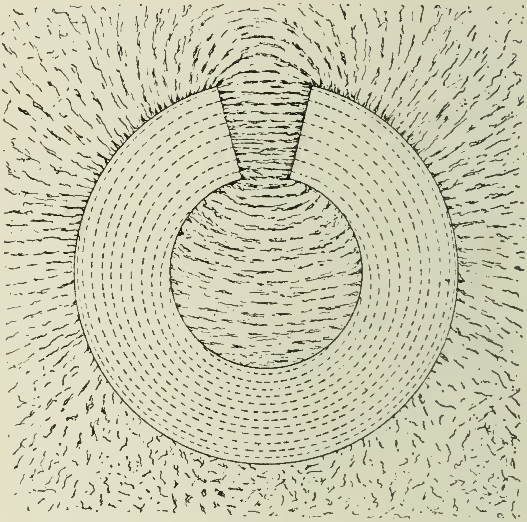

Fig. 8.

Diagram showing the paths of the Lines of Induction in and near an Iron Ring with wider air-gap in it, when magnetised by a solenoid wound on it. The dotted lines show the paths of the lines of induction in the iron core, and the exterior field is supposed to be delineated by iron filings.

Speaking generally, the subject of closed magnetic circuits is more easy to treat and deal with than that of open magnetic circuits. The particular reason why magnetic problems are more difficult to manage than the corresponding electrical problems is because the magnetic resistance even of a closed magnetic circuit is not a constant quantity. It is not only affected by temperature, but is also determined, within very wide limits, by the value of the magnetic induction itself, by the direction of the induction, in that it is not the same for increasing as for diminishing induction when the magnetomotive force, and therefore the induction, is periodic; and, in fact, it is dependent upon the whole past magnetic history of the iron. It is impossible, therefore, to draw up a table of exact specific magnetic resistances of magnetic metals as we can draw up a table of specific electrical resistances, and reduce the calculation of the induction produced by a given steady magnetomotive force to the same simplicity that results from the application of Ohm's law to electric circuits.

The reciprocal of magnetic resistance is *magnetic conductance.* Faraday used the phrase "conducting power for lines of magnetic force" to express the reciprocal notion to that of magnetic resistance. The reciprocal quality to specific magnetic resistance is also called *magnetic permeability.* This is generally denoted by the symbol $\mu$. Hence the magnetic permeability is the magnetic conductance of a magnetic circuit of unit length and unit cross-section. If $\rho$ stands for specific magnetic resistance and $\mu$ for magnetic permeability, we can say that

$$\frac{1}{\rho} = \mu. \quad \ldots \ldots \quad (13)$$

Consider a closed magnetic circuit of uniform specific magnetic conductance $\mu$ and of length $l$ and section $s$, and let a uniform magnetic force $\mathsf{H}$ act in this circuit, producing a uniform induction $\mathsf{B}$. Then by definition the magnetic resistance R of the circuit is given by

$$\mathrm{R} = \frac{\mathsf{H}\,l}{\mathsf{B}\,s} = \frac{\text{Magnetomotive force}}{\text{Total induction}},$$

where $\mathsf{H}\,l$ is the magnetomotive force, and $\mathsf{B}\,s$ the total induction; but

$$\mathrm{R} = \frac{\rho\,l}{s} \text{ and } \mu = \frac{1}{\rho};$$

therefore we have

$$\mathsf{B} = \mu\,\mathsf{H}. \quad \ldots \ldots \quad (14)$$

This equation (14) is not only a fundamental equation in magnetism, just as Ohm's law is in electro-kinetics, but it furnishes a definition of the method of measuring magnetic permeability.

Experiment shows, as stated, that the ratio **B** to **H**, expressed by the equation $\frac{\mathbf{B}}{\mathbf{H}}=\mu$, is not of a determinate character, and that the value of $\mu$, so far from being constant, is dependent on the whole previous magnetic history of the iron, on the value of **B**, and on the nature of the magnetic changes the iron is undergoing, viz., whether **H** is increasing or diminishing. In fact, $\mu$ for iron varies from a value not far from unity for strongly magnetised iron up to a value of 2,000 or 2,500 at a maximum for closed magnetic circuits of best soft iron, and down to a value of about 100 for very feeble magnetising forces.

In a cycle of magnetic operations, during which a bar of iron is exposed to increasing magnetising forces and then afterwards to decreasing ones, beginning and ending with the same force, the value of **B** is always greater on the descending than on the ascending course, and hence the value of $\mu$ which is given by the ratio of **B** to **H** is also different. This phenomenon, which is exemplified familiarly by the retention of magnetism in a bar after withdrawal of the magnetising force, has been named by Prof. Ewing hysterēsis (from ὑστερέω, to lag behind).

The magnetic permeability above defined is a quantity which is in magnetism the analogue of *specific inductive capacity* in electrostatics, and of the conducting power of a body for heat. It was spoken of by Faraday as *the conducting power of a magnetic medium for lines of force* ("Exp. Researches," Ser. XXVI., § 2,797 and § 2,846). More recently the reciprocal of $\mu$ has been called the *magnetic reluctance.* The term magnetic resistance has, however, become so sanctified by use that we shall continue to employ it in spite of certain objections which have been urged against it.

The magnetic resistance of a circuit, composed partly of iron and partly of air, is greater in proportion as the length of the air path is increased in proportion to that of the iron. This fact is strikingly shown in experiments on the magnetic induction produced in closed rings and short bars of soft iron. Thus from some curves given by Prof. Ewing we find that in a certain soft iron ring a magnetising force of 7 C.G.S. units produced an induction of 11,000 C.G.S. units; whereas, in the case of an iron rod, of which the length was 50 times the diameter, the same force produced an induction of only 4,000 units. In the first case the circuit was a complete iron circuit, and the resistance small. In the second case the magnetic circuit was partly iron and partly air, and therefore the total magnetic resistance was much greater.

The equation $\mathsf{B} = \mu \mathsf{H}$ is the expression of the fundamental relation between magnetic force and magnetic induction. For all ordinary non-magnetic materials the value of $\rho$ and also of its reciprocal $\mu$ is taken as unity. Hence, in these cases the magnetic force and magnetic induction have the same value and are in the same direction. In the case of such diamagnetic bodies as bismuth and phosphorus, $\mu$ is very slightly less than unity and $\rho$ very slightly greater, but so little as to be unappreciable except to refined experiments. In speaking of air circuits, or non-magnetic circuits generally, we can speak of lines of force or lines of induction indifferently, as the force and induction have everywhere the same numerical value and same direction, but in dealing with magnetic circuits of permeability greater than unity, the magnetic force and magnetic induction must be carefully distinguished, because they have not the same magnitude and may not have the same direction. In crystalline magnetic bodies the induction might have a very different direction and value to the force, and as we have seen in the case of soft iron, the induction may be, numerically, two or three thousand times greater than the magnetic force.

It should be borne in mind, therefore, that in the air-space outside a magnet or a mass of iron under induction the magnetic force and magnetic induction have the same direction and same numerical value, but inside a magnet or mass of iron under induction they must be distinguished. The magnetic force outside the magnet may be called the magnetic induction through the air, and generally in the non-magnetic material surrounding the magnet the magnetic force and magnetic induction are, for the purposes of measurement, one and the same. In the interior of a mass of iron under induction in a magnetic field, the magnetic force at each point is one compounded of that due to the external or original field and that due to the induced polarity acquired, and which acts to produce an opposing magnetic force. Hence the effect of the induced poles on any element in the interior of the iron is to tend to demagnetise it when the external magnetising force is withdrawn. In the inside of a straight uniformly magnetised bar the magnetic force due to the influence of the poles themselves is *from* the end which points to the north *to* the end which points to the south, both within the magnet and in the space outside. The magnetic induction, on the other hand, is from the north pole to the south pole *outside* the magnet, and from the south pole to the north pole *inside* the magnet. A line of induction followed round, moving always in the positive direction, is found to be a closed loop or endless line.

It is a fact of fundamental importance that a thin disc of iron or steel, magnetised so that at all points the magnetic axis of each small element of it is in a direction normal to its surface, produces a magnetic field identical with that produced by a wire conveying an electric current coinciding in form with the edge of the disc. In short, the magnetic field of a closed circuit conveying a current is identical with that of a *magnetic shell* filling up the aperture of the circuit. The magnetic field due to electric currents circulating in conductors is, however, of such a nature that each line of induction *embraces* or surrounds the axis of the circuit once or more. The magnetic field due to permanent or electromagnets is of such a nature that each line of induction *passes through* the magnet, giving rise to magnetic polarity at the places where it enters and leaves the iron or steel. Every line of induction either surrounds an electric current or passes through magnetised iron.

The *intensity of magnetisation* of any element of a magnet or mass of iron under induction is a term requiring definition.

The couple required to hold a very small magnet when placed with its axis of magnetisation perpendicularly across the lines of force of a uniform magnetic field in air of unit strength is a numerical measure of the *moment* of the magnet.

The moment of a magnet or of any element of a magnet may be considered numerically to be made up of two factors —one its volume, and the other its *intensity of magnetisation*, or, simply, its *magnetisation*; and hence, for a uniformly magnetised small linear needle, we may define the intensity of its magnetisation by saying that it is the magnetic moment of unit volume of it. Intensity of magnetisation is, like force and induction, a vector quantity.* In the case of a very long thin wire of soft iron placed along the lines of force of a uniform field the three quantities—the magnetic force, the magnetic induction, and the magnetisation—are all in alignment at any point.

It is important that the full meaning of the phrase "magnetic force at any point" should clearly be grasped. If there be any uniform magnetic field of strength $\mathsf{H}_0$, and in it is placed a mass of iron in the shape of an elongated bar, the configuration of this uniform field is disturbed and magnetic polarity is developed in the iron. At any point in the interior of the iron there is a magnetising force, henceforth denoted by the letter $\mathsf{H}$, which is due partly to the original magnetic field $\mathsf{H}_0$ and partly to the induced poles which create a force opposing $\mathsf{H}_0$. This resultant magnetic force is spoken of as the magnetising force in the iron, and it is the resultant of the external magnetic forces and the internal magnetic forces due to the polarity. If the form of this bar is so chosen that there are no magnetic poles, as in the case of a ring lapped over with an endless solenoid, then the magnetic force in the iron is easily calculable, and it is that due to the external magnetic force alone. If the bar is straight and very long the induced magnetic poles may exert so little effect at the centre of the bar that the induction there and the magnetic force also is that due to the external field alone.

At each point in the iron the *magnetic force* $\mathsf{H}$ must be thought of as producing *magnetisation* or magnetic displacement, just as in electrostatic phenomena electric force produces in a dielectric electric displacement or electric strain, or just as mechanical stress produces in an elastic body ordinary strain or displacement at every point. This magnetisation is not necessarily in the same direction as the force.

§ **4. Lines and Tubes of Magnetic Induction.**—Faraday and Maxwell have raised the conception of a line of magnetic induction from a simply directive notion to one which enables it to be used to convey a quantitative knowledge of the magnetic field—in other words, have enabled lines of induction to be used not only to show the direction of the induction, but also its magnitude in certain units. By this means the magnetic field can be mapped out into areas and volumes which have a definite dynamical signification.

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* A vector quantity is one which is only precisely fixed when we know its direction as well as magnitude.

In the twenty-eighth series of his "Researches on Electricity," Faraday lays stress on the fact stated above that every line of force (induction) is an endless loop (§ 3,117, "Exp. Res."):—"Every line of force must therefore be considered as a closed circuit passing in some part of its course through a magnet, and having an equal amount of force in every part of its course. There exist lines of force within the magnet of the same nature as those without. What is more, they are exactly equal in *amount* to those without. They have a relation in direction to those without, and are, in fact, continuations of them."

Let a magnetic field have drawn in it a number of closely contiguous lines of induction. None of these lines can cut each other, because the resultant magnetic induction at any point can have only one definite direction.

In any region it is possible to describe a surface perpendicular to all the lines of induction. Such a surface is called a level surface.

In the case of a straight infinite current, these level surfaces will be planes radiating out from the axis of the wire, and their traces on a plane perpendicular to the axis of the wire will be a series of radial lines cutting all the circular loops of induction normally. Let A (Fig. 9) represent such a level surface, and let B be another, both cutting the same sheaf of magnetic lines of induction.

On the level surface A let any unit of area $a$ be taken, and let this area $a$ be projected on to the adjacent level surface B by lines of induction drawn through its boundary. We have then a tubular surface, the ends of which are formed of portions of level surfaces, and the rest of the tubular surface may be conceived to be formed of lines of magnetic induction, supposed to be very closely drawn through the bounding line. Such a geometrical conception is called a *tube of induction.*

The characteristic quality of a tube of induction is as follows. If the areas of the sections made by the two level surfaces A and B be called $s$ and $s'$, and if $\mathbf{B}$ be the mean magnetic induction over $s$, and $\mathbf{B}'$ that over $s'$, then $\mathbf{B}\,s = \mathbf{B}'\,s'$, or the product of a normal cross-section of tube and mean magnetic induction over that section is constant for all sections of the tube.

If any level surface be drawn, and on this surface be marked off contiguous small areas such that the magnitude of the area is inversely as the mean value of the magnetic induction over that little area, and if $s$ and $\mathbf{B}$ are, as before, the numerical values of any small area and the mean induction over it, then the product $\mathbf{B}\,s$ may be made equal to unity for each of these portions of that level surface. From these small areas let tubes of induction be supposed to take their rise, the whole field will be cut up into contiguous tubes of induction. Each of these tubes is called a *unit tube of induction.* By their mode of description these tubes will have small cross-section at places where the field is strong and widen out in section at places where it is weak, and by the fundamental property of the tubes the value of the mean magnetic induction at any place is inversely as the cross-section of the tubes of induction force at that place.

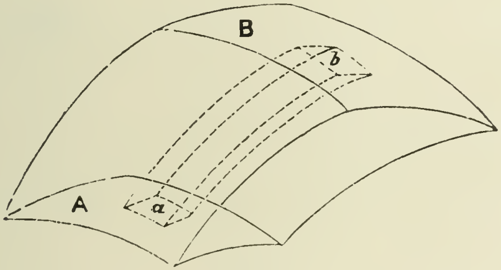

Fig. 9.

From Faraday's point of view, a magnet of any form must be mentally pictured as surrounded with and as having its whole field filled up by a closely packed arrangement of such unit tubes of induction, the tubes being intersected at right angles by the equipotential or level surfaces, and each having at any point a normal cross-section which is inversely as the magnetic induction at that point. This system of tubes must be supposed to be rigidly attached to the magnet, and to move with it wherever it goes. Furthermore, in accordance with Faraday's conception, each tube is an endless tube, or, as it were, a pipe returning into itself and passing in some part of its course through a magnet or round an electric current. In constructing what may seem to the student to be a highly artificial conception, we are not postulating *necessarily* any physical existence for these tubes. They should be regarded simply as a device for plotting out the space round a magnet according to a definite rule, and may, in the first place, be regarded as no more than analogous to such subdivisions of the earth's surface as we make by lines of latitude and longitude.

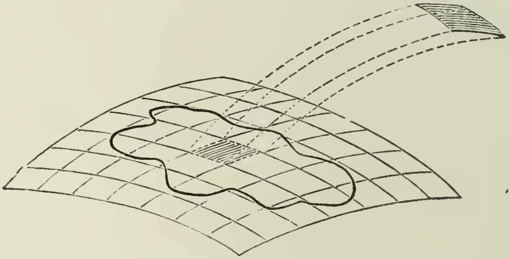

FIG. 10.

Having thus divided up a magnetic field into unit tubes of induction, it is simpler in thought to suppose a single line of induction to run down the axis of each tube, and then to mentally disregard the tubular system, and, instead of speaking of a unit tube, to speak of each as a single line of induction. If we imagine a system of induction tubes starting from an equipotential surface and draw any irregular curve on this surface, we shall find that this curve encloses a certain number of tubes or lines of induction (Fig. 10). Bearing in mind that the cross-section $s$ of the tube where it sprouts out from the equipotential surface is inversely as the magnetic induction $\mathsf{B}$ at the centre of this cross-section, it is at once evident that the greater the average induction over the area defined so much the more numerous will be the number of tubes or lines of induction which pass through it. If the cross-section $s$ of each tube should happen to be equal, and there be $n$ tubes passing altogether through an area equal to S, bounded by the black line, then by the very definition of a unit tube

$$\mathbf{B}s=1,$$

or

$$n\mathbf{B}s=n;$$

but

$$ns=\mathrm{S},$$

hence

$$\mathbf{B}\mathrm{S}=n;$$

and the number of tubes passing through any area S on such an equipotential surface in a uniform field is numerically equal to the product of the whole area and of the induction at any point on that area.

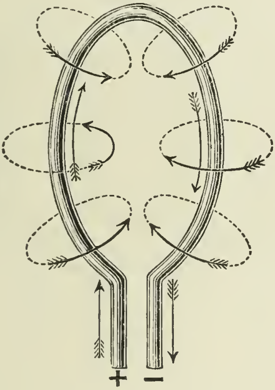

FIG. 11.

The characteristic quality of a tube of induction is that the flux of induction along it is constant throughout its length. That is to say, the product $\mathbf{B}s=\text{induction}\times\text{cross-section}$ is constant, and, since what is true of one tube is true of all, we may say that in a space wholly made up of tubes of magnetic induction the total magnetic induction or flux of induction is the same across all sections of this mass of tubes.

The lines of induction of a permanent steel magnet are to be thought of as closed loops which pass in their course partly through the steel and partly through the air. The lines of induction of a circuit conveying an electric current are closed loops entirely surrounding the axis of the wire. If this circuit is a straight wire, with the return wire at a very great distance, the lines of induction are concentric circles described on a plane perpendicular to the axis of the wire, and having their centre in that axis.

If a circuit is formed by coiling up into a circular coil a length of insulated wire, the coil having $n$ turns, then to a first degree of approximation we may say that each line of induction forms a closed curve embracing the circuit $n$ times. Thus, if the wire forms a coil of one turn (Fig. 11) each line of induction (represented by the dotted line) is a closed loop embracing or linked with the circuit once. If we take a circuit of two turns (Fig. 12), then nearly all loops of induction belonging to one single turn embrace not only that turn but the adjacent turn, and if the circuit could be supposed to be opened out straight (Fig. 13), without destroying the loop of induction, it would be found to be twisted *twice* round that circuit. By similar reasoning, if a loop of induction embraces or is linked with $n$ turns of a conducting circuit, it is in fact the same as linking each loop of induction $n$ times with the single circuit.

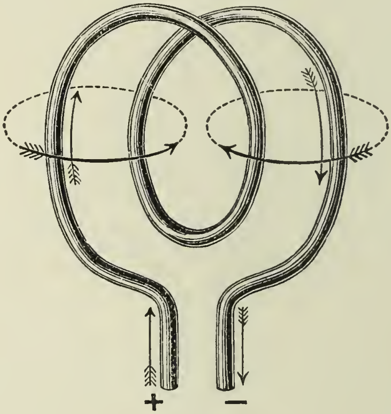

Fig 12.

Let a conducting circuit have the form of a helix (Fig. 14), then the lines of induction are closed loops, which embrace some or all the turns of the spiral, and, if the helix have $n$ turns, then each loop of induction, according to its length and position, in reality embraces that circuit 1, 2, 3......or $n$ times. If there be two circuits, in one or both of which currents are flowing, then each circuit is surrounded by lines or loops of induction, and of those belonging to one circuit some or all are linked in as well with the other circuit, so that a certain number of all the loops of induction are common to the two circuits, and are called the loops or lines of mutual induction.

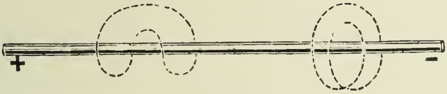

Fig. 13.

It is exceedingly convenient to think of these lines or tubes of induction as *linked* with various circuits. A line of induction is always linked with an electric circuit or else passes through a magnet in some part of its course. If any closed conducting circuit is placed in a field of magnetic induction, it may be thought of as linked with a certain number of unit lines or tubes of induction. If the circuit is moved or the field is changed, the *number of linkages* is altered or may be altered. If at any instant N unit tubes or lines of induction are linked with any circuit, and at a very short interval of time

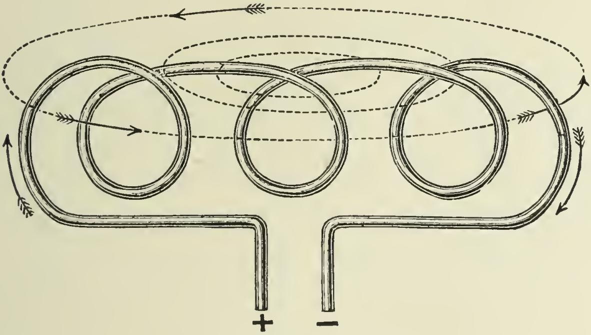

Fig. 14.

afterwards, say after an interval $t$, the number of linkages is N′, then N − N′ is the change in the linkage, and $\frac{N - N'}{t}$ is the rate of change of linkage, and this, by Faraday's law, is the numerical value of the electromotive force set up in the circuit.

If during any period of time a circuit is exposed to magnetic induction the rate of change of which varies, then from instant to instant the impressed inductive electromotive force varies and may be represented graphically as follows: Let the straight line O X (Fig. 15) be an axis on which lengths are marked off to represent intervals of time, and let ordinates perpendicular to it represent the instantaneous value of the flux of induction through, or lines of induction included by, a circuit; then, if the variation of induction is continuous, it may be represented by a curve drawn between these axes. This curve may be called a curve of induction. If at any point P a tangent P T is drawn to this curve, the trigonometrical tangent of the angle P T M represents the rate of variation of P M with respect to O M, and if P M represents at any instant the induction N, then the *slope* of the tangent at P represents $\frac{dN}{dt}$, or the rate of change of N with respect to time.

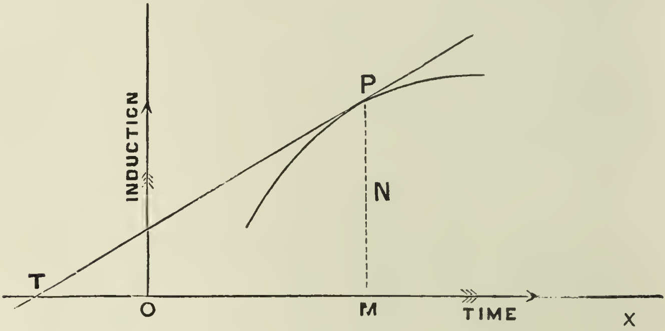

FIG. 15.

In the practical application of the above rule it must be borne in mind that if N, or the number of lines or linkages, is measured in units based on the centimetre, gramme, and second system, then the electromotive force is given in the same units.

Since one volt is $10^8$ C.G.S. electromagnetic units, to get the electromotive force in volts we must divide the time rate of change of induction by $10^8$. Thus, if the change of induction be such that N C.G.S. lines are removed uniformly from the circuit per *minute*, the electromotive force in volts set up will be

$$\frac{N}{60} \cdot \frac{1}{10^8} \text{ volts.}$$

§ **5. Curves of Magnetisation.**—One of the most important practical problems in magnetism is to discover the manner in which the induction $\mathbf{B}$ varies with the magnetising force $\mathbf{H}$ in different cases. Since this variation is often very complicated, it is best represented by a curve of which the ordinates are taken as proportional to the induction and the abscissæ as proportional to the magnetising force. Such curves are called *curves of magnetisation.* We shall consider a few special cases and describe the mode of practically determining these curves. The principal instance is the curve of magnetisation of a closed iron ring. Let an iron ring be made of circular cross-section radius $a$ and be circular in form, the radius of the mean circular central axis being R. Let $a$ be very small compared with R. Let such a ring be wound over uniformly with N turns of insulated wire, and let a current of strength A amperes be sent through this wire. The magnetic force creates an induction in the iron, and the lines of induction are circles lying wholly in the iron. Inside the circular solenoid the magnetic force is nearly the same in value everywhere, and is equal to $\frac{4\pi}{10} n$ A, where $n$ is the number of turns of the wire per unit length of the solenoid; or $n = \frac{N}{2\pi R}$. Hence we can calculate the magnetic force acting everywhere on the iron. Let the ring be also wound over with a second insulated coil of wire of N′ turns, and let this secondary coil be connected to a ballistic galvanometer or a galvanometer suitable for measuring quantity of electricity. If the current in the primary coil is altered in strength or reversed, the magnetic force undergoes a change, and the induction changes also. Since the lines of induction all pass through, or are linked with, the turns of the secondary coil, this change in induction will, by Faraday's law, create a flow of electricity in the secondary circuit. If the current in the primary coil is suddenly changed from one value A amperes to another value A′ amperes, a certain quantity of electricity will flow through the secondary coil and galvanometer, and it will cause the galvanometer needle to deflect through an angle $\theta$. If $k$ is the galvanometer constant (*see* Appendix, Note C), then the whole quantity of electricity which flows through the galvanometer is equal to Q, where $Q = k \sin\frac{\theta}{2}\left(1 + \frac{\lambda}{2}\right)$, $\lambda$ being the logarithmic decrement of the galvanometer.

If R′ is the total resistance of the secondary coil on the ring, together with that of galvanometer coils and connections, then, as above shown, the total change in the induction through the secondary circuit is equal numerically to R′ Q. If $\mathbf{B}$ is the mean induction density in the interior of the circular solenoid, S the mean cross-sectional area of the primary coil, and N′ the number of turns on the secondary coil, then it is clear that $\mathbf{B}$ S N′ represents the total induction passing through the secondary circuit, or the number of lines of induction which are linked with the galvanometer circuit.

Hence if the induction is suppressed by stopping the current the total change in induction is $\mathbf{B}$ S N′, and this is equal to R′ Q. If the current, instead of being simply stopped, is reversed, then the total change in induction is 2 $\mathbf{B}$ S N′. Therefore we have

$$\mathbf{B}\,S\,N' = R'\,Q \text{ for stoppage of current ;}$$

or
$$2\,\mathbf{B}\,S\,N' = R'\,Q \text{ for reversal of current.}$$

By our fundamental equation

$$\mathbf{B} = \mu\,\mathbf{H}.$$

In the above case the average magnetic force $\mathbf{H}$ is equal to $\frac{4\pi\,N\,A}{10\,l}$, in which formula N is the number of primary turns, A the primary current in amperes, and $l$ the mean perimeter of the primary solenoid, which last is equal to $2\pi$ R, where R is the mean radius of the primary solenoid. Hence we find that

$$S\,N'\,\frac{4\pi\,N\,A\,\mu}{10 \times 2\pi\,R} = R'\,Q, \quad . \quad . \quad . \quad . \quad (15)$$

or

$$S N' \frac{8\pi N A \mu}{10 \times 2\pi R} = R' Q, \quad . \quad . \quad . \quad . \quad (16)$$

according as the primary current is stopped or reversed. Taking the latter as the usual case, we find finally that

$$S N' \frac{8\pi N A \mu}{10 \times 2\pi R} = R' k \sin \frac{\theta}{2} \left(1 + \frac{\lambda}{2}\right),$$

or

$$\mu = \frac{5}{2} \frac{k \, . \, R' \, . \, R \, . \, \sin \frac{\theta}{2} \left(1 + \frac{\lambda}{2}\right)}{S \, . \, N \, . \, N' \, . \, A \, .}. \quad . \quad . \quad . \quad (17)$$

Hence $\mu$ is determined in terms of nine quantities, all of which can be easily and exactly measured. If all quantities are measured in centimetres and seconds, then it must be particularly noted that the galvanometer constant $k$ is the number by which the corrected sine of half the angle of throw has to be multiplied in order to obtain the quantity of electricity producing that throw, estimated *in absolute C.G.S. electromagnetic measure*, which has flowed through the galvanometer. Generally speaking, it is most convenient to find $k$ by determining the *throw* produced by a discharge through the galvanometer of a certain number of *microcoulombs*, obtained by charging a condenser of known capacity in *microfarads* with a certain number of volts. Then it must be remembered that a microcoulomb is $10^{-7}$ of an electromagnetic unit of quantity in C.G.S. measure.

By the help of the above formula we can deduce the values of $\mu$ for the iron ring corresponding to certain values of **B** or **H**, and tabulate them. For instance, taking a perfectly new iron ring, we can apply gradually increasing values of **H**, increasing by small steps, and obtain the corresponding values of **B**, and then draw curves representing the relation of **B** and **H** and of **B** and $\mu$. Such curves have been given by many observers: Rowland, Hopkinson, Ewing, Shelford Bidwell, and others. For the sake of showing what are the sort of values of **B** and $\mu$ corresponding to certain values of **H**, a table is given on the next page of results of an experiment by Shelford Bidwell on a soft iron ring, and in Figs. 16 and 17, page 55, are given diagrams showing the forms of the curves of induction and permeability for such a ring.

*Values of the Permeability and Induction corresponding to various Magnetising Forces for Closed Iron Magnetic Circuits.*

| H. Magnetising Force. | B. Induction. | $\mu$. Permeability. |
|---|---|---|
| 0·2 | 80 | 400 |
| 0·5 | 300 | 600 |
| 1·0 | 1,400 | 1,400 |
| 2·0 | 4,800 | 2,400 |
| 3·9 | 7,390 | 1,899 |
| 5·7 | 9,240 | 1,621 |
| 10·3 | 11,550 | 1,121 |
| 17·7 | 13,630 | 770 |
| 22·2 | 14,450 | 651 |
| 30·2 | 15,100 | 500 |
| 40 | 15,460 | 386 |
| 78 | 16,880 | 216 |
| 115 | 17,330 | 151 |
| 145 | 17,770 | 122 |
| 208 | 18,470 | 89 |
| 293 | 18,820 | 64 |
| 362 | 19,080 | 52·7 |
| 427 | 19,330 | 45 |
| 465 | 19,470 | 42 |
| 503 | 19,480 | 38·7 |
| 585 | 19,820 | 34 |
| 24,500 (Ewing) | 45,300 | 1·9 |

It will be noticed that the value of the permeability rises very rapidly to a maximum, but that with increasing magnetising force it finally diminishes again, and that in very strong magnetic fields iron becames scarcely much more permeable than air. Although there is apparently no limit to the induction or number of lines of force which can be forced through iron, yet it seems as if the excess of lines of force generated in the iron, over and above that which would exist if the iron were not there, is limited. Hence, if we consider the strength of the original field as numerically defined by a certain number, the number expressing the induction when a closed circuit of iron is placed in that field is obtained by using a certain multiplier, which becomes less and less as the magnitude of the field increases.

On looking at these permeability curves it will be seen that the permeability rises to a maximum for a certain value of the induction and then falls again. It has been shown by Lord Rayleigh that for very small magnetic forces soft iron has an initial permeability of about 100. Its maximum permeability is about 2,500, and for very great inductions it falls again to something not much greater than unity. Hence the specific magnetic resistance curve is the inverse of the above curves, and the specific magnetic resistance has a minimum value which for a soft iron ring appears to correspond to an induc-

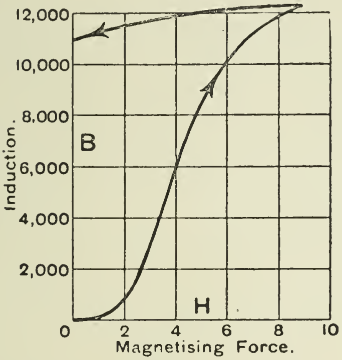

FIG. 16.
Curve of Magnetisation for rising and falling magnetisation in a Soft Iron Ring.

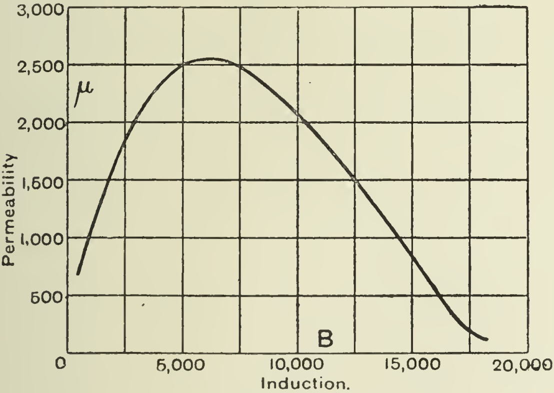

FIG. 17.
Permeability Curve for Soft Iron Ring (Rowland).

tion density of about 5,000 C.G.S. (lines per square centimetre). At less or greater induction densities the specific magnetic resistance is increased.

Experiments to determine $\mu$ can only be performed properly on very long bars or rings of iron placed in uniform predetermined magnetic fields by lapping over the ring with insulated wire or placing the bar in a helix, so that an electric current traversing this wire generates a field having known values at each point in the interior of the coil. Such experiments have been carried out extensively by various experimentalists, and the results embodied in curves called *permeability curves.** The form of these permeability curves is considerably affected

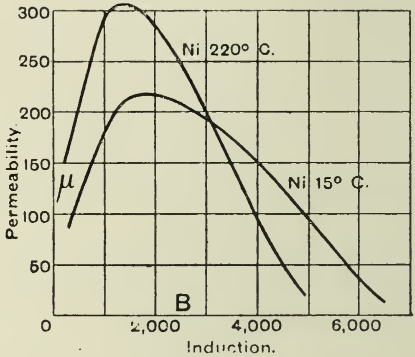

FIG. 18.

Permeability Curves for Nickel (Rowland).

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* Accounts of experiments and investigations on the form of the permeability and susceptibility curves for iron and other paramagnetic metals will be found in the following memoirs :—Weber, *Electrodynamische Maasbestimmungen*, Bd. III., § 26 ; Von Quintus Icilius, *Poggendorff's Annalen*, CXXI., 1864 ; Oberbeck, *Pogg. Ann.*, CXXXV., 1868 ; Riecke, *Pogg. Ann.*, CXLI., 1870 ; Stoletow, *Pogg. Ann.*, Ergbd. V., 1870 ; Rowland, *Phil. Mag.*, Ser. IV., 1873, p. 336, 1874, p. 254 ; Bouty, *Comptes Rendus*, 1875 ; Fromme, *Pogg. Ann.*, Ergbd. VII., 1875 ; Warburg, *Wiedemann's Annalen*, XIII., 1881 ; Bidwell, *Proc. Roy. Soc. Lond.*, No. 245, 1886 ; Bidwell, *Proc. Roy. Soc. Lond.*, Vol. XLIII., 1888 ; Bidwell, *Proc. Roy. Soc. Lond.*, No. 242, 1886 ; Ewing, *Trans. Roy. Soc. Lond.*, Part. II., 1885 ; Hopkinson, *Trans. Roy. Soc. Lond.*, Part II., 1885.

by temperature, and for each magnetic metal there appears to be a temperature at and beyond which it is not much more permeable than air. The permeability of nickel and cobalt varies very much with temperature. In Figs. 17 and 18 are shown the permeability curves for iron and for nickel for two very different temperatures. At about 750°C. iron, and at about 400°C. nickel, possess a permeability not much greater than air.* In cobalt, permeability appears to be increased up to about 150°C. and then diminished.

In Fig. 19, page 58, is shown the form of the magnetisation curve of a long soft iron rod. It will be seen that the curve is divided roughly into three parts: first, a part in which magnetic induction increases slowly with magnetising force; second, a part in which it increases very rapidly; and third, a part when it increases somewhat more slowly again. Generally speaking, it is difficult to force up the induction in a soft iron ring or long rod to a greater value than about 18,000 or 20,000 C.G.S. units, but there is no actual physical limit to the amount of induction to be created in iron; the sole limit seems to be the difficulty of obtaining sufficient magnetising force. If instead of operating on the best soft iron we had selected a hard iron or a steel, it would have been found that the magnetisation curves would not be quite of the same form, but that for a given magnetising force there would be less induction. As soon as the induction reaches the point where the magnetisation curve becomes approximately flat the iron is said to be *magnetically saturated.*

If instead of operating on an endless iron circuit or ring we select an iron rod, say 200 diameters long, and wind it over with a magnetising coil, we can, by means of the ballistic galvanometer, determine for it in the above way a magnetisation curve. For if we surround the centre of the rod with a secondary coil, and connect this to the galvanometer, we can, by making changes in the strength of the primary magnetising current, alter the induction through the secondary circuit, and so obtain throws of the galvanometer indicating certain quantities of electricity discharged through it. If the primary helix is very long compared with its diameter we can estimate the strength of its interior field, or the magnetising force, as everywhere approximately equal to $4\pi/10$ times the ampere-turns per unit of length of the helix. Hence we know the

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* See *Proc. Physical Soc*, *London*, Vol. IX., p. 187: Mr. Tomlinson on "The Temperature at which Nickel begins to Lose its Magnetic Qualities."

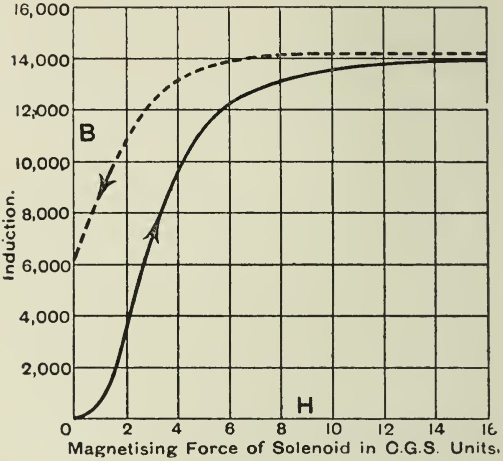

FIG. 19.

Curve of Magnetisation for rising and falling magnetism in an Iron Rod. Length = 200 diameters (Ewing).*

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* This curve is taken from Prof. J. A. Ewing's Paper, "Experimental Researches in Magnetism," *Trans. Royal Soc.*, Part II., 1885, p. 535. The treatise by Prof. Ewing on "Magnetic Induction in Iron and Other Metals," in "*The Electrician* Series" of Standard Books, will furnish the student with the most complete account of modern magnetic research, and hence it has not been considered necessary to amplify very much this section of the present work.

*See also* Dr. J. Hopkinson, "Magnetisation of Iron," *Trans. Royal Soc.*, Part II., 1885, p. 455.

A very complete summary of recent research in magnetism is to be found in Prof. Chrystal's article, "Magnetism," in the *Encyclopædia Britannica*, Ninth Edition.

*See also* "On the Lifting Power of Electro-Magnets and Magnetisation of Iron," Shelford Bidwell, *Proc. Royal Soc*, June 10, 1886.

Other references to valuable Papers on the magnetisation of iron are—

Lord Rayleigh, "On the Energy of Magnetised Iron," *Phil. Mag.*, August, 1886, p. 175.

Lord Rayleigh, "On the Behaviour of Iron and Steel under the Operation of Feeble Magnetic Forces," *Phil. Mag.*, March, 1887, p. 225.

Ewing and Low, "On the Magnetisation of Iron in Strong Fields" *Proc. Royal Soc.*, March 24, 1887, Vol. XLII., p. 200.

force, and the induction is ascertained when we know the quantity of electricity discharged through the galvanometer on a reversal of the primary current. If the secondary coil makes N′ turns round the iron, and if the whole resistance of the secondary circuit, including galvanometer, is equal to R′, and if S is the cross-section of the rod, then $2\,\mathsf{B}\,S\,N'$ is the total change in induction through the secondary circuit obtained on reversing the primary current, and this is equal to $R'\,Q$, where Q is the quantity of electricity discharged through the galvanometer. After finding in this way a series of values of **B** and the corresponding values of **H**, we have the means of determining the values of $\mu$ for varying values of **B**. If, as in the above case, the iron rod is not endless, the values of $\mu$ so determined will be smaller than the corresponding values of $\mu$ for an iron ring of the same iron, and will be less as the iron rod is made shorter, because a larger proportion of the magnetic circuit is then formed of air and a less portion of iron.

§ **6. Magnetic Hysteresis.**—If an iron ring or rod is subjected to a cycle of magnetising force in which the force beginning at zero rises up gradually to a maximum in one direction, and is then reversed and made a maximum in the other direction, and finally reduced again to zero, we find that the following phenomena exhibit themselves. The induction in the iron—and, therefore, its magnetisation—has a higher value at all points during the descent of the force than during its ascent. Hence, if a curve is plotted in which horizontal abscissæ represent magnetic force and vertical ordinates induction or magnetisation, we obtain a curve of the kind shown in Fig. 20, which is a loop or encloses an area. If at any point in the cycle we stop and reverse the magnetism a small or subsidiary loop (*see* Fig. 21) is formed on the principal curve. This phenomena is called *magnetic hysteresis*, because the magnetism or induction "lags behind" the magnetic force. If the induction **B** and the magnetic force **H** are the variables in terms of which the curve is plotted, the curve is called a **B H** curve of hysteresis. We may next consider the physical meaning of this curve. Consider as before a ring of iron of cross-section S and mean perimeter $l$ wound over with N turns of a magnetising coil. Let a current of A amperes be sent through this coil, there are then N A ampere-turns acting on it, and the magnetic force operating on the iron is $\frac{4\pi}{10}\,\frac{N\,A}{l}$. At any instant let the difference of potential at the ends of the magnetising coil be V volts, and let the coil have a resistance of R ohms. If the induction in the iron has a value $\mathbf{B}$, the total number of lines of induction linked with the coil is $\mathbf{B}\,S\,N$.

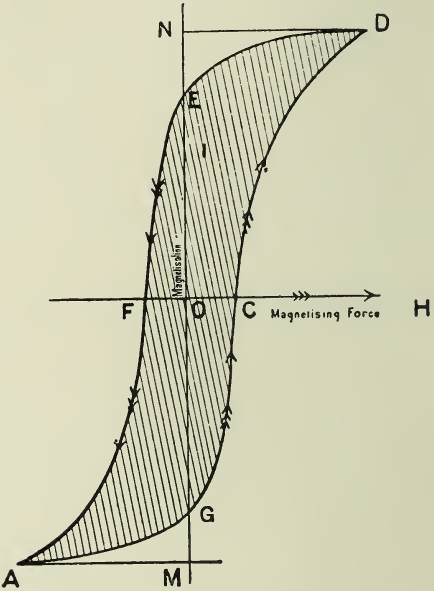

FIG. 20.

Complete Magnetisation Curve for Soft Iron Ring carried from strong positive to strong negative magnetisation. The arrows show the direction of the magnetising operation, and the shaded area the work done due to hysteresis.

If we make a change in the potential and increase the volts to $V + \delta V$, and at the same time increase the current from A to $A + \delta A$, and the induction from $\mathbf{B}$ to $\mathbf{B} + \delta \mathbf{B}$, in the small time $\delta t$ the following relations between these increments will exist: The time rate of change of the induction is in the limit equal to $\frac{d\mathsf{B}}{dt}$, and the time rate of change of the whole number of lines of induction linked with the circuit is $\mathrm{S\,N}\,\frac{d\mathsf{B}}{dt}$. Hence this last is numerically equal to the induced electromotive force set up in the circuit by this change; and by considering the direction of this induction it will be seen that this induced electromotive force is opposed in direction to the impressed electromotive force V producing the current. This induced electromotive force reckoned in volts is equal to $\frac{\mathrm{S\,N}}{10^8}\,\frac{d\mathsf{B}}{dt}$. Hence the current A in amperes must be equal at any instant to the resultant electromotive force divided by the resistance of the circuit; or

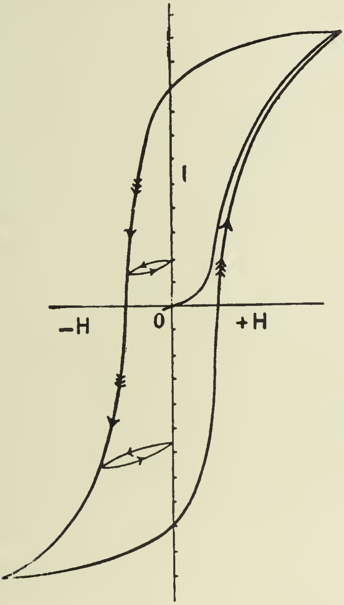

FIG 21.

Magnetisation Curve for Very Soft Iron Ring, showing loops due to hysteresis on the descending branch, and enclosed area due to a complete cycle (Ewing).

$$A = \frac{V - \frac{\mathrm{S\,N}}{10^8}\,\frac{d\mathsf{B}}{dt}}{R}.$$

Therefore, multiplying all through by A and $d\,t$, we have

$$\mathrm{V\,A}\,d\,t = \mathrm{R\,A}^2\,d\,t + \frac{\mathrm{S\,N}}{10^8}\,\mathrm{A}\,d\,\mathbf{B}.$$

But the magnetising force $\mathbf{H}$ in the iron at that instant is equal to $\frac{4\pi}{10}\,\frac{\mathrm{N}}{l}\,\mathrm{A}$. Hence

$$\mathrm{V\,A}\,d\,t = \mathrm{R\,A}^2\,d\,t + \frac{\mathrm{S}\,l}{10^7}\,\frac{1}{4\pi}\,\mathbf{H}\,d\,\mathbf{B}. \quad . \quad . \quad (18)$$

The first term of this equation represents the whole energy in joules given to the ring coil and core in the small time $d\,t$; the second term represents the energy wasted in that time in heating the copper magnetising coil; and the third term must therefore represent the whole energy, measured in joules, absorbed or wasted in the iron core in that same element of time. Accordingly it is easily seen that the integral

$$\frac{1}{4\pi}\int \mathbf{H}\,d\,\mathbf{B}, \quad . \quad . \quad . \quad . \quad . \quad (19)$$

taken between any limits, must represent the whole energy measured in ergs, dissipated by a unit of volume (viz., one cubic centimetre) of the iron core in the time limits of the integral. If the time limit is the interval of time occupied in making one complete magnetic cycle, then the above integral will represent the energy dissipated per unit of volume of the iron in this cycle estimated in ergs. But if the diagram of the magnetic cycle is drawn in terms of $\mathbf{B}$ and $\mathbf{H}$, the integral $\int \mathbf{H}\,d\,\mathbf{B}$ taken over the whole limits of the cycle is the value of the area enclosed by the induction curve. Finally, therefore, we reach this rule. If a mass of iron is taken once round a magnetic cycle, and an $\mathbf{H}\ \mathbf{B}$ diagram is drawn, showing the relation of induction to magnetising force during the cycle, one centimetre or one unit of length along the horizontal being taken as equal to one C.G.S. unit of magnetic force, and one centimetre or unit of length along the vertical being taken for one C.G.S. unit (one line) of induction; then $1/4\pi$ of the area of this closed loop in square units is equal to the energy wasted in such single magnetic cycle measured in ergs. The physical meaning of these loops or enclosed areas in magnetisation curves of complete magnetic cycles was first pointed out by Prof. E. Warburg,* but also independently by Ewing. In the first place, we may remark that however slowly the magnetic cycle may be performed, this waste of energy always takes place, and it is therefore seen to be dependent essentially on the reversal or change of magnetism and not upon the production of local electric currents in the iron. This energy waste is called the *hysteresis loss* in the iron, and it cannot be got rid of by any amount of lamination or division of the iron. Careful measurements have shown what is the value of this hysteresis loss in iron of various kinds.

In a Paper entitled "Researches in Magnetism" (*Phil. Trans.*, Part II., 1885), Prof. Ewing has given the values of the energy dissipated in ergs per cubic centimetre, experimentally determined for complete magnetic cycles performed on various samples of iron, as follows :—

| Sample of iron operated upon. | Energy dissipated in ergs per cubic centimetre during a complete cycle of doubly-reversed strong magnetisation. |
|---|---|
| Very soft annealed iron ................. | 9,300 ergs. |
| Less soft annealed iron ................. | 16,300 ,, |
| Hard drawn steel wire..................... | 60,000 ,, |
| Annealed steel wire ........................ | 70,500 ,, |
| Same steel, glass hard ............... ..... | 76,000 ,, |
| Pianoforte steel wire, normal temper.. | 116,000 ,, |
| Same, annealed .............................. | 94,000 ,, |
| Same, glass hard ........................... | 117,000 ,, |

If we make one hundred complete cycles of magnetisation per second, the power absorbed per cubic centimetre of metal estimated in watts is as follows :—

| Sample. | Power wasted in watts per cubic centimetre for 100 ( ∾ ) cycles per second. |
|---|---|
| Very soft annealed iron ..................... | 0·093 |
| Less soft annealed iron ..................... | 1 63 |
| Hard drawn steel wire........................ | 6·00 |
| Annealed steel wire .......................... | 7·05 |
| Same steel, glass hard ....................... | 7·60 |
| Pianoforte steel wire, normal temper ... | 11·600 |
| Same, annealed ................................ | 9·40 |
| Same, glass hard ............................. | 11·700 |

These numbers should be divided by 10 - except the first.

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* Wiedemann's *Annalen*, XIII., p. 141, 1881.

From the above we can deduce that, roughly speaking, it requires 18 foot-pounds of energy to make a double-reversal of strong magnetisation in a cubic foot of soft iron. The energy so expended can take no other form than that of heat diffused throughout the mass.

A similar table of experimental results has been given by Dr. J. Hopkinson (*Trans.* Roy. Soc., Part II., 1885, p. 463), in which Paper the chemical analysis of the samples operated upon is given. The highest value of specific hysteresial dissipation was found for Tungsten steel, oil hardened, in which the value of the energy in ergs per cubic centimetre dissipated in a complete magnetic reversal was 216,864.

Hysteresis is therefore a quality of iron in virtue of which reversal of magnetisation is accompanied by dissipation of energy. The energy so wasted is, of course, converted into heat. This dissipation of energy into heat during magnetisation is something quite apart from any production of heat by eddy (or so-called Foucault) electric currents induced in the mass, and would take place in iron so perfectly divided that no eddy currents could exist.

One result of Prof. Ewing's researches has been to show that if the iron is kept in a state of mechanical vibration hysteresis is greatly diminished, and the value of the energy dissipated in a complete cycle is much reduced. The removal of strong residual magnetism from soft iron by slight tapping or twisting has also been noticed and commented on by Prof. Hughes.*

Hysteresis, therefore, is a source of dissipation of energy in the armature of dynamos. For in this case we have a mass of soft iron, viz., the armature core, which has its direction of magnetisation reversed every revolution. Suppose the core has a volume of 9,000 cubic centimetres, and that it makes 15 revolutions per second. Taking the specific hysteresis for this sample of iron at 13,356 ergs, we find that the dissipation of energy in ergs per second is equal to $9{,}000 \times 15 \times 13{,}356 = 180 \times 10^7 = 180$ joules, or a loss of about a quarter of a horse-power-hour.

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* Prof. D. E. Hughes, "On the Cause of Evident Magnetism in Iron," *Proc. Soc. Tel. Engineers*, May 24, 1883, p. 3.

Experiments were at one time made by Joule and others to determine by direct observation the heating effect of magnetisation upon iron, but in these early experiments it is probable that the results were mostly impure, and qualified largely by the production in the iron of heat by local or eddy electric currents.*

Since about 10,000 ergs per cubic centimetre are dissipated by a double reversal of strong magnetisation of soft iron, it is not difficult to show that the consequent rise of temperature, even if all the heat is retained in the iron, is 0·000284°C., or that some 4,000 reversals would be required to raise the temperature 1°C., even provided all the heat generated is retained in the metal.

If the iron is subjected to very rapid reversals of induction, and if it is in one solid mass, then, in addition to the hysteresis waste of energy, eddy electric currents are set up in the mass of the iron and create heat. In such cases it must be noted that the source of waste of energy in the iron is twofold: first, that due to true magnetic hysteresis, and, second, that due to eddy currents. The last source of waste can be prevented by sufficiently and properly subdividing the iron into very thin plates or wires, which are rusted or painted so as to prevent the production to any degree of eddy currents. The first source of waste cannot be prevented by any such lamination. The question has been very much debated and considered whether the true hysteresis loss in iron depends upon the speed at which the magnetic cycle is described—whether the dissipation of energy at, say, 100 reversals per second is or is not more than one hundred times that of one slowly performed cycle.

The matter seems now decided as follows:—It appears evident from the researches of J. and B. Hopkinson,† that if the induction density is moderate in amount (for example, not more than 3,000 or 4,000 C.G.S. units) then, whether the reversal of magnetism or cycle is made very slowly or at the rate of a hundred complete cycles per second, the area of the hysteresis curve remains practically unaltered. The same fact has been noted by Messrs. Evershed and Vignoles,* and also by Mr. Steinmetz. We may say, therefore, as the result of the latest work, that the difference between slow cycle and quick cycle hysteresis, once supposed to exist, is found to be non-existent, and that, at any rate for such inductions as are used in transformers, we can apply the results of the hysteresis losses obtained by slow cycle methods to the cases of reversals at about a frequency of one hundred or more.

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* *See* Joule's "Scientific Papers," Vol. I., p. 123, "On the Calorific Effects of Magneto-Electricity," and "On the Mechanical Value of Heat.' Also *Phil. Mag.*, Series 3, Vol. XXIII., p. 263.

† *The Electrician*, September 9, 1892. *See* also *Proc. Roy. Soc.*, London, April 20, 1893, Vol. LIII., p. 352.

From the researches of the Messrs. Hopkinson it is also clear that, for higher induction densities, there is a difference between the hysteresis loops for very slow cycles and for rapid ones, and this difference is chiefly in that part of the the curve preceding the maximum induction. As Prof. J. A. Ewing has observed, after sudden changes of magnetising force the induction does not at once attain its full value, but there is a slight increase going on for some seconds. Dr. Hopkinson has remarked that this small difference between the curve as determined by very slow reversals and that as determined by very rapid reversals is a true time effect, the difference being greater between a frequency of 5 per second and 72 per second than between 5 per second and exceedingly slow cycles. There may be, therefore, a true time lag of magnetism at the higher speeds, but for all such frequencies as are employed in ordinary transformer work, we may take it that the hysteresis loss is constant per cycle and equal to that obtained by slow reversals.

The reader should carefully note that, if the hysteresis diagrams are taken for a solid iron ring and an equal sized ring made of iron wire in a way afterwards to be described with the wattmeter, when rapidly alternating currents are employed, the area of the diagram will be greater for the solid than for the divided ring. The area of the hysteresis diagram, then, gives us the energy loss due both to eddy currents set up in the iron and also that due to true hysteresis. Hence, in such experiments as above described, the greatest care has to be taken to eliminate all eddy current loss before we can draw any conclusions as to hysteresis proper. When a solid mass of iron is magnetised the eddy currents set up in the mass have to die away first before the iron attains its maximum magnetisation, because at every point in the interior of the iron the magnetic force due to the eddy current is in opposition to the external impressed magnetising force. Hence the effect of eddy currents is to delay the rise of magnetism. Over and above this, however, for strong inductions there seems to be a slow increase of magnetisation *after* the magnetising current has become constant. Prof. Ewing says: "I repeatedly observed that when the magnetising current was applied to long wires of soft iron there was a distinct *creeping up* of the magnetometer deflection after the current had attained a steady value."

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* *The Electrician*, September 16, 1892.

This time lag appears to be most manifest in the softest iron, and to be especially noticeable near the beginning of the steep part of the magnetisation curve.

In an investigation on the magnetisation of iron under feeble magnetic forces, Lord Rayleigh has also drawn attention to the fact that the settling down of iron when very soft or annealed into a new magnetic state is far from instantaneous.* He has shown that if the strength of the earth's horizontal magnetic field is called $h$, for unannealed iron and steel magnetising forces ranging from $\frac{1}{5}h$ to $\frac{1}{10000}h$ call forth proportional magnetisation—in other words, the susceptibility is constant over this range, and the value of the corresponding permeability is from 90 to 100, this small proportional magnetisation taking place independently of what may be the actual magnetisation of the iron, provided it is not very near the condition usually called saturation. The moment, however, that the magnetising force is pushed beyond these limits the phenomena of hysteresis and retentiveness make their appearance.

The subject of hysteresis in iron is by no means yet entirely explored. The chemical and physical states of the iron exercise the greatest influence on its magnetic hysteresis, and high specific electrical resistance seems in general to be an index of large hysteretic power in iron; and those elements, like manganese, which, when added to iron, increase its specific electrical resistance, have also an effect in increasing its hysteresis waste.

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* Lord Rayleigh, "On the Behaviour of Iron and Steel Under the Operation of Feeble Magnetic Forces," *Phil. Mag.*, March, 1887, p. 225.

One important point in connection with magnetic hysteresis is the effect of rise of temperature of the iron upon the hysteresis loss. Experiments made recently by W. Kunz,* are instructive on this point. This experimentalist investigated in 1892 and in 1894 the effect of rise of temperature in producing a diminution of magnetic hysteresis in iron, and the following are the results obtained from a long series of observations of this phenomenon:—

Four kinds of iron, two of steel, and one of nickel have been the subject of investigation. Special difficulties occurred in maintaining the wire samples at the high temperature for the required length of time, and in the measurement of these temperatures, and therefore the methods are more particularly described.

To measure the temperature of the wire, thermo-electric junctions were used, consisting of a platinum wire twisted for about 1·5 cm. of length round a wire of platinum containing 10 per cent. of rhodium. Two such couples were used, whose free ends were brought well insulated to a mercury switch, by means of which each couple could be connected to a Deprez-d'Arsonval galvanometer having about 200 ohms resistance. The temperature of the junctions at switch and galvanometer was always the same—about 20°C. The calibration of these couples up to 300°C. was done by means of an accurate mercury thermometer, plunged with one junction of the couple into oil, which was warmed up in large beakers surrounded with asbestos and kept well stirred. The galvanometer deflections corresponding to the temperatures from 40°C to 300°C were noted. From 300°C upwards, the calibration was done by utilising the known fusing points of certain substances melted by a gas furnace: the junction was placed in the molten material, the flame lowered, and the deflection noted when solidification began. Each couple was thus separately calibrated—the outside junctions being kept at a constant temperature by immersion in petroleum at 20°C.

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* See *Elektrotechnische Zeitschrift*, 1894, No. 14., p. 194.

The substances used and temperatures measured, or assumed from Börnstein-Landolt's tables, were as follows:—

| Substance Warmed or Melted. | Temperature or Melting Point. | Galvanometer Deflection. |
|---|---|---|
| | Deg. | |
| Oil | 40 | 16·0 |
| ,, | 73 | 27·2 |
| ,, | 104 | 38·0 |
| ,, | 140 | 50·8 |
| ,, | 200 | 77·6 |
| ,, | 300 | 125·4 |
| $K Cl O_3$ | 359 | 155·6 |
| Pb Cl | 498 | 244·4 |
| I K | 634 | 330·2 |
| K Cl | 734 | 393·6 |
| Na Cl | 772 | 417·8 |
| $Na_2 SO_4$ | 861 | 474·2 |

To heat the wire under test a method already employed by Ledeboer was adopted—namely, winding an insulated platinum wire round the test wire, and heating it by passing a current through the platinum wire. The iron wire was contained in a porcelain cylinder having a suitable opening to take the thermo-couple, and round this cylinder was wound (non-inductively) the platinum wire. The insulation between the couple and the platinum was tested before each observation. This platinum coil was surrounded by layers of asbestos, among which the wires of the thermo couple were led out. The tube thus formed was placed inside a glass tube and accurately centred by suitable packing. This tube was placed again in another glass tube with asbestos distance pieces. A current of hydrogen passing through the inner tube protected the wires from oxidation. To protect the magnetising coil against the high temperatures, a hollow tube of pure copper was placed between, and a stream of water kept passing through it, and this had to be insulated by asbestos from the glass tubes to prevent their breaking. Observation showed that no heat passed through this jacket. The two couples always agreed in their indications, thus showing that the wire was evenly heated.

As indicated above, the test wire was a long straight piece; its magnetic condition was observed by magnetometer, by the "single pole" method. The magnetising coil is placed vertically in the direction east and west from the magnetometer, and the top end of the magnetised wire is on a level with the instrument. The alteration in the position of the pole due to differences in induction density affect the distance between the magnetometer and wire so little as to be negligible. The vertical component of the earth's field must be taken into account. In magnetising, the cycle was always performed a few times, until the curve became regular. Then a series of observations at varying temperature of a certain cycle was taken, generally with maximum $B = 3{,}590$ (about), until at the high temperature (830° C. or so) the magnetism disappeared. Then another cycle after cooling. Suitable arrangements were made for compensating for the effect of the magnetising coil itself on the magnetometer, and for demagnetising by reversals of a gradually diminishing current.

The strength of field in the middle of the solenoid is calculated from the well-known formula,

$$H = \frac{4\pi}{10} \frac{N C}{l},$$

where $N$ = number of turns, $C$ = the current in C.G.S. units, and $l$ = the length of the solenoid; and the induction was calculated from the magnetometer deflection by means of the known value of the horizontal component of the earth's field.

The following Tables are for a maximum induction density of 3,590, and are the mean of four series :—

| Material. | Temp. | Hysteresis Loss in Ergs. | Material. | Temp. | Hysteresis Loss in Ergs. |
|---|---|---|---|---|---|
| | Deg. | | | Deg. | |
| German annealed charcoal iron | 20 | 2,350 | Swedish iron | 20 | 2,690 |
| | 290 | 1,600 | | 270 | 2,080 |
| | 470 | 1,204 | | 460 | 1,550 |
| | 656 | 710 | | 650 | 905 |
| | 728 | 550 | | 742 | 825 |
| | 836 | 316 | | 812 | 712 |
| | 20 | 2,107 | | 20 | Indefinite |
| Soft wrought iron | 20 | 3,420 | Puddled iron | 20 | 3,100 |
| | 284 | 2,480 | | 275 | 2,270 |
| | 468 | 1,750 | | 460 | 1,730 |
| | 656 | 821 | | 560 | 1,310 |
| | 744 | 800 | | 656 | 979 |
| | 20 | 900 | | 744 | 777 |
| | | | | 20 | 2,090 |

These values show, when plotted as curves, that the equation $L = a - b\ t$ (where L is the hysteresis loss and $t$ the temperature, and $a$ and $b$ constants) expresses fairly the law.

| Material. | Temp. | Hysteresis Loss in Ergs. | Material. | Temp. | Hysteresis Loss in Ergs. |
|---|---|---|---|---|---|
| | Deg. | | | Deg. | |
| Hard patent steel | 20 | 11,540 | Patent cast steel | 20 | 9,660 |
| | 309 | 11,580 | | 309 | 9,860 |
| | 526 | 6,040 | | 468 | 4,950 |
| | 660 | 2,200 | | 560 | 1,985 |
| | 790 | 1,180 | | 640 | 1,614 |
| | 20 | 5,230 | | 744 | 1,048 |
| | | | | 20 | 4,670 |

The above Tables are to be interpreted as follows: Taking a wire of the material named, it was subjected to a magnetic cycle of induction in which the maximum induction reached was 3,590 C. G. S. The wire being taken at a particular temperature, as given in the second column of the tables, a hysteresis loss, diminishing with rise of temperature, was found, the value of which per cycle is given in the third column.

The two kinds of steel referred to in the last two tables had, for ordinary temperatures, magnetic cycles in shape like a rhombus, altering in shape at about 300°, even increasing in area, and between this and 470° changing in form to that of an ordinary iron curve, and decreasing greatly in area. The character of the steel is lost after heating, as shown by the final observations, and it becomes also quite soft. There is, in these cases, no simple relation between hysteresis loss and temperature.

The following tables relate to charcoal iron :—

| B. | Temp. | Loss. | B. | Temp. | Loss. |
|---|---|---|---|---|---|
| | Deg. | | | Deg. | |
| 7,200... ... | 20 | 8,900 | 14,400 ... | 20 | 21,020 |
| | 270 | 6,690 | | 270 | 14,840 |
| | 468 | 4,660 | | 470 | 9,900 |
| | 570 | 3,340 | | 570 | 7,550 |
| | 668 | 2,270 | | — | — |
| | 744 | 2,168 | | — | — |

For higher temperatures than 570, B = 14,400 could not be reached.

The results show that the above law also holds good, generally speaking, for higher values of induction density.

The corresponding results for steel show that the characteristics described for the lower values of induction density hold also for the higher. In all the above cases a new wire was taken for each series of observations. It is shown, therefore, that when an iron wire is subjected to repeated cycles of temperature and magnetisation the hysteresis loss decreases up to the fourth cycle of temperature, and then becomes uniform; the results of each temperature cycle being expressible as a straight line, but of different inclination. The steel wire has the first temperature cycle as already described, and the remainder behave like the soft iron.

In the case of nickel subjected to a cycle of maximum B = 3,590, it was found that the hysteresis fell with the increase of temperature, at first rapidly, and afterwards increasingly slowly: it fell from 11,420 ergs at 20° to 4,700 ergs at 288°.

One of the most important results is the fact that repeated cycles of magnetism at a high temperature reduce the hysteresis loss in iron very considerably, and it would appear that this is also the result of one cycle at a very high temperature.

The above results show that in soft iron a very marked decrease in the hysteresis loss takes place as the temperature of the iron is raised. Further reference to this matter will be made in reference to the losses of energy in transformer working.

§ 7. **The Electromotive Force of Induction.**—We have in § 2 enunciated Faraday's law of induction in terms of the variation of a quantity called the flux of induction through the circuit. It is possible to express the fundamental rule in a more elementary manner, and in a way which adapts it to explain every fact yet observed. It is as follows:—If any element of a conducting circuit is so placed in a field of magnetic induction that a movement of that element of the conductor or change in the field of induction causes lines of induction to intersect it, it creates an electromotive force in the element of which the direction is perpendicular to the plane containing the lines of magnetic force and the direction of motion of the centre of the element.

This operation is called "cutting lines of magnetic force." We shall allude later to hypotheses which have been constructed to suggest in some degree an explanation of the nature of this effect.

The simplest possible case which can be considered is when a short linear element, such as a straight wire, is made to move in a uniform magnetic field, in a direction perpendicular to the plane containing the field lines and the linear conductor, the direction of the length of this last being also perpendicular to the direction of the field lines.

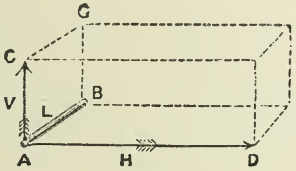

FIG. 22.

If A B (Fig. 22) is the element of length of the conductor of length L, and if A D represent in magnitude and direction one of the lines of induction of the uniform magnetic field, H, in which it is placed, A B being at right angles to A D, and if A C represent in magnitude and direction a displacement of A B taking place uniformly in one second, so that A B moves uniformly parallel to itself from position A B to position C G in one second, we have then three lines, A B, A C, A D, mutually at right angles, and representing respectively the length L, the velocity V, and the magnetic induction H.

The result of the motion is to generate in A B an electromotive force E, numerically equal to the product H L V in consistent units. But since the sides of the parallelopipedon, or solid rectangle, are taken to represent respectively H, L and V, their product E represents the volume of the solid and the magnitude of the electromotive force of induction.

If the directions of A B, A C and A D are not orthogonal, but inclined, the same still holds good.

For let the field lines A D (Fig. 23) be supposed to be inclined at an angle $\theta$ with the direction of the length of the conductor A B, and let the direction of motion of A B parallel to itself, represented by A C, be inclined at an angle $\phi$ with A B. The strength of field estimated perpendicular to A B is H sin $\theta$, and if A C represents the actual velocity of A B, or displacement in one second, then A C sin $\phi$, or V sin $\phi$, is its velocity in a direction perpendicular to its own length. The magnitude of the induced electromotive force E in A B is then numerically equal to L H sin $\theta$ V sin $\phi$, or to H L V sin $\theta$ sin $\phi$; but this expression also represents the volume of a doubly skew parallelopipedon or solid rhomboid; hence, as before, if vectors be drawn representing respectively the length and velocity of a conducting element, and also the field strength in which it is placed, the volume of the solid rhomboid described on these vectors as adjacent sides represents the magnitude of the electromotive force induced in the element.

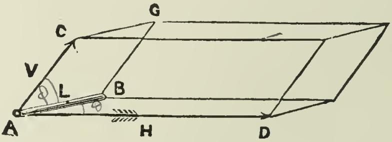

Fig. 23.

The magnitude of this induced electromotive force is not in any way dependent upon the nature of the material of which this conductor is made. Faraday experimentally proved this ("Exp. Res.," § 193-201) by taking a double conductor composed of an iron and a copper wire twisted together and united at one end. On passing this double conductor through a magnetic field no induced current was detected in it by a galvanometer. This proved that the electromotive forces set up in each separate conductor were equal and opposite, and hence, since the lengths, field, and velocities were the same, no factor entered into the production of the effect, which depended on the nature of the conductor. From further experiments with circuits partly metallic and partly electrolytic fluids he inferred that in all bodies, whether what are commonly called conductors or non-conductors, or electrolytic conductors, identically the same electromotive force is brought into existence by moving the same lengths in the same way in the same magnetic fields.

When a metallic disc is rotated in a uniform magnetic field so that its axis of rotation is parallel to the direction of the field, there is set up a difference of potential between the centre and the edge. In this case we can tap off a current by an external wire connected to the centre and the edge of the disc.

We can now show that, starting with the elementary law above stated, as to the magnitude of the induced E.M.F. in an element of a conductor, we can deduce the other principle of the relation of the induced E.M.F. to the rate of change of the induction through the circuit.

Let A B C D (Fig. 24) be a conducting rectangle, of which the plane is perpendicular to the induction lines of a uniform magnetic field of strength H, the same being shown in plan on the figure; let the circuit be capable of revolving about an axis O O in its own plane, and let it be displaced through any angle, $\theta$, as shown in elevation and plan in Fig. 20. If the frame is so displaced it is clear that the sides A C, B D "cut" across lines of magnetic induction, but that the upper and lower sides do not. During this displacement the vertical sides alone will be the seat of electromotive forces. Imagine this frame to revolve round the vertical axis with a uniform angular velocity $\omega$, and at any instant $t$ to have a position such that its plane makes an angle $\theta$ with the plane normal to the lines of force. Let the length of the side A C be L and that of A B be R: the actual velocity of the side A C is $\frac{\omega R}{2}$, and the strength of the field, in a direction perpendicular to its length and its direction of motion at that instant, is $H \sin \theta$. Hence the electromotive force of induction in the side A C is $\frac{\omega R}{2} L H \sin \theta$, and an equal and oppositely directed electromotive force acts in the side B D at the same instant. Hence the total electromotive force acting round the frame is equal to $\omega$ H R L sin $\theta$. If the area of A B C D is denoted by A we may write the above as $\omega$ H A sin $\theta$. The angular velocity $\omega$ may be expressed as the time rate of change of $\theta$, or as $\frac{d\theta}{dt}$; hence the expression for the total electromotive force of induction round the frame is $HA \sin\theta \frac{d\theta}{dt}$, or $-\frac{d}{dt}(HA\cos\theta)$.*

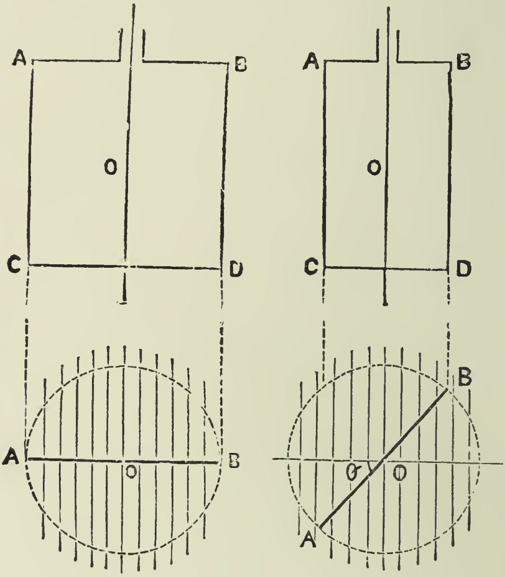

FIG. 24.

The expression A cos $\theta$ denotes the apparent size of the frame as looked at from a considerable distance along the direction of the lines of induction, and the quantity H A cos $\theta$ is the numerical value of the number of lines of magnetic induction passing through or traversing the frame in its position when its plane is inclined at an angle $\theta$ to the normal position. We assume that these lines are spaced out according to the rule proper for such distribution, viz., that the number passing through a unit of area whose plane is taken normal to the direction of these lines is numerically equal to the magnetic induction over that area.

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* We here suppose the circuit to be formed of a single loop of wire having a practically negligible self-induction. The above statements would require some modification for a circuit of many turns of wire.

Writing N for this number of lines so piercing through the frame at any instant, we have, as the expression for the total electromotive force acting round the frame at any instant, the quantity $-\frac{dN}{dt}$; that is, the electromotive force of induction is numerically equal to the rate of change (decrease) of the included lines of induction. It is customary to speak of this induced electromotive force as generated either by the "cutting of lines of force" by the various elements of the conductor or by a change in the number of lines of force piercing through the aperture of the circuit; but they are merely two different geometrical ways of viewing the same phenomena. The actual results are capable of receiving a physical explanation on the assumption that the act of intersection of a line of force and a portion of a conducting circuit is productive of an electromotive force. We see that the total electromotive force is the resultant effect due to a summing-up of all the forces acting on each element of the circuit, each elemental E.M.F. being measured by the product of the length of that element, the field strength around it, and its normal velocity in that part of the field. The result is concisely expressed by the number which expresses the time rate of change of the whole number of the lines of induction traversing the circuit. This same may be extended to any circuit of any form moving in any way in any field.

If a circuit of any form which is traversed by an electric current is placed in a magnetic field due to other neighbouring currents or magnets, there is a flux of induction through that circuit due partly to the current in the conductor and partly to the external field of the other currents or magnets. If there be M lines of induction due to the external field passing through it, and N lines of induction due to its own current, any variation of the external induction, of which the rate of change at any instant is represented by $\frac{dM}{dt}$, will produce an impressed electromotive force in such a direction that taking lines of induction out of the circuit induces an electromotive force in the clockhandwise (+) direction, as seen from that side of the circuit at which the lines enter. When a current is flowing in any conductor, the relation between the direction of the current and that of its own lines of induction is the same as the relation between the thrust and the twist of a corkscrew. Hence, it is evident that, if we consider a circular current (Fig. 25) with the current flowing in it clockhandwise (+), as seen from one side, its own lines of induction pass through the circuit in the positive direction, or away from the eye.

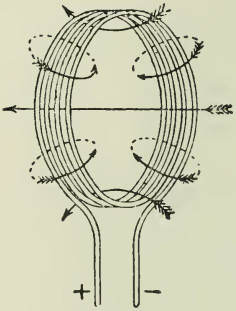

Fig. 25.

Accordingly, a little reflection shows that, if the current in the conducting circuit is made to increase, an opposing electromotive force is created by the increasing induction of the current on its own circuit. The current in the act of increasing crowds its own circuit more full of lines of induction, and creates an electromotive force of induction during the period of this increase equal numerically at any instant to its own rate of increase, and directed in opposition to the impressed external electromotive force which is driving the current.

# CHAPTER III.

## THE THEORY OF SIMPLE PERIODIC CURRENTS.

§ **1. Variable and Steady Flow.**—In the following pages we shall be chiefly concerned in considering the properties and uses of currents of electricity which are periodic in character; that is, which are changing in strength from instant to instant in a cyclic or periodic manner. An electric current or an electromotive force may either be *steady*, in which state it remains uniformly at the same value, or it may be *variable*, in which case it is changing in value from instant to instant. In this last case we can consider two separate conditions. The current strength or electromotive force may be *periodic* or *non-periodic* in value. A non-periodic variable current or electromotive force is one which changes in value from instant to instant accordingly to any assigned law or mode, but in which the same series of values are not regularly repeated. A periodic current or electromotive force is one which runs through a regular cycle of values, returning after a certain period to the same value. It is accordingly said to vary in a cyclic manner, because it changes through a cycle of values. We may take illustrations of these three states from the flow of fluids. A stream of fluid may exist in a steady state; in this case the motion of each particle of the stream has settled down into a uniform condition as regards velocity. If we imagine a small short tube open at both ends, held anywhere in that flowing fluid, the same volume of fluid would flow through that tube in every unit of time. We may, however, find the fluid in such a condition that the velocity of each particle of the fluid at any point is changing, and the flow is then in a variable condition

If that change is of such a character that the motion is regular in its mode of change, then the flow is said to be periodic. Thus, in a non-tidal river the water flows in general uniformly in one direction; it is in a steady state. At the time of a flood its speed at any point may be rapidly increasing, and in this case its flow is variable. In the case of a tidal river the flow of water is regularly reversed, a cycle of fluid motion is repeated at any point, and the motion is said to be cyclic or periodic in character.

In considering the motion, either of actual fluids or of electric currents, we can, then, distinguish three states—the *variable*, the *periodic*, and the *steady* condition. In the first case the strength or direction of the electric currents or of the fluid velocity is changing at every instant; in the latter cases the flow has settled down into a permanent state. The questions involved in dealing with the variable or periodic states present rather more difficulties than do problems in steady flow, for the reason that the notions of *time* and *inertia* enter into these in a way in which they do not when that flow has reached a steady condition. We shall proceed to examine in an elementary manner some features of electrical flow when variable or periodic. We must, however, prepare the way by considering some purely geometrical properties of certain curves, and also some modes of motion which have special reference to the kind of electric current to be considered subsequently. When a mass of water is in motion, a particle of water selected for examination has at any instant a certain velocity in a certain direction. This may be represented graphically by a straight line drawn from that particle representing its velocity in direction and magnitude. Similarly, if electricity is flowing through the mass of a conductor in any manner, it is possible at any point to draw a *vector* or line representing at that instant the direction and magnitude of the current at the point from which the line is drawn. Lines drawn within the mass of a fluid at any points such that the flow at that instant is along or tangential to these lines are called *flow lines.* In the first place, let us make the supposition that the flow has reached a steady condition. The flow lines are then fixed. When this is the case each line of flow becomes the actual path of a fluid particle, and is called a *stream line.* A surface may be supposed to be described in the mass of the fluid everywhere perpendicular or orthogonal to the stream lines; such a surface is called an equipotential or level surface. We may also suppose such a level surface drawn in the mass of a conductor through which a current is flowing. Let any area be drawn on the equipotential surface, and let it be divided up into units of area. If the quantity of fluid or of electricity flowing through each unit of area is the same, and if, moreover, it is the same for each unit during each succeeding instant of time, the current is said to be *steady* and to be uniformly distributed. The quantity flowing per unit of time through any area is the numerical measure of the mean strength of current over that section of the conductor, and the quantity flowing per unit of time through a unit of area is the measure of the mean density of current over that unit of area. If the distribution of current and strength is not uniform, we can only express them at any time and place by calling to aid the language of the differential calculus. If $ds$ be a small area described on an equipotential surface, and if $dq$ be the quantity of electricity which flows in a small time $dt$ through that area $ds$, and if $i$ is the strength of the current at the centre of that small area at any instant, then in the limit

$$i=\frac{dq}{dt}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

§ **2. Current and Electromotive Force Curves.**—To fix our ideas, let us now suppose the electric flow to take place through a thin cylindrical conductor, such as a wire, in which, at positions sufficiently remote from the ends, the stream lines will be parallel to the axis of the wire and the equipotential surfaces perpendicular to it. Consider any one section, and let the flow across this section be variable both in strength and direction—that is to say, let it vary in the quantity of electricity which flows across that section in each succeeding instant, and let the flow be first one way and then the other, changing in any manner, however irregular. We can represent graphically the state of things as regards electric flow at that section by means of a curve called a *current curve.*

Take a horizontal line (Fig. 26) to represent the uniform flow of time. At successive instants let ordinates be drawn to this line, representing the strength of current flowing past that section, and let them be drawn above (+) or below (−), according as the direction of the flow is to the right or to the left. Thus, if time begins to reckon from O, after the lapse of a time O T the current is positive, and is represented by a line T I. After the lapse of a time O T′ the current is negative, and is represented in strength by a line T′ I′.

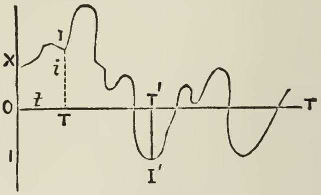

Fig. 26.

This current curve is obviously a single-valued function—that is to say, corresponding to a given instant of time the current can only have one value. The curve can never cut itself or double back.

We may here remind the student of the distinction between single and multiple-valued functions. A single-valued function is one which, when represented graphically by a continuous curve, presents only one value of the ordinate for each value of the abscissa.

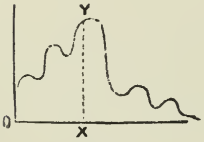

Fig. 27.
Single-valued function.

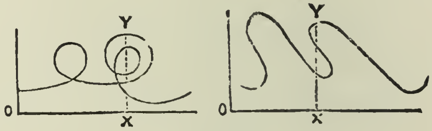

Fig. 28.
Multiple-valued functions.

In Fig. 27 is represented graphically a single-valued function, having only one value of the ordinate X Y corresponding to a given value of abscissa O X. In Fig. 28 is represented a curve such that there are five different values of the ordinate of the curve corresponding to one value of the abscissa O X. This curve represents a multiple-valued function.

Amongst single-valued functions, or single ordinate curves, there is one which is particularly important, because it proves to be the constituent element of every single-valued function. This curve is called a *simple periodic curve*, or simple sine curve, or simple harmonic curve. This curve may be described as follows :—Let a circle (say a coach wheel) roll with uniform speed along a straight line, A B : a point P on its circumference will mark out a curve called a *cycloid*, represented in Fig. 29 by the thick line, A E P B. If the point P be projected at every instant on the vertical diameter of the circle, then the point M will mark out a curve (represented by the dotted curve) as the circle rolls along which has been sometimes called "the companion to the cycloid." It is also called a harmonic curve, a sine curve, or a simple periodic curve. Draw a line O S N through the centre of the circle and parallel to the base line A B. Let it cut the dotted curve at the point O. The mathematical student will see that if the point O is taken as origin, and O C is called $x$, and C M called $y$, then also, if the radius C P of the circle is R, and the angle M P C = P C N is called $\theta$, it is clear that

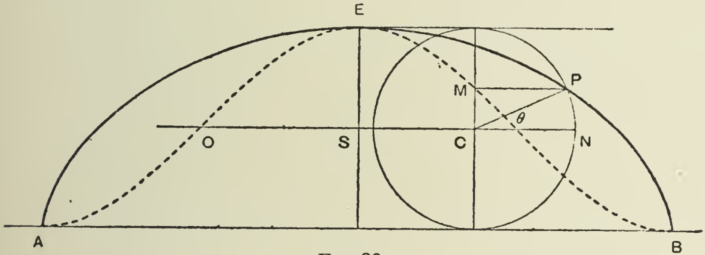

FIG. 29.

$$x = \mathrm{R}\,(180 - \theta)$$

and $$y = \mathrm{R}\sin\theta.$$

or, $$y = \mathrm{R}\sin\left(180 - \frac{x}{\mathrm{R}}\right).$$

If $l$ is the circumference of the circle, then $l = 2\pi\mathrm{R}$, and, by substitution,

$$y = \frac{l}{2\pi}\sin\frac{2\pi}{l}x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This last is the equation to the dotted curve O E M B, and it is the equation to a simple periodic or sine curve. The quantity $l = AB$ is called the wave length, and $R = SE$ is called the amplitude of the harmonic curve. It will be seen that this simple periodic curve is a smooth wavy curve which has points of maxima above and below the axis O C.

§ 3. **Simple and Compound Periodic Curves.**—If on one common axis we draw two simple periodic curves of any wave lengths and any amplitudes, and having any relative position with regard to each other, we may obtain another curve, called a complex periodic curve, by adding together the ordinates of the two simple curves.

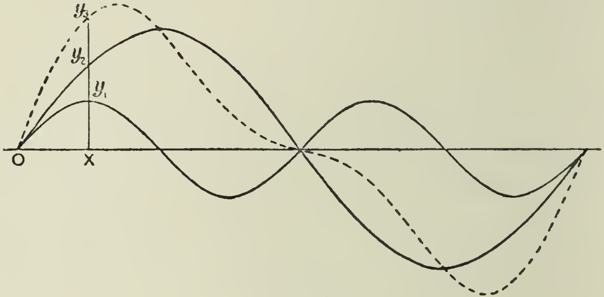

Fig. 30.

As an example, in Fig. 30 are shown two simple sine curves, represented by the firm lines, of which one has double the wave-length and about two and a-quarter times the amplitude of the other. If these curves are superimposed, and a new curve, represented by the dotted line, formed by adding the ordinates $Xy_1$, $Xy_2$, of a common abscissa, O X, into a third, $Xy_3$, then we obtain, by repeating this at all points, a new curve, which is called a complex periodic curve, because it is compounded of two simple sine curves. The dotted curve is the complex sine curve, and the two firm-line curves are its two components.

We may in this way add together any number of simple periodic curves and obtain an exceedingly complicated complex periodic curve, which is, however, always, like a simple periodic curve, a single-valued function. It is clear, also, that just as we can compound simple periodic curves into a complex one, so we can resolve a complex single-valued function into a set of simple periodic components, suitably situated with respect to one another.

§ 4. **Fourier's Theorem.**—One of the most attractive and important of all mathematical discoveries is that of Jean Baptiste Fourier, who in his "Théorie Analytique de la Chaleur," published in 1822, gave a demonstration of the above theorem, viz., that any periodic curve, however complex, provided it is a single-valued function, can be resolved into a series of simple periodic curves, of suitable amplitudes and wave-lengths, and be placed in a certain relative position to each other. In mathematical language, any single-valued periodic function can be expressed analytically as a sum of a series of terms the first of which is an arbitrary constant, and each of the following terms is the sine or cosine of an angle multiplied by a constant. Take such a case as that of a zig-zag line, made up of lines inclined at an angle of 60deg., like the teeth of a saw (Fig. 31). We can, by Fourier's theorem, express the equation to this periodic line in terms of a series of sine or cosine terms. Thus the equation to the zig-zag line in Fig. 31 is

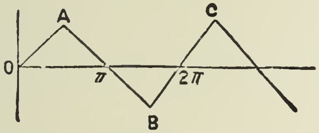

FIG. 31.

$$y=\frac{4}{\pi}\left\{\sin x-\frac{1}{9}\sin 3x+\frac{1}{25}\sin 5x-\&c.\right\}.$$

Hence, by adding together the ordinates of a number of sine curves suitably chosen and placed, we can obtain a complex periodic curve which imitates in form any given single valued periodic curve, however complex it may be, provided only that it is periodic, and that the curve does not cut itself.

This very remarkable theorem has applications in all departments of physics. In acoustics it shows that any continuous sound may be resolved into a series of simple harmonic sounds. In alternating current investigations it demonstrates that any curve of current, however complex, can be resolved into a series of simple periodic currents. If, then, any single function is graphically represented—that is to say, any such curve as in Fig. 30—we see that this curve may be described by a point which moves horizontally with a uniform velocity, whilst at the same time it executes in a vertical direction a movement which is the sum of a number of simple harmonic motions superimposed upon one another. The combination of these two rectangular motions causes the point to describe the curve considered.

In subsequent chapters we shall be examining effects which are due to periodic or fluctuating electric currents. Fourier's theorem gives us, when applied to these cases, a simplification of immense value, in that it enables us to see that, however complicated may be the fluctuation of current in a conductor, it can always be resolved into the sum of a series of simultaneous currents varying in a simple manner, and each of which can be graphically represented by a simple harmonic curve. The general consideration of periodic currents must, then, be preceded by an examination of the elementary theory of electric currents of a periodic character, in which the variation is of the most simple kind.

Fourier's theorem applies also to many other physical phenomena of great importance. In acoustics it shows, for instance, that however complicated may be the motion of an air particle in a mass of air through which sound waves are being transmitted, it can be resolved into the sum of a series of motions such as would be produced by the action of tuning forks, each of which gives rise to a motion in the air particles approximately of the nature of a simple harmonic vibration. Helmholtz actually realised this in his synthesis of vowel sounds.

Physically interpreted, Fourier's theorem means that any variation of motion which can be represented by the changing ordinate of a single-valued periodic curve can be expressed as the sum of a series of simultaneous motions, each one of which is called a simple harmonic, or simple periodic, or simple sine motion. It becomes important, then, to start by examining the simplest form of periodic motion. Suppose a circular disc (Fig. 32), having a pin at its centre, O, to be pivoted so as to revolve round an eccentric point, C. Let a T bar, moving in guides and having a slot in the cross-piece, be so fixed that the centre pin O is constrained to move in the slot. Furthermore, let the point C round which the disc moves be fixed to some support in the line of the bar A B produced. If the eccentric is compelled to move round C, the extremity of the bar A will move backwards and forwards with a motion called a simple harmonic motion or a simple periodic motion.

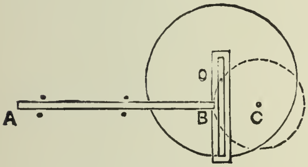

FIG. 32.

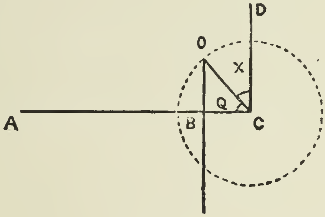

FIG. 33.

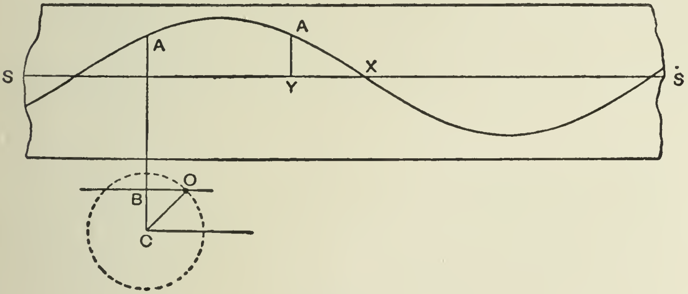

FIG. 34.

For it is clear the point O (Fig. 33) is compelled to move in a circle round C as a centre, and hence the distance of the point A from C at any instant is the length of the bar A B *plus* the length B C, which is the projection of O C on the line A C. The point B, therefore, executes a simple vibration to and fro along the line A C as O moves round, and the point A imitates the motion of B. If the angle O C D is called $x$ and the radius O C is $a$, then the length B C is $a \sin x$, and the displacement of A at any instant from its mean or middle position has the same value. The motion of A is called a simple harmonic motion, and the above eccentric and T bar is a mechanical device for compelling a point to describe a simple harmonic motion (abbreviated into S.H.M.). If such a harmonic motion be executed by point A (Fig. 34), whilst at the same time a strip of paper, S S′, is caused to move uniformly in a direction perpendicularly to the line A B, a tracing point fixed to A will describe on the paper a curve of

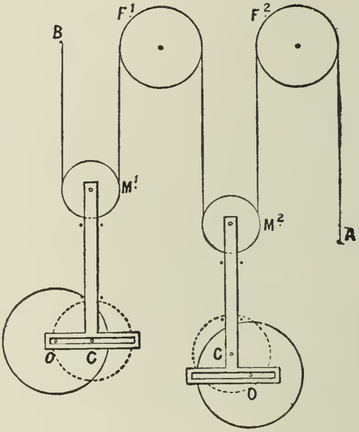

Fig. 35.

which the ordinate A Y is proportional to the sine of the abscissa X Y, or the equation to the curve will be of the form $y = a \sin x$, $a$ being some constant quantity. Hence a simple periodic curve is also called a *sine* curve.

By combining together two similar pieces of mechanism it is possible to construct a machine which can add together graphically two simple harmonic motions in the same line, but of which the phase angles $x$ and the amplitudes $a$ are different. Machines for doing this have been devised by Lord Kelvin, Mr. Stroh, and others. Apart from complications the general principle is as follows.

Let a cord pass over four pulleys (Fig. 35), two of which, $F^1$ $F^2$, are fixed in space, and two, $M^1$ $M^2$, can be made to rise and fall in vertical lines with a simple harmonic motion by being attached to T bars and eccentrics. If the cord has one end, B, fixed, and the other end, A, free, it is easy to see that, if either the pulley $M^1$ or $M^2$ rises and falls along a vertical line and the cord is just kept tight, the free end A will be displaced by an amount equal to twice the displacement of $M^1$ or $M^2$, and as $M^1$ or $M^2$ moves up and down with a S.H.M., the free end of A will also execute similar vibrations. If $M^1$ and $M^2$ move together the displacement of A at any instant is equal to the sum of the displacements of $M^1$ and $M^2$. By providing the end A with a tracing point, and moving under it uniformly a sheet of paper in a direction perpendicular to the direction of motion of A, it will describe a curve of which the equation will be of the form

$$y = a \sin x + a' \sin x',$$

$a$ and $a'$ being the amplitudes and $x$ $x'$ the phase angles of the two motions of $M^1$ and $M^2$ respectively. This apparatus, or one of similar principle, has been devised and employed by Lord Kelvin in his researches on the tides. It will be evident from the foregoing explanation that a machine can be constructed capable of causing a tracing point to move to and fro across a uniformly flowing sheet of paper, with a motion compounded of any number of simple harmonic motions of different amplitude and phase taking place in the same straight line.

§ 5. **Mathematical Sketch of Fourier's Theorem.**—Without going into a complete proof of Fourier's theorem, for which we must refer the advanced student to mathematical text-books, we propose to indicate to the student how it is practically employed in the analysis of any complex curve into a series of simple harmonic constituents. At a later stage the student will find that this analysis is of use in discussing certain current and electromotive force curves obtained from transformers.

We start with the assumption, for the propriety of which we must refer the reader to more advanced treatises, that if $y$

is the magnitude of the ordinate of any complex periodic single-valued curve, we can always express $y$ as follows:—

$$y = A_0 + B_0 + A_1 \sin p\,t + B_1 \cos p\,t + A_2 \sin 2\,p\,t$$
$$+ B_2 \cos 2\,p\,t + A_3 \sin 3\,p\,t + B_3 \cos 3\,p\,t + \&c.$$

The problem is, given any complex periodic curve, to find the A's and B's in the above equation for its ordinate at any point. To do this we need a preliminary lemma in the integral calculus. It is as follows:—

The integrals, $\int \sin p\,t \sin q\,t\,d\,t,$

and $\int \cos p\,t \cos q\,t\,d\,t,$

when integrated between the limits 0 and $\pi$, are equal to zero, if $p$ and $q$ are unequal integers; and equal to $\frac{\pi}{2}$, if $p$ and $q$ are equal integers. For, since

$$2 \sin p\,t \sin q\,t = \cos (p-q)\,t - \cos (p+q)\,t,$$

and
$$2 \cos p\,t \cos q\,t = \cos (p-q)\,t + \cos (p+q)\,t;$$

therefore,
$$\int \sin p\,t \sin q\,t\,d\,t = \frac{\sin (p-q)\,t}{2\,(p-q)} - \frac{\sin (p+q)\,t}{2\,(p+q)},$$

and
$$\int \cos p\,t \cos q\,t\,d\,t = \frac{\sin (p-q)\,t}{2\,(p-q)} + \frac{\sin (p+q)\,t}{2\,(p+q)}.$$

Hence, if $p$ and $q$ are unequal integers, both these integrals between the limits $t=0$ and $t=\pi$ are zero. If $p=q$ they both become equal to $\frac{\pi}{2}$. Again, if $y$ is the ordinate of a periodic curve, and if $l$ is the half-wave length, then the integral $\frac{1}{l}\int_0^l y\,d\,l$ represents the *mean value* of $y$ during half the period; because it is obvious that, if the mean or average value of $y$ is called M, the area enclosed by the periodic curve and the base line between the two limiting ordinates is $M\,l$, and this area is also expressed by the integral $\int_0^l y\,d\,l$. Hence the above equality results. From these two simple lemmas it follows that we can easily determine the values of the constants in the harmonic expansion. Let us assume a simple case as an example. Let

$$y = A_0 + A_1 \sin x + A_2 \sin 2x.$$

To determine $A_2$, multiply all through by $\sin 2x$ and integrate between the limits $x = 0$ and $x = \pi$,

$$\int_0^\pi y \sin 2x\, dx = \int_0^\pi A_0 \sin 2x\, dx + \int_0^\pi A_1 \sin x \sin 2x\, dx$$
$$+ \int_0^\pi A_2 \sin^2 2x\, dx.$$

All the integrals on the right-hand side of the equation vanish except the last, which is equal to $\frac{A_2 \pi}{2}$.

Hence $$A_2 = \frac{2}{\pi} \int_0^\pi y \sin 2x\, dx.$$

In other words, $A_2$ is equal to *twice* the mean value of the product of $y$ and $\sin 2x$ throughout the half period. In the same way all the other constants may be found. The process of analysing a complex function into its simple harmonic constituents is then reduced to little more than mere arithmetic.

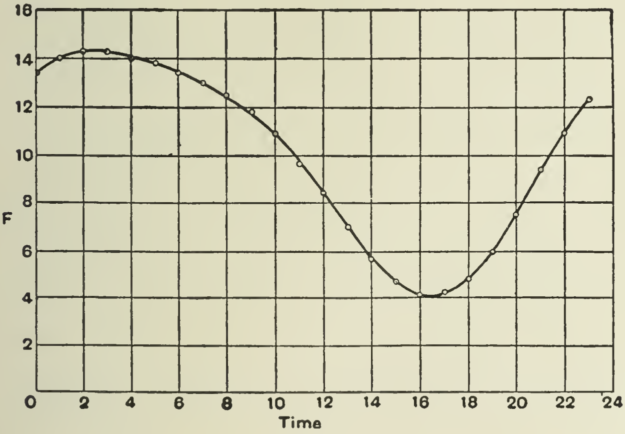

Fig. 36.

A single example will make this clear.* There is a certain complex periodic curve, one period of which is represented in Fig. 36. The problem is to find the simple harmonic or sine curves of which it is composed. Call $y$ the ordinate of the curve. Divide the whole period into twenty-four equal parts. Let T be the whole periodic time, and let $p$ stand for $\frac{2\pi}{T}$. Let $t$ be any fraction of the periodic time, so that $pt$ is the angular magnitude of the abscissa corresponding to any ordinate $y$. Since we have divided the period into twenty-four equal parts each of these corresponds to an angular interval of 15°. Hence, $pt$ is successively 15°, 30°, 45°, 60°, &c.

Measure from the curve the value of $y$ corresponding to each of these intervals, and tabulate them as follows:—

| $y$ | $pt$ | $y$ | $pt$ | $y$ | $pt$ |
|---|---|---|---|---|---|
| 13·3660 | 0 | 11·7940 | 135 | 4·8030 | 270 |
| 14·0355 | 15 | 10·8660 | 150 | 5·9645 | 285 |
| 14·3300 | 30 | 9·7060 | 165 | 7·5000 | 300 |
| 14·3295 | 45 | 8·3660 | 180 | 9·3940 | 315 |
| 14·1340 | 60 | 6·9645 | 195 | 10 8660 | 330 |
| 13·8295 | 75 | 5·6700 | 210 | 12·2940 | 345 |
| 13·4640 | 90 | 4·6705 | 225 | 13·3660 | 360 |
| 13·0355 | 105 | 4·1330 | 240 | | |
| 12·5000 | 120 | 4·1705 | 255 | | |

Proceed then to make a second table as follows:—

| I.<br>$t$ | II.<br>$y$ | III.<br>$pt$ | IV.<br>$\sin pt$ | V.<br>$y \times \sin pt$ | VI.<br>$\cos pt$ | VII.<br>$y \times \cos pt$ |
|---|---|---|---|---|---|---|
| 0 | 13·3660 | 0 | 0 | 0 | 1 | 13·3660 |
| 1 | 14·0335 | 15 | 0·2588 | 3 6324 | 0·9659 | 13·5569 |
| 2 | 14·3300 | 30 | 0·5000 | 7·1650 | 0·8660 | 12 4098 |
| 3 | 14·3295 | 45 | 0 7071 | 10·1324 | 0·7071 | 10·3124 |
| 4 | &c. | | | | &c. | |

Similarly in Column VIII. put the values of $\sin 2pt$; in Column IX. put the values of $y \times \sin 2pt$; and in Columns X. and XI. put $\cos 2pt$ and $y \times \cos 2pt$. Then the value of the constant term $A_0 + B_0$ is the mean or average value of all the 24 numbers in Column II.

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* The example above given is taken almost verbatim from a letter by Prof. John Perry in *The Electrician* of February 5, 1892, Vol. XXVIII., p. 362.

$A_1$ is *twice* the average of all the 24 numbers in Column V.

$B_1$ is *twice* the average of all the 24 numbers in Column VII.

$A_2$ is the same for Column IX., and $B_2$ for Column XI.

Any number of columns may be calculated corresponding to the multiple angles, $3\,p\,t$, $4\,p\,t$, &c., for higher terms of the Fourier series.

When we have all the sine and cosine terms it is easy to express $y$ in the form

$$y = A_0 + B_0 + \sqrt{A_1^2 + B_1^2}\sin(p\,t + \theta) + \sqrt{A_2^2 + B_2^2}\sin(2\,p\,t + \theta') + \&c.,$$

by grouping together the sine and cosine terms.

In the example calculated above it is found that the value of $y$ is approximately

$$y = 10 + 5\sin(p\,t + 30^\circ) - \sin(2\,p\,t - 60^\circ),$$

and this shows us that the given periodic curve is made up of two sine curves of amplitudes, 5 and 1 respectively, which differ in phase by 30°. The student will find it to be a useful exercise to take two or three simple periodic curves and add their ordinates into a complex periodic curve, and then by the Fourier analysis to re-discover the simple harmonic constituents again, and see if he can find the amplitudes correctly.

**§ 6. Simple Periodic Currents and Electromotive Forces.**—Returning, then, to electric currents, we may consider how a complex periodic current is made up of simple periodic currents superimposed. It is necessary to examine, in the first place, how a simple periodic current or electromotive force may be generated. Let A B C D (Fig. 37) be a rectangular frame or conductor, able to revolve round a vertical axis, O O′, in a uniform magnetic field. The adjacent figure represents the same in plan. If the frame revolve round the axis O O′, the total electromotive force acting round the circuit at any instant is numerically equal to the time rate of change of magnetic induction or number of lines of magnetic force passing through the circuit. If H is the field strength in C.G.S. units, $l$ the length of the side A C, and $k$ the length of the side C D, and $x$ the angle which at any instant the plane of the frame makes with a plane drawn at right angles to the lines of the field, then the magnetic induction or number of lines of force through the frame is the product of H, and the apparent size of the frame, as seen along the direction of the lines of force of the field, is equal to H $l$ $k$ cos $x$.

If the area of the frame is A square centimetres, the magnetic induction through it is H A cos $x$. The effective electromotive force acting to produce a current in the circuit is numerically equal to the time rate of change (decrease) of the magnetic flux or induction, or to

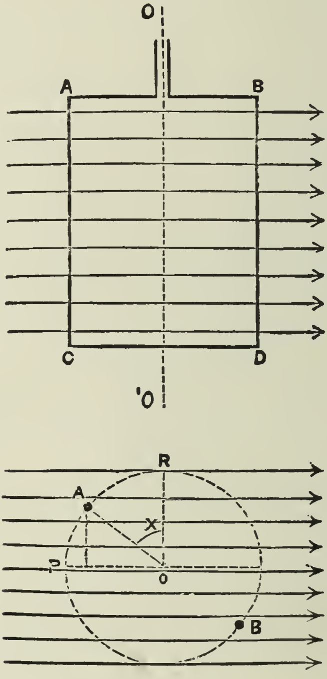

FIG. 37.

$$-\frac{d\,(\mathrm{H\ A}\cos x)}{d\,t} = -\mathrm{H\ A}\sin x\frac{d\,x}{d\,t}.$$

This last equation is merely a symbolic statement of the fact that, if such a frame of area A revolve round an axis perpendicular to the lines of force in a uniform magnetic field, H, with an angular velocity $\frac{dx}{dt}$, then the integral electromotive force acting round the frame at any instant corresponding to an angular displacement $x$ is $\mathrm{H\,A}\frac{dx}{dt}\sin x$.

If the angular velocity remains constant, the effective electromotive force will be simply proportional at any instant to the sine of the angular displacement of the frame from its initial position. Such a frame produces by its uniform revolution a simple periodic variation of electromotive force in its own circuit. If we suppose such a frame to have a closed circuit, then this periodically varying electromotive force will produce in the circuit an electric current which varies in strength very nearly as the sine of the angle of the displacement of the frame from its zero position when no lines of force penetrate through its area. Hence, graphically represented, the current varies according to a simple harmonic law, or is a simple sine current. We can then synthesise by the superposition of such simple harmonic electric currents any form of variable current, however complicated. Let a series of such sine inductors be joined up on one circuit (Fig. 38), each capable of being regulated as to angular velocity, and imagine these to revolve in magnetic fields of equal strength. These sine inductors are originally set with the plane of their frames at certain different but fixed angles to the planes at right angles to the fields of force in which they revolve, and they must be supposed to maintain these relative positions during their revolution. Accordingly, the effective electromotive force in the whole circuit, when they are all joined up in series and set revolving at fixed speeds, is represented by a function

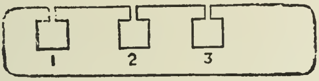

FIG. 38.

$$e = \mathrm{A}\sin x + \mathrm{A}'\sin x' + \mathrm{A}''\sin x'' + \text{\&c.};$$

and by Fourier's theorem any possible periodic variation of $e$ which, graphically described, is a single-valued function, can be produced by suitable values of the speeds and phase angles of these sine inductors.

The converse of the above proposition is also true. Let there be any periodic current-generating machine producing in a circuit an electromotive force, and therefore a current varying periodically according to any law. This kind or form of current could be exactly imitated by removing the given machine and substituting a series of sine inductors coupled in series and arranged so as to each produce a simple sine varying E.M.F., the respective sine currents having different phases and amplitudes, but being superimposed upon one another. That is to say, however complicated may be the nature of the periodic current which traverses a circuit, provided the same electric motions are repeated at regular intervals, we may build up this current by suitably superimposing in the same circuit a number of simple periodic currents of certain amplitude and wave-lengths and fixed difference of phase.

The above remarks may be taken as an outline of the analysis of any single-valued continuous function into a series of simple harmonic functions. To simplify language, we shall in future speak of a curve whose equation is of the form $y = \mathrm{A} \sin x$ as a *simple periodic curve*, and if such curve graphically represents the continuous variation of the flow of electricity past any section of a conductor, or the fluctuation of electromotive force in any circuit, we shall speak of such as a *simple periodic current* or a *simple periodic E.M.F.*

Any other mode of variation of these quantities which, graphically represented, would be a single-valued curve repeating the same form, will be spoken of as a complex periodic curve, current, or E.M.F., and, by the foregoing analysis, a complex periodic function can be analysed into a sum of simple periodic functions.

§ 7. **Description of a Simple Periodic Curve.**—The following method affords a very easy means of drawing a simple periodic curve. Take a cylinder or tube of pasteboard (*see* Fig. 39) and cut it through obliquely with a sharp knife, taking care to make the cut in one plane. The section of this cylindrical tube by an oblique plane will be an ellipse. Slit the tube open along the line A B and unfold it. Lay it down on another sheet of paper and draw a pencil line guided by the curved edge A Y Y′. Draw a dotted line, X X′, so that its vertical distance below the highest point Y on the curve is equal to its vertical distance above the points A and Y′, or make O Y equal to A X. Then move this cardboard template forward through a distance equal to its own width, and draw another piece of curve, repeating the first, and similarly placed (*see* Fig. 40).

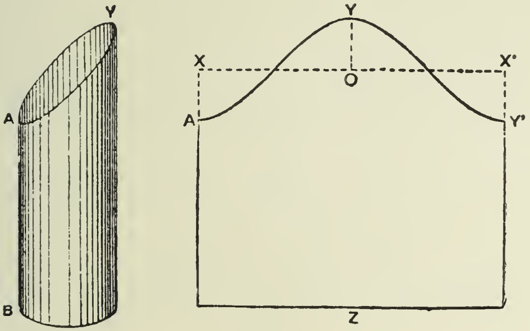

FIG. 39.

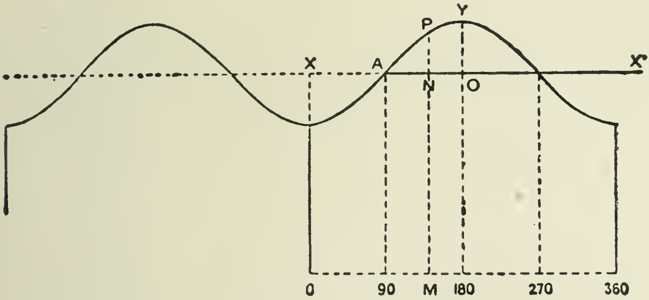

FIG. 40.

The resulting curve is a simple periodic or simple sine curve.* The distance X X′, equal to the circumference of the tube or to the width of the template, is the *wave length.* The distance O Y of the highest point above the mean line is the *amplitude*. If the bottom edge of the template is divided into 360 parts or units, then the distance O′ M, measured in such units of the foot of the perpendicular, let fall from any point P on O′ M, is the *phase* of the point P, measured in degrees.

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* "Elements of Dynamics" (Clifford), p. 22.

It is, perhaps, more convenient to reckon the phase of the point P by the magnitude of the line A N, or the distance of the foot of the perpendicular, let fall from P on X X′ from the point A, where the curve crosses the mean line. The phase of the maximum ordinate O Y is then 90deg.

§ 8. **The Value of the Mean Ordinate of a Sine Curve.**—Let Fig. 41 represent the semi-wave of a simple periodic curve; we shall proceed to prove some geometric properties of such a curve. Considering this curve as bounding an area of which the other including line is the datum line X X′, we shall first find the value of the mean or average ordinate. Let X X′ be divided into equal and very small intervals, such as N N′, of which the length is $d\,x$, and let X N be called $x$. Assume as a unit of length the radius of the cylinder of which X X′ is the semi-circumference. At each of these small elements raise ordinates, such as P N, to touch the curve. We require to find the mean value of all these equi-spaced ordinates when they are infinitely close. The arithmetic mean value of a number of things is the sum of them divided by their number. If $y$ denote the length of one such ordinate P N, and $\Sigma\,y$ the sum of all such ordinates when ruled at $n$ equal and exceedingly small intervals, each of length $d\,x$, then the average value of these infinitely numerous ordinates is

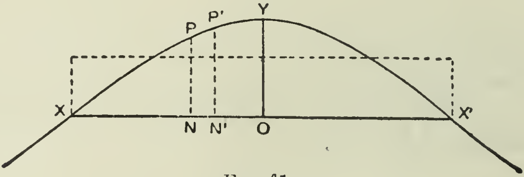

FIG. 41.

$$\frac{\Sigma\,y}{n}, \quad \text{or} \quad \frac{d\,x\,\Sigma\,y}{n\,d\,x}, \quad \text{or} \quad \frac{\Sigma\,y\,d\,x}{n\,d\,x};$$

but the sum of all such quantities as $y\,dx$, or P N . N N′ is the sum of all the areas of the little rectangular slips into which these infinitely numerous ordinates divide the area bounded by the curve and X X′, and $n\,dx$ is the length X X′; hence we have

$$\text{mean ordinate} = \frac{\text{area X Y X}'}{\text{length X X}'}.$$

The area X Y X′ is obtained by integrating the equation to the curve. Calling the maximum ordinate O Y, A, and the distance X N, $x$, the unit being the radius of the cylinder of which X X′ is the semi-circumference, we have as the equation to the curve

$$y = \mathrm{A} \sin x,$$

and therefore $\int y\,dx$, or $\mathrm{A}\int \sin x\,dx$,

between the limits 0 and $\pi$, is the value of the area of the curve. But

$$\mathrm{A}\int \sin x\,dx = -\mathrm{A}\cos x,$$

and this, between the limits $x = 0$ and $x = \pi$, is equal to 2 A. On the same scale, the length

$$\mathrm{X\,X}' = \pi;$$

hence, the average value of the infinitely numerous and equi-spaced ordinates is $\frac{2\,\mathrm{A}}{\pi}$, or the average ordinate of a simple sine curve is equal to $\frac{2}{\pi}$ times the maximum ordinate. The value of

$$\frac{2}{\pi} = 0{\cdot}6369.$$

Therefore, the average value of the equi-spaced ordinates of a simple periodic curve, or the true mean ordinate, is 0·6369 of the maximum ordinate; and, if the current or electromotive force varies according to a simple periodic law, the true mean current or the true mean E.M.F. is 0·6369 of the maximum current or E.M.F. during the phase.

We have here made use of one simple integration, and it is generally easier to master the elements of the infinitesimal calculus than to construct or follow proofs which aim at avoiding its use. We shall, however, indicate how the value of this mean ordinate may be found from first principles. If we call the length of the base line X X′ $l$, and divide it into $n$ equal and very small parts of length $\delta x$, then $n\,\delta x = l$. Erect at each interval an ordinate whose height is $y$, then the equation to the curve is $y = \mathrm{A} \sin \frac{\pi}{l} x$, where $x$ is, as before, the distance X N. The mean value, M, of the ordinate is the sum of all the values of the ordinates divided by their number, or is equal to $\frac{1}{n}(y_1 + y_2 + y_3 + \&c.)$.

$$\therefore \mathrm{M} = \frac{1}{n} \mathrm{A} \left\{ \sin 0 + \sin \frac{\pi}{l} \delta x + \sin \frac{\pi}{l} 2\, \delta x + \ldots \right.$$

$$\left. + \sin \left( \frac{\pi}{l} \overline{n-1}\, \delta x \right) \right\}.$$

The sum of the sine terms in the bracket is known by trigonometry to be equal to

$$\frac{\sin \left( \frac{\overline{n-1}}{2} \frac{\pi}{l} \delta x \right) \sin \frac{n}{2} \frac{\pi}{l} \delta x}{\sin \frac{\pi}{l} \frac{\delta x}{2}}.$$

Hence $$\mathrm{M} = \frac{\mathrm{A}}{n} \left\{ \frac{\sin \left( \frac{\overline{n-1}}{2} \frac{\pi}{l} \delta x \right) \sin \frac{n}{2} \frac{\pi}{l} \delta x}{\sin \frac{\pi}{l} \frac{\delta x}{2}} \right\},$$

which may be otherwise written—

$$\mathrm{M} = \frac{\mathrm{A}}{n\, \delta x} \frac{l}{\pi} \frac{\frac{\pi}{l} \delta x}{\sin \frac{\pi}{l} \frac{\delta x}{2}} \left\{ \sin \left( \frac{n\, \pi\, \delta x}{2\, l} - \frac{\pi\, \delta x}{2\, l} \right) \sin \frac{n\, \pi\, \delta x}{2\, l} \right\}.$$

When $n$ becomes infinite and $\delta x$ becomes zero, $n\, \delta x$ remains still equal to $l$; hence the above expression in this case reduces to the following :—

$$\mathrm{M} = \frac{\mathrm{A}}{\pi} \,.\, 2 \,.\, \sin^2 \frac{\pi}{2} \,.\, = \frac{2}{\pi} \mathrm{A},$$

for the value of $\dfrac{h}{\sin \frac{h}{2}}$ is 2 when $h$ becomes zero.

Accordingly the mean value of the ordinates, when they are infinite in number and equi-spaced, is $\frac{2}{\pi}$ times the magnitude of the maximum ordinate.

§ 9. **The Value of the Mean of the Square of the Ordinates of a Simple Periodic Curve.**—We require in the next place to find the value of the mean of the square of the ordinates to the same curve, assuming them to be equi-distant and infinite in number. If $y_1, y_2$, &c., are the ordinates, and $n$ the number, we require to find the value of

$$\frac{1}{n}\left(y_1^2 + y_2^2 + y_3^2 + \&c.\right),$$

the value of any ordinate being, as above,

$$y = \mathrm{A} \sin \frac{\pi}{l} x.$$

If X X′ or $l$ is divided into $n$ intervals, each equal to $\delta x$, so that $n \, \delta x = l$, we have to find the value of

$$\frac{\mathrm{A}^2}{n}\left(\sin^2 0 + \sin^2 \frac{\pi}{l} \delta x + \sin^2 \frac{\pi}{l} 2 \, \delta x \quad . \quad . \quad . \right.$$
$$\left. + \sin^2 \frac{\pi}{l} \overline{n-1} \, \delta x\right);$$

but, since

$$\sin^2\theta = \frac{1}{2}\left(1 - \cos 2\,\theta\right),$$

the series in the bracket can be replaced by

$$-\frac{1}{2} \cos 0 + \frac{1}{2} - \frac{1}{2} \cos \frac{\pi}{l} 2 \, \delta x + \frac{1}{2} - \frac{1}{2} \cos \frac{\pi}{l} 4 \, \delta x + \&c.$$
$$. \quad . \quad . \quad . \quad + \frac{1}{2} - \frac{1}{2} \cos \left(\frac{\pi}{l} 2 \, n \, \delta x - \frac{\pi}{l} 2 \, \delta x\right)$$

for $n$ terms. Hence the mean value M is

$$\mathrm{M} = \frac{\mathrm{A}^2}{n} \frac{n}{2} - \frac{\mathrm{A}^2}{2n}\left(\cos 0 + \cos \frac{\pi}{l} 2 \, \delta x + \&c.\right)$$

for $n$ terms.

The cosine series forms a progression of terms which begins with unity, since $\cos 0° = 1$, and passes down through zero to $-1$, and then up from $-1$ through zero to unity again,

for

$$\cos\left(\frac{\pi}{l} 2 \, n \, \delta x - \frac{\pi}{l} 2 \, \delta x\right) = +1,$$

when $n \, \delta x = l$ and $\delta x$ becomes infinitely small.

Since the angles are in arithmetic progression, we can pick out from this series pairs of cosine terms such that they are equal in magnitude but opposite in sign, and, when taken pair and pair, cancel each other out. The sum of the cosine series in the bracket is thus equal to zero, and, therefore,

$$M=\frac{A^2}{2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (22)$$

that is, the mean of the values of all the ordinates squared, taken equi-distant and infinite in number, is half the square of the maximum value.

We have, therefore, this result: If the current in a linear conductor varies in strength and direction in a manner which geometrically would be represented by the ordinate of a simple sine curve, the true mean value of the current strength is $\frac{2}{\pi}$ or 0·637 of its maximum value, and the mean value of the square of the current strength, taken at equal and very small intervals, is half the value of the square of the maximum value.

Since $\frac{2}{\pi}=0{\cdot}637$ and $\frac{1}{\sqrt{2}}=0{\cdot}707$, and since the difference $=0{\cdot}07$, the true mean current is less than the square root of the mean of the squares at each instant by an amount which is very nearly 10 per cent. of the latter.

If we proceed by the ordinary rules of the integral calculus, we can find the value of the mean of the squares of the ordinates of the sine curve as follows:—

Let $$y=A\sin x$$

be the curve; then

$$y^2=A^2\sin^2 x$$

$$=\frac{A^2}{2}(1-\cos 2x).$$

The mean value of the square of the ordinate between the limits 0 and $\pi$—that is, during the half-wave length—is

$$M=\frac{1}{\pi}\int_0^{\pi} y^2\,dx.$$

Therefore,

$$M = \frac{A_2}{2\pi}\int_0^\pi (1 - \cos 2\, n)\, d\, x,$$

$$M = \frac{A^2}{2\pi}\Big[_0^\pi x - \tfrac{1}{2}\sin 2\, x\Big]$$

$$= \frac{A^2}{2}.$$

Hence we reach the same result as above. In order to avoid repeating constantly the clumsy phrase *the square root of the mean of the squares of all the equi-spaced ordinates of a curve*, we may call this, in speaking, *the mean-square* value of the ordinate, and express it by the symbol $\sqrt{\text{mean}^2}$. Hence, $\sqrt{\text{mean}^2}\, y$ stands for the above particular kind of mean of $y$.

In practice, in alternating-current work, we hardly ever require to concern ourselves with the true mean of the ordinates of a simple or complex periodic curve. Chiefly we require to know or find the *square root of the mean of the squares of the ordinates* of a periodic curve taken at equi-distant positions throughout the period. Hence the $\sqrt{\text{mean}^2}$ value of the ordinate of a simple periodic curve is equal to the quotient of the maximum ordinate by the $\sqrt{2}$, for the mean of the squares of the equi-distant ordinate is equal to the value of $\frac{A^2}{2}$, as shown above, and hence the $\sqrt{\text{mean}^2}$ value is $\frac{A}{\sqrt{2}}$. Since $\sqrt{2} = 1{\cdot}414$ nearly, we see that the maximum ordinate of a simple sine curve is $\sqrt{2}$ times the $\sqrt{\text{mean}^2}$ ordinate. In the practical measurement of alternating currents, the value given by the instruments is nearly always the $\sqrt{\text{mean}^2}$ value of the instantaneous values throughout the period.

§ **10. Derived Curves.**—Let the curve in Fig. 42 represent the complete period of a simple periodic curve of which the equation is $y = A \sin \frac{\pi}{l}\, x$. Let P be any point on the curve. Then $PN = y$, $OY = A$, $XX' = l$. At P draw a tangent PT to the curve, and let it meet the datum line at T.

We shall call the trigonometrical tangent of the angle P T N, the *slope* of the tangent at the point P, hence $\frac{\text{P N}}{\text{T N}}$ = the slope.

If two points, P P′ (Fig. 43), are taken on the curve very near together, and a secant, P′ P T, is drawn through them,

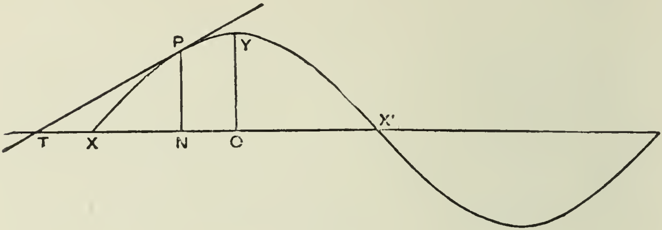

Fig. 42.

this secant will become a tangent when the points P P′ move up into contact. The ratio of $\frac{\text{P}'\,\text{M}}{\text{P M}}$ will then, in the limit, become the slope of the tangent. If now X N $= x - \frac{\delta x}{2}$, and

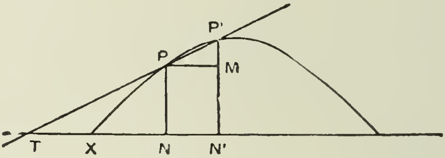

Fig. 43.

X N′ $= x + \frac{\delta x}{2}$, and P M = N N′ is $\delta x$, we have the equations

$$\text{P N} = \text{A} \sin \frac{\pi}{l}\left(x - \frac{\delta x}{2}\right)$$

and

$$\text{P}'\,\text{N}' = \text{A} \sin \frac{\pi}{l}\left(x + \frac{\delta x}{2}\right);$$

hence, $\frac{\text{P}'\,\text{M}}{\text{P M}} = \frac{\text{A}}{\delta x}\left[\sin \frac{\pi}{l}\left(x + \frac{\delta x}{2}\right) - \sin \frac{\pi}{l}\left(x - \frac{\delta x}{2}\right)\right]$.

The quantity in the bracket is identically the same as

$$2 \cos \frac{\pi}{l} x \sin \frac{\pi}{l} \frac{\delta x}{2};$$

and hence

$$\frac{P'M}{PM} = \frac{A}{\delta x}\frac{\pi}{l}\frac{\delta x}{2}\, 2 \cos \frac{\pi}{l}x \left[ \frac{\sin \frac{\pi}{l}\frac{\delta x}{2}}{\frac{\pi}{l}\frac{\delta x}{2}} \right]$$

$$= \frac{A\pi}{l} \sin \left( \frac{\pi}{l} \left( \frac{l}{2} - x \right) \right) \left[ \frac{\sin \frac{\pi}{l}\frac{\delta x}{2}}{\frac{\pi}{l}\frac{\delta x}{2}} \right].$$

When $\delta x$ is made infinitely small, the quantity in the square brackets is unity, and we have

$$\text{slope} = \frac{A\pi}{l} \sin \left( \frac{\pi}{l} \left( \frac{l}{2} - x \right) \right).$$

If we plot a curve whose ordinates at any point are the slope of the primal curve at the corresponding points, the above equation shows us three things—first, that it is a sine curve or simple periodic curve of the same type as the curve from which it is derived; second, that its maximum value is $\frac{\pi}{l}$ times the maximum value of the original; and third, that its zero ordinate corresponds to the maximum one of the original, and *vice versâ*.

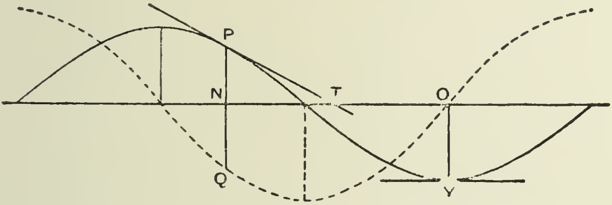

FIG. 44.

In Fig. 44 the firm line curve is a curve of sines

$$y = A \sin \frac{\pi}{l} x\,;$$

the dotted line is a curve of sines, whose ordinate QN at any point represents the *slope* of the tangent at P on the original curve. Accordingly, at Y, where the original curve is at its maximum, and the slope of its tangent is zero, the derived curve cuts the datum line, or has its phase shifted 90deg. backwards relatively to the original curve. In the language of the differential calculus, the firm line curve is the plotting of the curve $y = A \sin \frac{\pi}{l} x$, and the dotted curve is the plotting of $\frac{dy}{dx}$ as ordinates for the same abscissæ.

We may regard it from another point of view. Let the simple sine curve be supposed to be generated or marked out by a tracing point, P, which moves to and fro along a line P N P′ with a simple harmonic motion, whilst the point N moves uniformly along a straight line X X′. (*See* Fig. 45.)

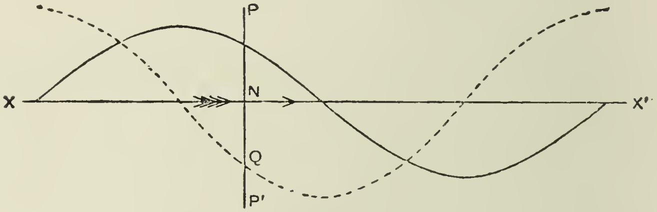

FIG. 45.

Draw as before the dotted curve whose ordinate Q N at any point represents the slope of the firm curve at the corresponding point P. Then the magnitude of N Q will represent the *rate* at which the ordinate P N is increasing or decreasing. For, in this case, distances such as X N, measured along the mean line, are proportional to time, and hence N makes a small movement forward in a small time $dt$; there is a corresponding decrease in the ordinate P N, which we may denote by $dy$, and accordingly $\frac{dy}{dt}$ represents the rate of decrease of P N. If the small forward movement of N causes N to advance through a space $dx$, $dx$ is proportional to $dt$, as the motion is uniform, and accordingly $\frac{dy}{dx}$ is proportional to $\frac{dy}{dt}$; hence $\frac{dy}{dt}$ is at any instant graphically represented by the slope of the tangent at P—that is, by the ordinate Q N. The dotted curve represents, therefore, the *rate of change* of the ordinates of the firm curve at that same instant. We shall call the dotted curve the *derived curve.*

If the ordinates of the original curve represent the instantaneous values of a simple periodic current flowing in a conductor, then the ordinates of the curve called above the derived curve will represent the rate of change of that current at the corresponding instants. The derived curve is a similar curve, but shifted backwards by one quarter of a wave length.

§ **11. Inductance and Inductive Circuits.**—Before we can proceed to discuss the laws of periodic current flow in circuits of various kinds, we must call attention to some of the fundamental properties of electric circuits. Every electric circuit in which a flow of electricity, whether continuous or periodic, can take place possesses three primary qualities, viz., *Resistance*, *Inductance*, and *Capacity.* The resistance of the circuit is a quality of it, in virtue of which a dissipation of energy takes place when an electric current flows through it. This specific quality is affected by change of temperature and by other alterations of physical condition. In the case of *pure* metals it has been shown* that, if the metal could be reduced to the absolute zero of temperature, its electrical resistance would vanish.

It is generally assumed that, apart from the change due to temperature or other altered physical conditions, the electrical resistance of a body is a constant quantity, which is independent of the current flowing through it. It is evident from experience that this is approximately, even if not accurately, the case. It would require very careful and extensive experiments before we should be entitled to say that the resistance of any circuit of any metal, when all corrections have been made for change of volume and temperature, is *exactly* the same when a thousand amperes are flowing through it as when one-thousandth of an ampere is flowing through it. Still less can we generalise and lay it down as absolutely and universally true. Careful experiments made by Prof. Chrystal at the Cavendish Laboratory (B.A. Report, 1876) showed that the resistance of a metallic circuit of one ohm is not different for currents of one ampere and for infinitely small currents by as much as $10^{-12}$ part.

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* Dewar and Fleming, *Phil. Mag.*, Sept., 1893.

There is, therefore, a strong probability that the specific electrical resistance of a body is a quality which is not dependent upon the current flowing through it, but is only affected by the temperature and physical condition of the body. According to Joule's law the rate of dissipation of energy when a current flows through a conductor is proportional to the square of the strength of the current. The total resistance of a circuit may, therefore, be numerically defined by the *rate* at which energy is dissipated by it when unit current flows in that circuit. In the practical units a circuit which, when traversed by one ampere of current, dissipates energy at the rate of *one joule* per second, or has a dissipation rate of one watt, is said to have a resistance of one ohm. The energy required to heat one gramme of water one degree centigrade in the neighbourhood of its maximum density is 4·2 joules. Since the rate at which energy is being dissipated at any instant in a circuit is measured by the numerical value of the product of the strength of the current flowing in it and the fall in potential down that conductor, it follows that the resistance of the circuit, or of any part of it, is also measured by the ratio between the numerical values of the fall of potential down the circuit or down that part of it and the current strength in that circuit, provided that the inductance of that circuit is negligible. The resistance of a circuit is, therefore, the energy-dissipating quality of it, and the *specific resistance* of any material is the resistance of one cubic unit of it between opposed faces of the cube.

In addition to the quality of resistance every circuit possesses also *inductance.* This quality of a circuit is one in virtue of which a current of finite value cannot be instantaneously produced even in a circuit of negligible resistance by a finite electromotive force, and when produced cannot be instantaneously destroyed. On account of the fact that all bodies possess *mass,* and therefore inertia, a finite force cannot generate a finite velocity in any material body in an infinitely small time. We see this fact exemplified in every falling body or starting train. A time element due to inertia comes into play which causes the motion of the mass to be acquired gradually, even under the action of a constant finite force. Experience shows that in all electric circuits there is a physical quality present which is related to current and electromotive force, just as the *mass* of a material body is related to velocity and dynamical force. In virtue of the *mass* of a body time is required for a finite moving force to generate a finite velocity, and in virtue of inductance of a circuit time is required for a finite electromotive force to generate a finite current. The inductance of the circuit bestows on it a quality which may be called its *electrical mass* or *electrical inertia.* The mass of a material body enables it in some way to become the vehicle of energy when in motion, and this energy of motion is called its *kinetic* energy. This kinetic energy is capable of being removed from the moving body, and the moving body can be brought to rest again only by taking away from it the kinetic energy it possesses as a whole, and transferring that energy to some other body or bodies, or to the molecules of the body itself. In like manner the inductance of a circuit may be said to cause it to be capable of being the vehicle of electrical energy when traversed by an electric current. A current cannot be instantaneously produced in finite value by any electromotive force, and when produced cannot be destroyed except by transforming that energy into some other form. Hence we have a very complete dynamical analogy between material bodies set in motion by what may be called materio-motive force and the flow of electric currents in circuits which possess inductance under the action of electromotive force.

These qualities may be compared as follows :—

| Motion in matter | corresponds to | Electric flow in circuits, |
|---|---|---|
| Mass $=m$ ............... | ,, | Inductance $=\mathrm{L}$, |
| Velocity $=v$ ............ | ,, | Current strength $=i$, |
| Momentum $=m\,v$ ...... | ,, | Electromagnetic momentum $=\mathrm{L}\,i$, |
| Kinetic energy of motion $=\frac{1}{2}m\,v^2$ ... | ,, | Electromagnetic energy $=\frac{1}{2}\mathrm{L}\,i^2$, |
| Rate of change of momentum $=m\frac{d\,v}{d\,t}$ | ,, | Rate of change of electromagnetic momentum $=\mathrm{L}\frac{d\,i}{d\,t}$. |

The force acting on a body which is being expended in making change of momentum is numerically measured at any instant by the rate of change of its momentum existing at that instant. So also the electromotive force which is being exerted to produce change of current strength or change of electro-magnetic momentum in a circuit is measured at any instant by the rate of change of electro-magnetic momentum.

There is an exact analogy between a heavy body being set in motion against inertia and friction and between an electric current being generated against inductance and resistance. For in the first case one part of the impressed force is being expended to overcome friction and the remainder to accelerate the mass against inertia, and in the second case one part of the impressed electromotive force is expended to overcome resistance and the remainder to increase the current against electrical inductance.

A circuit possessing inductance is called an *inductive circuit*, and a circuit whose inductance is negligible is called a *non-inductive circuit.* A truly non-inductive circuit can no more be realised in practice than a mass-less material body. The clear recognition that an electric circuit possessed a quality in virtue of which kinetic energy is associated with it when a current is flowing through it was first reached by Joseph Henry. In 1832 Henry made the observation that if the poles of a single galvanic cell are united by a short thick wire, then on breaking the circuit there is little or no spark; but if the uniting wire is a very long one, and, better, if it is coiled into a spiral, then there is a considerable spark at the contact on opening the circuit. In 1835 he expanded and continued these observations,* and noticed that if the wire is coiled round an iron core, and thus forms an electro-magnet, the spark and shock at breaking circuit are still more marked. Henry still further elaborated these observations in 1835.† Later still Faraday attacked the same problem, and devoted to its consideration the Ninth Series (§ 1048) of his "Electrical Researches."

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* *Journal of Franklin Institute*, March, 1835, Vol. XV., pp. 169-170.

† *Phil. Mag.*, 1840; *see also* Scientific Writings of Joseph Henry, pp. 87-97.

The chain of experiments which led to this inquiry was apparently started by a question addressed to Faraday by a Mr. Jenkin, one Friday evening, at the Royal Institution, as to the reason why a shock was experienced when a circuit containing an electromagnet was broken, the observer retaining in his two hands the ends of the circuit, but no shock was felt if the circuit contained neither magnet nor long coils of wire. Faraday seems speedily to have arranged an organised attack on the subject, and to have returned from his investigation burdened with the spoils of victory in the shape of the following facts:—

1. If a battery circuit is closed by a short thick wire, then, although there may be a very strong current existing in this wire, on breaking contact at any point little or no spark is seen, and if the two ends of the circuit are grasped in the two hands, and the interruption takes place between the hands, then little or no shock is experienced.

2. If a very long wire is used instead, then, although the absolute strength of the current may be less, yet the spark and shock at interruption are more manifest.

3. If this length of insulated wire is coiled up into a helix on a pasteboard tube, then, although the length of wire and strength of current are the same, yet the spark and shock are still more marked.

4. If the above helix has an iron core placed in it, both these effects are yet more exalted.

5. If the same length of wire is doubled upon itself, being, however, insulated, then the effects nearly vanish, and, whether straight or coiled, this doubled wire with current going up one side and down the other is no better in respect of spark and shock on interruption than a very short wire.

The first observation which Faraday makes upon the above results is that electricity would seem to circulate with something like *momentum* or *inertia* in the wire, and that the greater the length and strength of the current, so much the more power is there to run on and jump over the obstacle presented by the first thin layer of air which is introduced between the contacts as they are separated, giving rise to a spark. He saw, however, at once that, since the form of this circuit is an important factor, the idea of inertia *in the current* *itself* was fallacious, or else the mere doubling the wire could not nullify all the effects. He did not at that time see that the idea of momentum was exceedingly appropriate, but its allocation in the electric current itself was wrong.

The observation, however, which led him to a consistent theory was as follows. A bobbin was prepared, having wound on it two insulated wires, 1 and 2. The ends of 2 being left unconnected, the wire 1 was used to complete a circuit, and gave a spark on interrupting a current traversing it. As we have seen (Chap. I.), Faraday* had three years previously established the fact that the commencement and cessation of a current in one circuit would produce in another circuit, if closed, an inverse or a direct induced electric wave or transitory current. Now, on closing the second circuit through a galvanometer or loose contact, and interrupting a steady current flowing in the first circuit, he found that when circuit 2 was completed, so that an induced or secondary current could be generated in it, little or no spark happened at the place of interruption in 1; but, if circuit 2 was opened, then the interruption of circuit 1 gave rise to a bright spark at the contact. Faraday therefore inferred that when circuit 2 was closed adjacent to circuit 1, the current in 1 exerted its full inductive effect in generating secondary currents in 2; but that, if circuit 2 was open, then, there being no adjacent conductors, the current in 1 expended its inductive effect in producing induced currents in its own circuit, and this *self-induction* manifested itself by temporarily diminishing the strength of the current at starting and assisting or increasing it momentarily at the interruption. He was thus able, from this point of view, to picture to himself the circuit of 1 as occupied by a steady current, superimposed on which was another current he called the *inverse extra current*, lasting but a very short time at starting the steady current; and a *direct extra current* which flowed on and produced the effects of the spark or shock at the interruption of the circuit. These extra currents, or currents of self-induction, he found could be removed from the circuit itself and exhibited in a neighbouring circuit when that adjacent circuit was closed, and so fitted to be the seat of induced currents due to the *mutual induction* of the primary on this secondary circuit. Faraday then placed this theory under test by requiring it to furnish an explanation of the following experiment:—

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* Faraday's "Exp. Res.," § 1,090.

M and N (Fig. 46) were two mercury cups* which formed the terminals of three circuits—a battery circuit, B, a galvanometer circuit, G, and a circuit consisting of an electromagnet or helix, C. The needle of the galvanometer was blocked in such a way that the tendency to deflect under the steady current was prevented and the needle kept at zero; but it was free to deflect in the opposite direction under an oppositely directed current. This being the case, the raising of the battery wires out of the mercury cups was accompanied by a violent "kick" or deflection of the needle in the free direction.

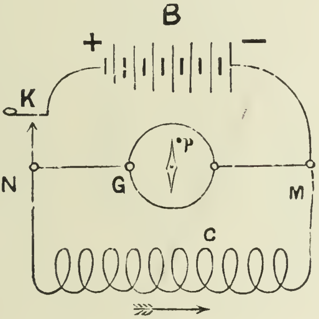

FIG. 46.

The action could clearly be explained by supposing that after the electromotive force of the battery is removed from the coil C, the current in it does not at once stop dead, but runs on like a heavy body and makes a backwash of current through the galvanometer in the direction from M to N. An illustration of the electromagnetic inertia of a coil on interrupting the current may be shown in a more modern form, thus: Let E (Fig. 47) be an electromagnet, and let L be an incandescent lamp of which the resistance is very large compared with that of E. Let S be a few cells of a storage battery supplying current, and let K be a key. On depressing the key the current flows both in the magnet and in the lamp arranged as a shunt on the magnet. This current, however, is, by assumption, not strong enough to illuminate the lamp. On raising the key and stopping the steady current through the lamp the electric inertia of the coil sends a momentary powerful current through the lamp, which causes it to flash up. Again, if a small shunt-wound dynamo be occupied in supplying current to a few incandescent lamps, and the two hands be employed to raise simultaneously the brushes from the armature, the momentary rush of current from the field-magnet due to this *extra current* will disagreeably impress the phenomenon upon the mind of the observer if the experiment is made with any but a very small dynamo. With a large dynamo this experiment is very dangerous to perform.

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* Faraday s "Exp. Res.," Vol. I., § 1,079.

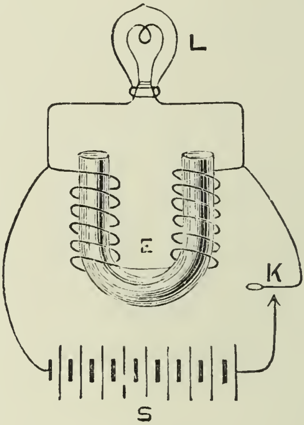

Fig. 47.

Neither of these experiments is well fitted to illustrate the extra current at the closing of the circuit or the effect of electric inertia on starting the current in a helix. The arrangement most suited to exhibit the whole effect is that of the differential galvanometer as used by Edlund, or that employing Wheatstone's bridge, due to Maxwell.

In Edlund's arrangement* a differential galvanometer is employed, of which the two coils $G_1$ $G_2$ are so placed and wound that when equal and oppositely-directed currents are sent through them the needle is unaffected. The coils are then connected, as shown in Fig. 48, to a battery, B, an electromagnet or helical coil, L, and a wire, R, of equal resistance to L, but wound double. The galvanometer coils are so connected to the circuits L and R that when the steady current from the battery flows through the divided circuit the needle remains at zero. On closing the circuit it is then found that the needle makes a sudden deflection in a direction indicating a brief current passing in coil $G_2$, and on breaking the circuit it makes another deflection, indicating a transitory current passing through $G_1$. In other words, the balance is destroyed at the instant of breaking and making, but restores itself again when the currents become steady. This experiment, therefore, most clearly shows that the electromagnetic helix L, although of exactly the same electrical resistance as the coil R, differs from it in possessing a peculiar quality, which it has in virtue of being in the form of a coil or helix, and to which the name self-induction or *inductance* has been given. We are able to define this term as follows:—The self-induction or inductance of a circuit is, speaking generally, a quality of it in virtue of which a finite and steady electromotive force applied to it cannot at once generate in it the full current due to its resistance, and when the electromotive force is withdrawn time is required for the current strength to fall to zero.

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* *See* Poggendorff's *Annalen*, 1849.

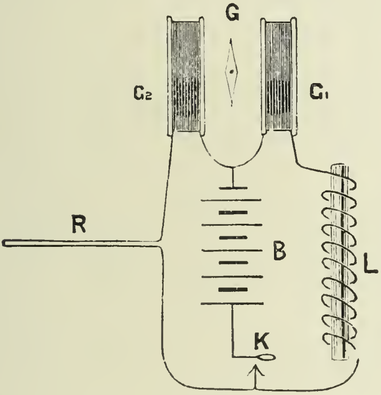

FIG. 48.

It must, however, be noticed that not only does the inductance of a circuit depend upon the geometrical form of the circuit, but it depends upon the magnetic permeability of the region which surrounds the circuit and on the magnetic permeability of the conducting circuit itself. If, in the arrangement with the differential galvanometer, the steady balance is obtained by using a copper wire helix wound on a cardboard tube and balanced against a non-inductive but equal resistance, it is found that the insertion of a soft iron core into the helix greatly increases the "kick" on making contact, indicating the passage of a greater quantity of electricity through the opposite galvanometer coil, and therefore a greater delay in the time of establishing the steady balance.

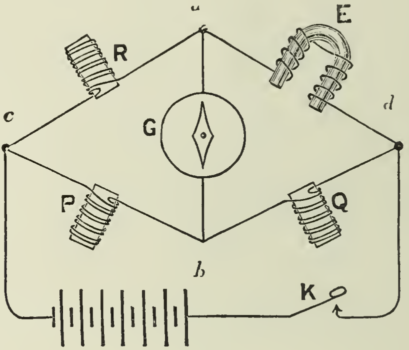

Fig. 49.

Maxwell's method of exhibiting the effect of inductance is a preferable arrangement.

Four conductors are arranged in a rectangle joining the points $a, b, c, d$, and the diagonals are completed by a galvanometer and battery (Fig. 49). P, Q and R are non-inductive resistances, and E is an electro-magnetic helix. If R and E are equal in actual resistance and P : Q=R : E, then the permanent closing of the battery circuit does not finally affect the galvanometer indication, and these circuits (battery and galvanometer) are then said to be *conjugate circuits*.

When, however, the battery key is first put down the galvanometer receives an impulse in one direction; when the key is kept down the galvanometer soon returns to zero, or to its original position. On raising the key the needle receives an impulse in the opposite direction. Examination of these impulses shows that if the current enters the quadrangle at *d*, on closing the key the potential rises at *b* faster than it does at *a*, and that on raising the key the potential dies down at *b* faster than at *a*; but that, if the "balance" is properly obtained, the points *a* and *b* reach finally the same potential when the key is kept closed.

An electromagnetic helix with or without a core of soft iron, behaves itself, therefore, towards an external electromotive force to which it is submitted as if it had an internal counter-electromotive force which gradually disappears—allowing the full current due to its resistance to be established in it more or less slowly, and behaves also, at the removal of this external electromotive force, as if a direct internal electromotive force suddenly made its appearance within it, this also gradually dying away.

The reader will see, therefore, that every electric circuit can not only dissipate electric energy in virtue of its resistance, but can conserve energy in virtue of its inductance. The resistance is measured by the rate of dissipation of energy which takes place when unit current (one ampere) flows through the circuit, and this rate of dissipation varies as the square of the current strength. The inductance is measured by the electromagnetic momentum associated with the circuit when unit current flows in it. Since, dynamically considered, the rate of change of momentum is a numerical measure of the force producing it, we must define electromagnetic momentum as that quantity the rate of variation of which numerically measures the electromotive force. We have already seen that if lines of magnetic induction (or force) perforate through and are linked with a circuit, then any variation of the number of these lines of induction or linkages gives rise to an induced electromotive force in the circuit equal in numerical magnitude to the rate of change of the included lines of induction. When an electric circuit is removed from all other circuits and magnets and is traversed by a current, the turns of this circuit are linked with and include the lines of magnetic induction created by itself. Hence we are able to connect the quantity we have called the electromagnetic momentum with the number of lines of magnetic induction which are linked with the circuit and which are created by the current flowing in that circuit. If a unit current is flowing in any circuit, there are a certain number of lines of magnetic induction at any instant linked with or perforating that circuit, and the number of these linkages defines the inductance of that circuit.

§ **12. Electromagnetic Momentum.**—The justification for the use of the term electromagnetic momentum is as follows:—When a heavy body is in motion it possesses at any instant *momentum*, in virtue of its inertia. Numerically the momentum of a heavy particle is obtained by taking the product of its mass and its velocity, each measured in appropriate units. The time rate of change of a body's momentum in any direction is, by the second law of motion, the measure of the force acting upon it in that direction, or, in the notation of the calculus,

$$\frac{d\,(m\,v)}{d\,t}=f.$$

We have seen that the induced electromotive force in a circuit depends on the time rate of change of the magnetic induction through it, and hence the magnetic induction at any instant through a circuit bears the same relation to the induced electromotive force in it that a body's momentum does to the mechanical force acting on it. Maxwell has accordingly employed the term *electromagnetic momentum* to represent the flux of magnetic induction or the number of lines of magnetic induction passing through a circuit, because it is upon the rate of change of this quantity that the induced electromotive force depends. Faraday very early recognised that induction effects depend on a *change* of some quantity. He makes frequent mention of the *electrotonic state*, and he spoke of a conductor in a magnetic field, when traversed by lines of induction, as in the electrotonic state, and he considered that when the electrotonic state was either assumed or disappeared its commencement or end was marked by the production of the induced electromotive force. Maxwell identified Faraday's electrotonic state with the total induction passing through the circuit or linked with it. Consider, then, the operations which go on when a conducting circuit—say a simple loop of wire—is subjected to a steady electromotive force. The instant that force is applied, a current begins to flow in the circuit; the instant that current begins, lines or rings of induction spread out from the circuit; and the loop at any instant encloses a certain number of lines of induction which are increasing at that instant at a certain rate. A counter or opposing electromotive force exists in that circuit numerically equal to the time rate of increase of this induction. In circuits which do not enclose or surround iron or other magnetic metal, or which are immersed wholly in a medium of constant permeability, the magnetic induction at any point in the neighbourhood of the circuit is numerically proportional to the strength of the current at that instant flowing in the circuit. This is the fact which lies at the root of the operation of most galvanometers, viz., that the field at any point in the neighbourhood of the coil is simply proportional to the strength of the current flowing in the coil. If, then, $i$ represent the strength of the current at any instant in the circuit, and L be a certain constant quantity such that $\mathrm{L}\,i$ represents the induction through the coil or circuit due to the current $i$ in it, then $\mathrm{L}\,i$ is the measure of the electromagnetic momentum of that circuit. This quantity L is a coefficient which, in this case, is dependent only upon the geometrical form of the circuit, and, under the assumption that there is no magnetic material in or near the circuit through which the lines of induction can pass, it is a constant quantity.

This quantity L is called the constant *coefficient of self-induction* of the circuit, or, more shortly, *the inductance* of the circuit.

*The inductance*, or the *coefficient of self-induction*, is thus defined:—In the case of circuits conveying electric currents which are wholly made of non-magnetic material and wholly immersed in a medium of constant magnetic permeability, the total magnetic induction through the circuit per unit of current flowing in that circuit when removed from the neighbourhood of all other magnets and circuits is the numerical measure of the inductance or of the coefficient of self-induction. Otherwise, the ratio of the numerical values of the electromagnetic momentum of such circuit and the current flowing in it when totally removed from all other currents and magnets is the numerical value of the *inductance* of that circuit.

§ 13. **Electromagnetic Energy.**—Let us confine our attention first to one circuit of constant inductance or self-induction in which a current is being generated by a constant electromotive force applied to it. Each increment of strength of the current creates an electromotive force opposing the impressed or external electromotive force. Hence this external electromotive force has to do work against an opposing force of its own creating all the time the current is rising in strength. When a mechanical force overcomes a resistance through a certain distance, mechanical work is being done, and, accordingly, we may ask—What is the electromotive force doing all the while it is increasing a current against an opposing electromotive force? The answer is, it is doing electrical work. The result of causing a current having a strength $i$ at any instant to flow for a small time, $d\,t$, against an opposing E.M.F. at any instant equal to $e$, is that a quantity of work, represented by $e\,i\,d\,t$, is done in the time $d\,t$. If $e$ is the instantaneous value of the opposing electromotive force of self-induction, it is measured at any instant by the rate of change of electromagnetic momentum $\mathrm{L}\,i$, or by $\mathrm{L}\dfrac{d\,i}{d\,t}$.

Hence the work done in raising the current from a strength $i$ to a strength $i+d\,i$ against the counter-electromotive force of self-induction is $\mathrm{L}\dfrac{d\,i}{d\,t}i\,d\,t=\mathrm{L}\,i\,d\,i$, and if this is integrated between limits zero and I, we get the whole quantity of work so done against self-induction alone in bringing up a current from zero to its full value, I, in the conductor, but

$$\int_0^{\mathrm{I}} \mathrm{L}\,i\,d\,i=\tfrac{1}{2}\,\mathrm{L}\,\mathrm{I}^2.$$

Exactly in the same way it may be shown that the work done in bestowing a velocity V upon a mass M is measured by the quantity $\frac{1}{2}\,\mathrm{M}\,\mathrm{V}^2$.

The total work done against the electromotive force of self-induction in creating a current I in a conductor of constant inductance L is, then, numerically equal to half the square of the final current strength, multiplied by the value of the constant inductance or coefficient of self-induction.

The equivalent of this work is found in the magnetic field formed round the conductor, and hence the formation of a magnetic field represents so much energy, measurable in foot-pounds per cubic inch, or in any other similar units, such as ergs or kilogrammetres, per cubic centimetre of field.

Next let us consider the case of two circuits. Let the constant coefficient of self-induction of the first be L, and let it be traversed at any instant by a current $i$. Let the inductance of the other be N, and let it be traversed by a current $i'$. Let the *coefficient of mutual induction* be M.

The definition of this last quantity is as follows:—If both circuits be traversed by unit currents, and if there be no other field than that due to these currents, the number of lines of induction which traverse *both* circuits, or are linked with both circuits, is called the constant coefficient of mutual induction. It will be a quantity constant for a given form and position of the two circuits on the assumption that the lines of induction flow in a medium of constant magnetic permeability. Hence, if we consider the work done, $dE$, in raising the currents $i$ and $i'$ by small increments, $di$ and $di'$, in a small time, $dt$, we find it consists of four parts—a part, $L\,i\,d\,i$, representing work done by the current $i$ against its own counter-electromotive force, and a similar part, $N\,i'\,d\,i'$, for the other circuit, then a portion, $M\,i\,d\,i'$, representing the work done by the current $i$ in its own circuit against the induced electromotive force, due to the increment of the current $i'$ in the other, and lastly, a similar part, $M\,i'\,d\,i$, for the second circuit. Hence, we have

$$dE = L\,i\,d\,i + M\,i\,d\,i' + M\,i'\,d\,i + N\,i'\,d\,i'.$$

Integrating this between the limits zero and I for one circuit, and zero and I′ for the other, we find the whole energy represented by the two currents I and I′ flowing in the circuits to be

$$E = \tfrac{1}{2}\,L\,I^2 + M\,I\,I' + \tfrac{1}{2}\,N\,I'^2. \quad . \quad . \quad . \quad (24)$$

The electro-kinetic energy is said to be a quadratic function of the currents and the inductances.

§ **14. The Unit of Inductance.**—*The Henry.*—The practical unit of inductance is called *one henry.* The henry is the unit of inductance which is in consistent relation with the ohm, the volt, the ampere, the watt, and the joule. A circuit has an inductance of one henry when there are $10^9$ C.G.S. lines of magnetic induction linked with the circuit, or when there are $10^9$ linkages of current and magnetic lines of induction, under the condition that one ampere of current traverses the circuit, and that no other lines of induction than those due to itself perforate or are linked with the circuit. If the circuit is a coiled circuit of wire, and the wire makes $n$ turns round a total number N lines of magnetic force or induction, then there are $n$ N linkages of circuit and induction. Suppose that we have a circular solenoid formed by winding thin, closely placed, covered wire on a wooden ring of circular cross section. Let the mean cross section of the circular solenoid be S, and let the induction density in the interior of the solenoid be **B**, when one ampere is sent through the wire windings. Then there are **B** S lines of induction in all round the interior of the solenoid. Let there be N turns of wire in all on the ring, then there are N S **B** linkages of current and magnetic lines of force. The inductance of this solenoid, or, its self-induction measured in henrys, is $10^8$

$$\frac{\mathbf{B}\,S\,N}{10^9} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

If we consider the above circular solenoid or very long straight solenoid to be wound on a wooden or non-magnetic core, the value of the induction **B** in the interior is numerically the same as that of the magnetic force in the interior, viz., $\frac{4\pi}{10}\frac{A\,N}{L}$ units, where A is the ampere current in the coil, N the number of windings, and L the mean length of the coil. Hence the self-induction of such a coil in henrys is $\frac{4\pi}{10^{10}}\frac{A\,S}{L}N^2$, or is proportional to the square of the total number of windings N.

An enormous number of wire windings are, therefore, necessary to obtain any sensible fraction of a henry of inductance in a circuit in which the path of the lines of magnetic force is wholly in air, or in some body of unit magnetic permeability.

In the case of such air or non-ferric magnetic circuits the inductance is a constant quantity which depends only on the geometrical form of the circuit.

The moment, however, that we introduce an iron core we alter the state of affairs. The inductance is then no longer the same for all values of the induction, because the induction varies with the magnetising force, but not proportionately to it. Hence, we cannot speak generally of the inductance of such an electric circuit when linked with an iron, or partly iron, magnetic circuit, except to define its value corresponding to one particular value of the current. We can, however, always refer to the *instantaneous value* of the inductance when we have occasion to mention a particular value which it has when varying from instant to instant. For very low or very high degrees of magnetisation, however, the inductance of such a circuit will be constant, but very different.

The following table taken from figures obtained by Mr. A. E. Kennelly and Prof. Ayrton* will furnish the reader with an idea of the approximate magnitude of the inductances of various well-known instruments, measured in henrys and fractions of a henry :—

| | |
|---|---|
| Cardew voltmeter ........................... | about 1 microhenry. |
| Ordinary telegraph sounder .............. | 25—50 millihenrys. |
| Astatic mirror galvanometer, about 5,000 ohms.............................. | 2 henrys. |
| Mirror speaking galvanometer, 2,250 ohms .............................. .. ... | 3·6 henrys. |
| Single coils of Morse receiver ..... ...... | 93 millihenrys. |
| Induction coil (giving 2in. spark) secondary circuit .............. ...... | 51 . . henrys. |
| Shunt dynamo (100 volts, 35 amps.) armature................................ | 5 henrys. |
| Field magnets of the above dynamo in series .................... ........... | 13·6 henrys. |
| Ordinary electric bell, 2·5 ohms resistance ................................. | 12 millihenrys. |

§ **15. Current Growth in Inductive Circuits.**—We see, therefore, that when electric energy is spent on a conductor in the production of a current, in addition to the energy taken up in the performance of any chemical or external mechanical work, part of it is dissipated as heat by an irreversible process, and part is associated with the circuit in a recoverable form, and is taken up in the establishment of the energy of the magnetic field, which then exists round the conductor. This last portion of the energy, however, dissipates itself as soon as the impressed electromotive force is withdrawn.

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* See *The Electrician*, Vol. XXVI., p. 290, also pp. 267 and 305.

A mechanical operation analogous to that of starting a current in a wire may be found in the process of starting from rest, or increasing the speed of, a heavy fly-wheel which runs in bearings with friction. On applying a twisting force or *torque* to the axle of the wheel we get up its speed. To maintain the speed, force has to be continually applied to the wheel, and the work so done against friction is frittered away irreversibly into heat in the bearings. The friction is analogous to the electrical resistance; it may be called the frictional resistance.

When the speed of the wheel is constant there is, however, associated with the wheel a certain quantity of energy in a kinetic form measured by $\frac{1}{2} I \omega^2$, where I is the moment of inertia, and $\omega$ the angular velocity of the wheel. As soon as the maintaining force is withdrawn this accumulated energy dissipates itself in heat by friction, or is utilised in some other way. During the time that the speed of the wheel is being increased, force must be applied to it for two purposes: firstly, to increase its angular momentum, and, secondly, to overcome the friction at the bearings. Suppose that, instead of revolving on bearings with friction, the fly-wheel revolves in a more or less viscous fluid, and that the bearings are truly frictionless; in such case the frictional resistance to motion would be fluid resistance, and would for low speeds be approximately proportional to the angular velocity. If I is the moment of inertia and $\omega$ the angular velocity of the wheel at any instant, then it is shown in treatises on dynamics that the product of the moment of inertia and the rate of change of the angular velocity at the instant, or $I\frac{d\omega}{dt}$, is the numerical measure of the torque or twisting force acting on the wheel to increase its angular velocity, friction being neglected. If we call the constant frictional coefficient B, so that $B\omega$ is at any instant the measure of the force necessary to maintain the motion against friction, the total torsional or twisting force acting on the wheel to maintain its angular velocity against the force of friction, and to increase it against the force of inertia, is

$$F = B\omega + I\frac{d\omega}{dt}.$$

A precisely similar equation may be found connecting the electromotive force, electric current, electrical resistance, and inductance in the case of current starting in a wire. The above equation gives us a value for the instantaneous angular velocity, or enables us to find the angular velocity after any time when F, B, and I are given. When a current of strength $i$ is flowing steadily in a linear conductor, such as the wire under consideration, the energy associated with it in the form of a magnetic field is measured by the quantity $\frac{1}{2}Li^2$, where L is the quantity called the inductance of the circuit. Since this quantity L bears to electromagnetic energy a relation similar to that which the moment of inertia of a wheel does to the energy of its rotation, it might be called the coefficient of electromagnetic inertia; but, as this would be a cumbersome name, it has been called the inductance, or, frequently, the self-induction of the circuit. The numerical product of the moment of inertia and the angular velocity of the wheel is called the angular momentum, and, analogously, the product of the inductance of a circuit and the current flowing at that instant through it is called the electromagnetic momentum.

The rate at which the angular momentum of a wheel is increasing or diminishing at any instant is a measure of the rotational force, or the couple acting on it at that instant. So also the rate of change of the electromagnetic momentum of a circuit is the measure of the electromotive force acting on it as far as mere change of current strength is concerned, and omitting, for the present, that part of the electromotive force required to overcome the true resistance. We have, then, the following parallel between a fly-wheel, with moment of inertia I, revolving frictionlessly, and having an angular velocity $\omega$ at any instant, and an electric circuit of inductance L, having a current of strength $i$ flowing in it at any instant:—

Angular kinetic energy of the wheel, or energy of rotation $= \frac{1}{2}I\omega^2$

Electromagnetic energy of the circuit ........................ $= \frac{1}{2}Li^2$

Angular momentum of wheel................ .................. $= I\omega$

Electromagnetic momentum of circuit ..................... $= Li$

Rate of change of angular momentum of wheel = couple or torsional force causing rotation .. ..................... $= I\dfrac{d\omega}{dt}$

Rate of change of electromagnetic momentum = electromotive force employed in changing current strength $= L\dfrac{di}{dt}$

The symbol (=) must in the above be understood as equivalent to the phrase "is measured by."

In the electric circuit, over and above the electromotive force which is required to change the electromagnetic momentum, there is an amount required to overcome the frictional resistance of the wire, and which is defined and measured by Ohm's law $E = Ri$. Hence, at any instant, if E is the impressed electromotive force acting on the circuit, we may divide E into two parts, one part equal to $Ri$ by Ohm's law, which is sometimes called the effective electromotive force, and which is that part of the impressed electromotive force which is operating to overcome the true resistance of the circuit, and another part equal to $L\dfrac{di}{dt}$, which is the part operating to change the strength of the current at that instant, producing a small change, $di$, in the current strength $i$ in a time $dt$. Hence, in mathematical language, we have

$$E = Ri + L\frac{di}{dt} \quad \ldots\ldots \quad (26)$$

This is the fundamental equation for varying or periodic currents, when the periodicity is not so rapid as to affect the uniform distribution of the current over the cross section of the wire, and when the electrostatic capacity of the circuit may be neglected. The part $L\dfrac{di}{dt}$ is often called the counter-electromotive force of self-induction, and the above equation might be read in words—

$$\left.\begin{array}{c}\text{Total}\\ \text{Impressed}\\ \text{Electro-}\\ \text{motive}\\ \text{Force.}\end{array}\right\} = \left\{\begin{array}{l}\text{Electromotive Force}\\ \text{employed in over-}\\ \text{coming resistance, or}\\ \text{the Effective Electro-}\\ \text{motive Force.}\end{array}\right\} + \left\{\begin{array}{l}\text{Electromotive Force}\\ \text{employed in chang-}\\ \text{ing strength of cur-}\\ \text{rent, or the Inductive}\\ \text{Electromotive Force.}\end{array}\right.$$

We might arrive at this fundamental equation otherwise thus :—The total rate of expenditure of energy in the circuit is at any instant measured by the product of the current at that instant existing in the wire and the difference of potential between its ends. The energy expended in the circuit is at any instant being partly dissipated *at a rate* equal to $\mathrm{R}i^2$, R being the ohmic resistance and $i$ the current, and partly being stored up in the field *at a rate* equal to the rate of change of the quantity $\frac{1}{2}\mathrm{L}i^2$. Hence we have :—

$$\left.\begin{array}{c}\text{Rate of supply}\\ \text{of energy}\end{array}\right\} = \left\{\begin{array}{l}\text{Rate of dissi-}\\ \text{pation of energy}\\ \text{as heat}\end{array}\right\} + \left\{\begin{array}{l}\text{Rate of absorption}\\ \text{or storage of energy}\\ \text{in the magnetic field,}\end{array}\right.$$

and this in symbols is—

$$\mathrm{E}i = \mathrm{R}i^2 + \frac{d}{dt}(\tfrac{1}{2}\mathrm{L}i^2),$$

or

$$\mathrm{E} = \mathrm{R}i + \mathrm{L}\frac{di}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

which is our fundamental equation.

At this stage we must particularly caution the student to note one thing. The quantity L, which is called the inductance of the circuit, is a constant and definite numerical quantity for any given form of circuit only as long as this circuit consists of non-magnetic material and is immersed in a non-magnetic medium. If, however, the circuit embraces or is embraced by iron, as in the case of an electromagnet, or is immersed in a medium which is not diamagnetic but magnetic like iron, then it is no longer a constant quantity, but the inductance varies from instant to instant with the strength of the current flowing in the circuit. In this chapter we suppose ourselves dealing only with circuits of constant inductance, and in which the value of L is fixed by the form of the circuit alone.

§ **16. Equation for Establishment of a Steady Current.**—We return to our discussion of equation (26) (§ 15). When a current is flowing in a conductor, we may picture it as surrounded by its lines of magnetic induction *properly* mapped out. That is, so that the number of the lines of induction passing perpendicularly through a small unit of area taken at any point in the field is equal to the numerical value of the mean strength of the magnetic field over the area. If the circuit has the form of a loop (Fig. 50) lying on a horizontal plane, with the current circulating round it in the opposite direction to that in which rotate the hands of a watch, then the lines of induction must be considered as springing out from the upper surface, and turning outwards and over the conductor, so as to re-enter the loop from the under surface. The closed circuit is, therefore, linked with a certain number of lines of induction, which, if the circuit is composed of non-magnetic material, are proportional in number to the strength of the current at that instant. Any increase in strength of the current causes more lines of induction to grow out from the circuit, and packs the loop fuller of lines of induction. By Faraday's law, any increase of the number of lines of induction traversing or linked with a circuit creates an induced electromotive force numerically equal to the rate of increase of that number at that instant. Hence, if 100 million lines of induction—C.G.S. measure—are put or inserted at a uniform rate in one second into a circuit, it will create an induced E.M.F. of one volt in it. If lines of induction are thrust into a circuit, the direction of the current induced is counter clockwise, as seen from that side of the circuit at which they are thrust in (*see* Fig. 50).

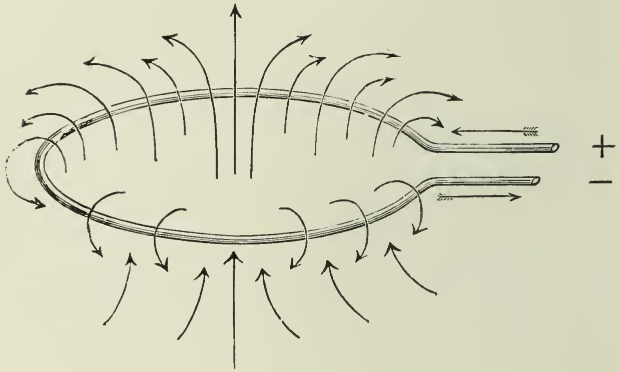

FIG. 50.

Applying this to the case before us, it is easily seen that any increase of current strength in the circuit in Fig. 51 crowds the space with more lines of force, and therefore creates in it an electromotive force of self-induction opposed to the impressed electromotive force which is acting to increase the current; and, so long as the current is increasing, this counter E.M.F. is at each instant proportional to the rate of growth of the current strength.

We can cast our equation (26)—

$$E = Ri + L\frac{di}{dt}$$

into another form, thus:—

$$\frac{E}{R} - i = \frac{L}{R}\frac{di}{dt},$$

where $\frac{E}{R}$ is the maximum value which the current can attain.

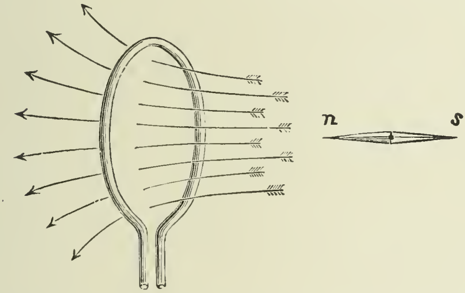

Fig. 51.—Lines of force being crowded into a circuit, inducing a counter-clockwise E.M.F., as seen from the side at which they are put in.

Let us call this value I. The quantity $\frac{L}{R}$, or the ratio of the inductance to the resistance of the circuit, is called the *time-constant* of the circuit; let this quantity be denoted by T. We then have

$$I - i = T\frac{di}{dt},$$

which, in words, is a statement that if a steady E.M.F. is made to act on any circuit whose time-constant is T, the amount by which at any instant the current falls short of its full value is equal to its rate of growth at that instant, multiplied by the time-constant.

§ 17. **Logarithmic Curves.**—A curve such that the rate of growth or shrinkage of the ordinate or slope of the curve is proportional to the ordinate itself is called a logarithmic curve.

Let a curve (Fig. 52) be described by the extremity P of an ordinate, P M, which moves uniformly along O X, parallel to itself, and let P M *shrink* in height at a rate proportional to its height at any instant. The differential equation to such a curve is then

$$y = -A\frac{dy}{dt},$$

and since $e^{-Bx}$ (where $e =$ the base of Napierian logarithms $= 2{\cdot}71828$) is a function which fulfils this condition of having

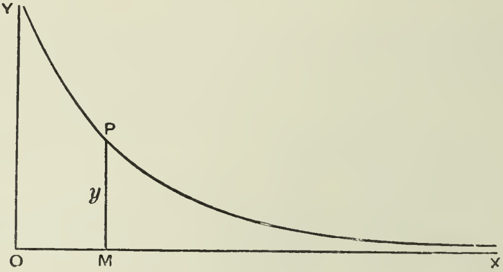

Fig. 52.

a differential coefficient proportional to itself, we can write the solution of the above

$$y = e^{-\frac{t}{A}} + \text{a constant},$$

for it is at once seen that by differentiating the equation

$$y = e^{-\frac{t}{A}} + \text{a constant},$$

we obtain

$$\frac{dy}{dt} = -\frac{e^{-\frac{t}{A}}}{A}.$$

and therefore

$$y = -A\frac{dy}{dt}.$$

Returning to our equation for the current, we can write, as an equivalent for the equation

$$I - i = T\frac{di}{dt},$$

the equation $$I - i = -T\frac{d(I-i)}{dt},$$

or $$\frac{dt}{T} = -\frac{d(I-i)}{I-i}.$$

Integrating this we have as a solution

$$-\frac{t}{T} = \log (I - i) + \text{a constant.}$$

The constant has to be determined by the condition that, when $t = 0$, $i = 0$, which gives constant $= -\log I$. Hence the complete solution is

$$-\frac{t}{T} = \log (I - i) - \log I,$$

or $$I - i = I e^{-\frac{t}{T}}.$$

This last equation expresses the fact that the amount by which the current falls short of its full value, I, at any time, $t$, after applying the E.M.F., is a fraction of its full value equal to $e^{-\frac{t}{T}}$. When $t = 0$, or at the instant of closing circuit, $I - i = I$, or the current $i = 0$; when $t = T$, $I - i = \frac{I}{e}$, or the deficit from full current is equal to $\frac{1}{2{\cdot}718}$ × the maximum current. Hence we may define the *time-constant* of a circuit as the time reckoned from the instant of closing the circuit in which the current rises up to a value equal to $\frac{e-1}{e}$ of its full value, or to about 0·632 of its maximum value. Approximately we may define the time-constant as the time from closing the circuit in which the current rises up to two-thirds of its maximum value $\frac{E}{R}$.

The rise of current strength in a wire of inductance L and resistance R, when a steady external electromotive force, E, is applied to the circuit, can be represented by a current curve, as shown in Fig. 53. Let OX be a time line on which we mark off time as lengths reckoned from O; let lines drawn vertically to this represent the current strength at any instant in a circuit of time constant T, inductance L, and resistance R; and let

$OY=I\frac{E}{R}$ represent the maximum current which is finally found in the circuit. On applying the electromotive force E to the circuit, the current strength grows up in the wire as graphically represented by the curve, the law of growth being that the rate of growth at any instant, multiplied by the time-constant, is equal to the difference between the actual current at that instant and the maximum current strength finally attained, or, symbolically,

$$I-i=T\frac{di}{dt},$$

the solution of the above differential equation being

$$I-i=Ie^{-\frac{t}{T}},$$

or

$$i=I\left(1-e^{-\frac{t}{T}}\right). \quad . \quad . \quad . \quad . \quad . \quad (27)$$

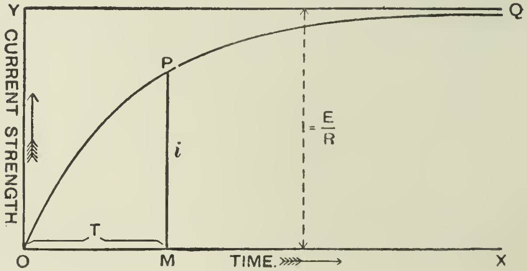

FIG. 53.

This last equation gives us the value of the current strength at any time $t$ seconds after closing the circuit, in terms of the time-constant, and the maximum current, I, which is finally attained.

The maximum current, I, would be produced at once in the circuit if its inductance were zero, so that we may finally formulate the law of growth of current in a circuit of constant inductance L, resistance R, and no sensible capacity, by saying that *the current strength at any instant, added to the rate of growth of the current strength at that instant multiplied by the time-constant, is equal to the current which would exist in the circuit if its inductance were zero.*

§ 18. **Instantaneous Value of a Simple Periodic Current.**—The application of these principles to the case of simple periodic currents will lead to another important equation. Let there be a circuit which has an inductance L and resistance R, and let a simple periodic electromotive force act upon it; let the maximum value of this E.M.F. be E, and let $p$ stand for $2\pi n$, where $n$ is the frequency of the oscillation, or $\frac{1}{n}$ is the duration of one single complete period. $p$ is a quantity of the nature of an angular velocity, and may be called the *pulsation*. Then, if $t$ is the time which has elapsed from the commencement of the wave of E.M.F. and $e$ is the actual value of the E.M.F. at that instant,

$$e = \mathrm{E} \sin p\,t.$$

In this case the impressed electromotive force varies from instant to instant, passing from zero to a maximum E, then to zero again, and then to a negative maximum $-\mathrm{E}$. Accordingly, our fundamental equation for the current strength at any instant is expressed thus:

$$\frac{d(\mathrm{L}i)}{dt} + \mathrm{R}\,i = e = \mathrm{E} \sin p\,t. \quad . \quad . \quad . \quad (28)$$

For, the total rate of expenditure of work on the circuit at any instant when the current has a value $i$ is $e\,i$, and this must be equal to the rate at which electrical work is being dissipated as heat, or to $\mathrm{R}\,i^2$ by Joule's law, and to the rate at which work is being stored up in the magnetic field, which is

$$\frac{d}{dt}(\tfrac{1}{2}\,\mathrm{L}\,i^2).$$

Hence
$$\frac{d}{dt}(\tfrac{1}{2}\,\mathrm{L}\,i^2) + \mathrm{R}\,i^2 = e\,i,$$

or,
$$\mathrm{L}\frac{di}{dt} + \mathrm{R}\,i = \mathrm{E} \sin p\,t. \quad . \quad . \quad . \quad . \quad (29)$$

In order to solve this differential equation, and obtain the value of the current $i$ in the circuit at any instant under the periodic electromotive force, we may adopt a well-known algebraic device, and substitute for the value of $\sin p t$ its equivalent in exponential terms. It is shown in treatises on trigonometry that

$$\sin\,\theta = \frac{e^{k\theta} - e^{-k\theta}}{2k},$$

where $k = \sqrt{-1}$, and $e$ is now the number 2·71828, which is the base of the Napierian logarithms.

Also that $$\cos\theta = \frac{e^{k\theta} + e^{-k\theta}}{2};$$

hence $\cos\theta + k\sin\theta = e^{k\theta}$.

These are called the exponential values of the sine and cosine.

Taking the equation (29),

$$\mathrm{L}\frac{d\,i}{d\,t} + \mathrm{R}\,i = \mathrm{E}\sin p\,t,$$

we divide both sides by L, and, writing T as before for the *time-constant* $\frac{\mathrm{L}}{\mathrm{R}}$, we get

$$\frac{d\,i}{d\,t} + \frac{i}{\mathrm{T}} = \frac{\mathrm{E}}{\mathrm{R\,T}}\sin p\,t.$$

Multiply both sides by $e^{\frac{t}{\mathrm{T}}}$ ($e$ being here the exponential base, not impressed E.M.F.), and we have

$$\frac{d\,i}{d\,t}\cdot e^{\frac{t}{\mathrm{T}}} + \frac{i}{\mathrm{T}}\,e^{\frac{t}{\mathrm{T}}} = \frac{\mathrm{E}}{\mathrm{R\,T}}\,e^{\frac{t}{\mathrm{T}}}\sin p\,t.$$

The left-hand side of this equation is the complete differential of $i\,e^{\frac{t}{\mathrm{T}}}$, and may be written $\frac{d}{d\,t}\left(i\,e^{\frac{t}{\mathrm{T}}}\right)$; and on substituting the exponential value for $\sin p\,t$ and putting $k$ for $\sqrt{-1}$, we have

$$\frac{d}{d\,t}\left(i\,e^{\frac{t}{\mathrm{T}}}\right) = \frac{\mathrm{E}}{2\,k\,\mathrm{R\,T}}\left\{e^{\left(\frac{t}{\mathrm{T}} + k\,p\,t\right)} - e^{\left(\frac{t}{\mathrm{T}} - k\,p\,t\right)}\right\}. \quad (30)$$

The right-hand side of this last equation is the differential with respect to $t$ of

$$\frac{\mathrm{E}}{2\,k\,\mathrm{R\,T}}\left\{\frac{e^{\left(\frac{1+k\,p\,\mathrm{T}}{\mathrm{T}}\right)t}}{\frac{1+k\,p\,\mathrm{T}}{\mathrm{T}}} - \frac{e^{\left(\frac{1-k\,p\,\mathrm{T}}{\mathrm{T}}\right)t}}{\frac{1-k\,p\,\mathrm{T}}{\mathrm{T}}}\right\};$$

and this last becomes by simplification

$$\frac{\mathrm{E}}{2\,k\,\mathrm{R}}\,e^{\frac{t}{\mathrm{T}}}\left\{\frac{e^{k\,p\,t}}{1+k\,p\,\mathrm{T}} - \frac{e^{-k\,p\,t}}{1-k\,p\,\mathrm{T}}\right\}.$$

Hence, equating both sides of equation (30), when integrated we have

$$i=\frac{\mathrm{E}}{2\,\mathrm{R}k}\left\{\frac{e^{kpt}}{1+kp\mathrm{T}}-\frac{e^{-kpt}}{1-kp\mathrm{T}}\right\}.$$

Substituting back into sine and cosine terms, and recollecting that

$$e^{kp}=\cos pt+k\sin pt,$$

and

$$e^{-kpt}=\cos pt-k\sin pt,$$

we get finally

$$i=\frac{\mathrm{E}}{\mathrm{R}}\left\{\frac{\sin pt-p\mathrm{T}\cos pt}{1+p^2\mathrm{T}^2}\right\}.$$

This equation gives us a value $i$ for the current at any instant, and at a time $t$ reckoned from the instant when the impressed electromotive force is zero. The value of $i$ is accordingly called the *instantaneous value* of the periodic current, and the instantaneous value runs through a certain cycle of magnitudes, ranging from zero at one particular instant to a maximum value I at another instant.

We can, however, put the above equation in a more intelligible form. Replace T by $\frac{\mathrm{L}}{\mathrm{R}}$, and let $\theta$ be an angle whose tangent is equal to

$$\frac{\mathrm{L}p}{\mathrm{R}};$$

hence

$$\tan\theta=\frac{\mathrm{L}p}{\mathrm{R}}=p\mathrm{T}.$$

It follows by an easy transformation that

$$\cos\theta=\frac{\mathrm{R}}{\sqrt{\mathrm{R}^2+p^2\mathrm{L}^2}} \text{ and } \sin\theta=\frac{\mathrm{L}p}{\sqrt{\mathrm{R}^2+p^2\mathrm{L}^2}}.$$

We have, then, for the value of the current $i$, the equation

$$i=\frac{\mathrm{E}}{\mathrm{R}}\left\{\frac{\sin pt-p\mathrm{T}\cos pt}{1+p^2\mathrm{T}^2}\right\};$$

or, by substitution,

$$i=\frac{\mathrm{E}}{\sqrt{\mathrm{R}^2+p^2\mathrm{L}^2}}\left\{\sin pt\cos\theta-\sin\theta\cos pt\right\}.$$

$$\therefore i=\frac{\mathrm{E}}{\sqrt{\mathrm{R}^2+p^2\mathrm{L}^2}}\sin(pt-\theta). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

This is called the *particular* solution of the equation

$$L\frac{d\,i}{d\,t}+R\,i=E\sin p\,t=e,$$

and it shows us three things:—First, that the phase of the current $i$ is retarded behind that of the impressed electromotive force by an angle $\theta$—such that $\tan\theta=p\,T=\frac{L\,p}{R}$; second, that the maximum value of the current is obtained by dividing the maximum value of the electromotive force by a quantity equal to $\sqrt{R^2+p^2L^2}$; and, third, that the current curve is a simple periodic curve. The quantity $\sqrt{R^2+p^2L^2}$ is called the *impedance* of the circuit.

The mathematical student will, however, remember that the complete solution of the equation

$$L\frac{d\,i}{d\,t}+R\,i=E\sin p\,t$$

involves a constant of integration, and this is obtained by adding to the particular solution above obtained the complementary function which is obtained by taking the solution of equation (29) when $E\sin p\,t=0$.

Now, since the solution of

$$L\frac{d\,i}{d\,t}+R\,i=0 \text{ is}$$

$$i=C\,\epsilon^{-\frac{R}{L}t},$$

where C is a constant of integration and $\epsilon$ in this last equation is the base of the Napierian logarithms, we have, then, the complete solution of the differential equation

$$L\frac{d\,i}{d\,t}+R\,i=E\sin p\,t$$

given by the equation

$$i=\frac{E}{\sqrt{R^2+L^2p^2}}\sin(p\,t-\theta)+C\,\epsilon^{-\frac{R}{L}t}. \quad . \quad (32)$$

The complementary function dies out rapidly as time increases at a rate depending on the value of $\frac{L}{R}$. Physically, the meaning of this is that the current does not settle down into its regular periodic state until a shorter or longer time after the closing of the circuit depending on the value of the time-constant $\frac{L}{R}$.

We shall return again to discuss the complete solution of the above equation (29), and show how to determine the value of the constant C in equation (32).

We have seen from the explanations on previous pages that the mean-square value of a simple periodic quantity is equal to its maximum value divided by $\sqrt{2}$. Hence, if we write Im for the impedance, we can put the equation, giving the instantaneous value of the current produced by a simple periodic impressed electromotive force of maximum value E, operating on a circuit of resistance R, inductance L, with a pulsation $p$, in the form

$$i = \frac{\mathrm{E}}{\mathrm{Im}} \sin(p\,t - \theta);$$

or, if we denote the maximum value of the current during the phase by the letter I, and since $\mathrm{I} = \frac{\mathrm{E}}{\mathrm{Im}}$, we have

$$i = \mathrm{I} \sin(p\,t - \theta).$$

Hence, we may write this in words as follows:—

$$\left.\begin{array}{r}\textit{the maximum value}\\ \textit{of the current}\\ \textit{strength}\end{array}\right\} = \left\{\frac{\textit{the maximum value of the impressed electromotive force}}{\textit{impedance}}\right.$$

and

$$\left.\begin{array}{c}\textit{the mean-square}\\ \textit{value of the}\\ \textit{current strength}\end{array}\right\} = \left\{\frac{\textit{the maximum value}}{\sqrt{2}}\right..$$

We see, then, that, in the case of simple periodic electromotive force, the quantity called the impedance appears to be related to the impressed E.M.F., just as does the resistance to the steady E.M.F. in the case of continuous currents, and the above may be called the equivalent of Ohm's law for simple periodic currents. Compare as below

$$\left.\begin{array}{c}\text{For steady}\\ \text{or}\\ \text{continuous}\\ \text{currents}\end{array}\right\} \quad \left.\begin{array}{c}\textit{current}\\ \textit{strength}\end{array}\right\} = \left\{\frac{\textit{electromotive force}}{\textit{resistance}}\right. \text{(Ohm's law).}$$

$$\left.\begin{array}{c}\text{For simple}\\ \text{periodic or}\\ \text{alternate}\\ \text{currents}\end{array}\right\} \quad \left.\begin{array}{c}\textit{mean-square}\\ \textit{value of the}\\ \textit{current}\\ \textit{strength}\end{array}\right\} = \left\{\frac{\begin{array}{c}\textit{mean-square value of}\\ \textit{the electromotive force}\end{array}}{\textit{impedance}}\right..$$

Impedance is a quantity which is measured, like resistance, in ohms, and has for that reason been sometimes, but erroneously, called the *virtual resistance*.

§ **19. Geometrical Illustrations.**—The current equation, expressing the current strength in terms of the impressed electromotive force, the resistance, inductance, and phase angles, which holds good when a circuit of constant inductance and no sensible capacity is subjected to moderately great pulsations of electromotive force has been in the previous pages arrived at algebraically from first principles.

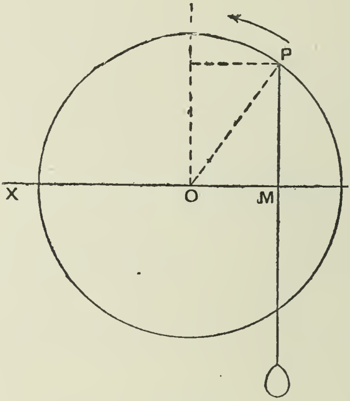

Fig. 54.

It is, however, possible to elucidate its meaning by geometrical methods. Let a circular disc (Fig. 54) be pivoted at the centre O, and at any point P on the circumference let a plummet line be attached. In front of the circle is a fixed horizontal line X X′. Let the disc move round counter-clockwise at a uniform rate, the time of one revolution being T. As the disc goes round, the length of plummet line P M above X X′ fluctuates. Since P M = O P sin P O M, it follows that, if the magnitude of P M be taken at small equal intervals of time during one revolution, and such heights be plotted off as off-sets at equal distances above and below a datum line, the extremities of these ordinates will lie on a simple periodic or sine curve. In other words, P M grows and shrinks in height in accordance with a simple periodic law. We can, therefore, represent any quantity which fluctuates in magnitude according to a simple sine law of growth by representing it as the projection of a point on the circumference of a circle revolving uniformly, taken on a horizontal or vertical fixed line drawn through the centre. Hence, if O P represents the maximum value of an electromotive force fluctuating periodically, P M will represent its various magnitudes during the complete period. The magnitude of P M at any instant is known when we know O P, which is called the *amplitude*, or maximum value, and P O M the *phase angle* of the motion.

A diagram, in which the projection on any other line of a radial line revolving round one extremity is made to represent a simple periodic function, is called a *clock-diagram*. In clock-diagrams radial lines are taken to represent in magnitude the *maximum values* of the quantities which are to be represented as periodically varying. Any line through the centre may be taken as the line on which projections are taken, and the projections in this line give us the instantaneous values of the periodic quantity whose maximum value is represented by the radius. If different radii are drawn from one centre, representing currents or electromotive forces, then the angular interval between these radii represent the *phase difference* of these quantities.

§ 20. **Graphic Representation of Periodic Currents.**—On such a diagram let a radius be drawn to any scale representing by its vertical projection the periodic fluctuation of an impressed electromotive force, varying according to a simple sine law, and acting on a circuit of given inductance and resistance with a fixed periodicity; the problem is to draw on the same diagram another radius, of which the vertical projection shall represent the actual current strength in the circuit at the corresponding instant. The impressed electromotive force at any instant balances, or is equal to, the sum of two others, viz., the effective electromotive force driving the current, which is equal to the product of the ohmic resistance of the circuit and the current at that instant in it; and the inductive or counter-electromotive force, which is equal to the rate of variation of the flux of force or number of lines of force traversing the circuit. The phases, or times of maximum, of these two components are not identical. They differ by 90°, since the effective electromotive force has the same phase as the actual current, and the inductive electromotive force, depending on the *rate of variation* of the current, comes to a maximum at the instant when the current is zero, or is changing sign.

By the proposition in § 10, these two periodic quantities can therefore be represented by sine curves, one of which is shifted backward relatively to the other, so that the crest of the wave of one coincides with the zero point of the other. We shall first proceed to show that the sum of two simple periodic motions of the same periodic time, but different phases and amplitudes, will, when added together, produce a simple periodic motion of the same periodic time.

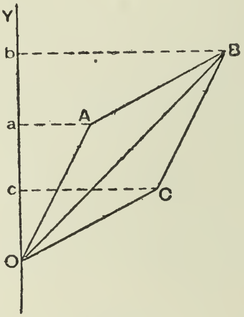

Fig. 55.

Let a parallelogram of cardboard, O A B C (Fig. 55), be cut out and pivoted by a pin at the angle O, so as to turn freely clockhand-wise. Let a vertical line, O Y, be drawn through O, and in any position let the sides O A, O C, A B be projected on to O Y. The projection of lines equal and equally inclined are equal; hence, since A B is equal and parallel to O C, the projection of A B—viz., $a\,b$—is equal to that of O C—viz., O $c$. But $O\,b = O\,a + a\,b$ always for any position of the card; hence $O\,b = O\,a + O\,c$. The projection of the diagonal is therefore equal to the sum of the projections of the adjacent sides. As the card moves uniformly round the magnitudes of the projections fluctuate at each instant, according to a simple periodic law. Hence the sum of the simple periodic motions of which O A, O C are the amplitudes, and which have a fixed difference of phase represented by the angle A O C, is the simple periodic motion represented by O B in amplitude and relative phase. If, then, a point be subjected to two simultaneous simple periodic motions of given amplitudes, and of which the phases differ by 90°, the actual motion will be represented, as to amplitude and phase, by the diagonal of the parallelogram of which these two form the adjacent sides. Returning in thought to electric motion, consider the motion of a *particle of electricity* (if we may be allowed the expression) in the wire subjected to two simultaneous simple periodic motions of unequal amplitude and fixed difference of phase equal to 90°. The displacement at any instant due to the two together is equal to the sum of each separately. If the individual motions are represented by the vertical projections of two lines, O A, O B, fixed like hands of a toy clock at right angles (Fig. 56), the resultant motion is that indicated by the projection of the diagonal O C on the same vertical. We have seen (in § 10) that, if the variation of a quantity is represented by a simple sine curve, the variation of its *rate of change* is represented by a sine curve of different amplitude

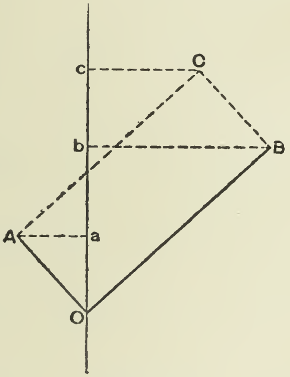

Fig. 56.

shifted backward by 90° of phase, or by a quarter of a wave length. It follows from this proposition that if we add together at every instant the motions or the ordinates representing them on a diagram of two simple periodic motions, one of which is the curve representing the rate of change of the ordinate of the other, we shall get a new sine curve, of which the maximum value falls between that of the other two, and of which the amplitude is different, but wave length or periodic time the same. In Fig. 57 the thick line sine curve represents one wave of a simple periodic motion. The fine continuous line is a sine curve of equal wave length, of which the ordinate P M at any point represents or is proportional to the rate of change of the ordinate Q M of the thick curve at the same instant. Adding together the ordinates of the thick and thin curves, we get a new dotted line sine curve, of which the ordinate R M is equal to Q M + rate of change of Q M. If we substitute for the sine curve diagram a clockhand diagram (Fig. 56), then the projection of O B—viz., O *b*—corresponds to the ordinate Q M of the thick curve; that of O A—viz., O *a*—corresponds to P M, the ordinate of the thin curve; and that of O C, the diagonal of the rectangle O A, O B, corresponds to R M, and is the resultant of the motion O B, and the rate of change of that motion, viz., O A. If, then, O B represents the amplitude or maximum value of the actual periodic current in a circuit, a line, O A, drawn at right angles to O B, will represent to a suitable scale the rate of change of that current.

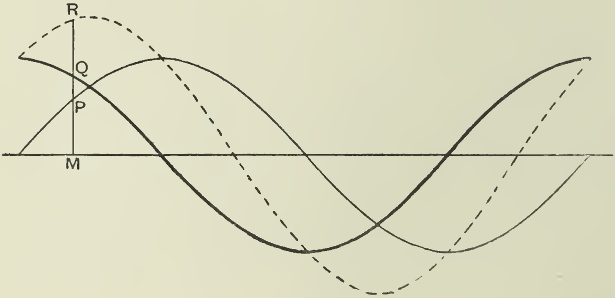

Fig. 57.

We are, then, led to this converse proposition, that we can resolve any simple periodic curve into a pair of component periodic curves of equal periodic time, but of which the maximum value happens for one before and for one after that of the original.

**§ 21. Impressed and Effective Electromotive Forces.**—If at any instant a current of which the instantaneous value is $i$ is flowing in an inductive circuit of which the true resistance is R, the quantity $R\,i$ represents the voltage necessary to make this current flow, and this part of the impressed electromotive force is called the *effective electromotive force* in the circuit. If the circuit is an inductive circuit, there will be another electromotive force equal in magnitude to $L\,\frac{di}{dt}$, which acts either with or against the total applied or impressed electromotive force. This is called the *inductive electromotive force.* The electromotive force which is at any instant applied to the circuit is called the *impressed electromotive force.* The effective electromotive force is always the resultant of the impressed and inductive electromotive forces. Hence, if these last two electromotive forces are represented in a clock diagram in magnitude and relative phase by the two sides of a parallelogram, the effective electromotive force will be represented in magnitude and phase by the diagonal of that parallelogram. A further condition is that, since the inductive electromotive force depends upon the rate of change of the current, it is always at right angles as regards phase with the effective electromotive force. This last is always in step or in synchronism as regards phase with the current. Hence, if we require to draw a clock diagram of electromotive forces for an inductive circuit in which a simple periodic impressed electromotive force is acting, we see that the proper construction is as follows:—Take any line, O P (Fig. 58), to represent the magnitude of the maximum value of the impressed electromotive force; on O P describe a semi-circle, O M P; let O P be supposed to revolve round the point O in the contrary direction to the hands of a watch, and let the projection of O P, at any instant on any line O Y drawn through O, be taken. Then O P represents the maximum value, E, of the impressed electromotive force, and O $q$, or the projection of O P, represents the magnitude of the instantaneous value of the impressed electromotive force at an instant when O P has completed such part of one revolution as is represented by the angle P O X.

Let us suppose O P to start from the position O X, and let time be reckoned from that instant of starting. Then, if T be the time of one complete revolution, and if $t$ be the time in which O P passes through the angle P O X,

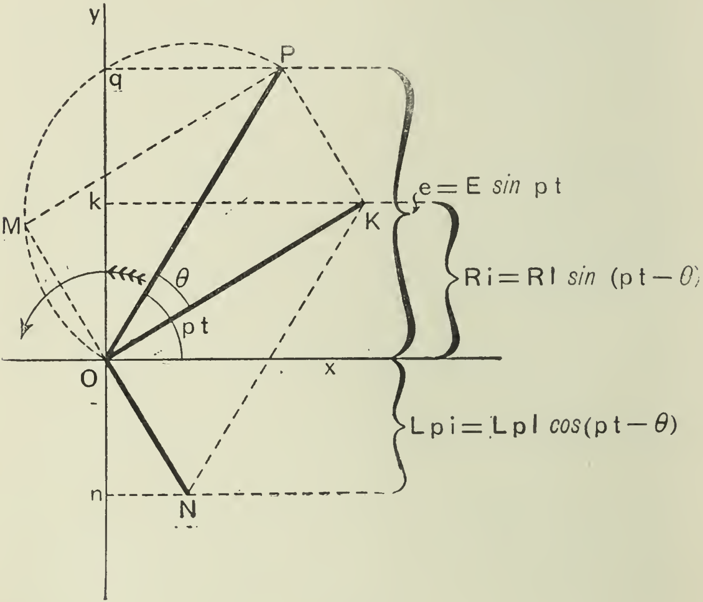

FIG. 58.

the angle P O X is the same fraction of four right angles of $2\pi$ that $t$ is of T. Hence the angle P O X is, in magnitude, equal to $\frac{2\pi t}{T}$. For shortness, $2\pi/T$ is written $p$. Therefore $p$ is a quantity of the nature of an angular velocity. Hence, if the magnitude of O $q$, which is the projection of O P, is denoted by $e$, and if O P is denoted by E, we see that

$$e = E \sin pt,$$

or the instantaneous value of the impressed electromotive force runs through a cycle of values represented by the ordinates of a sine curve.

Next, on O P describe a semi-circle, O M P, on that side of O P which is towards the direction in which O P is rotating. Take a point M on this circumference, such that O M is to M P in the ratio of $Lp$ to R, where L is the inductance and R the resistance of the circuit. Through O draw O K parallel to M P. Produce M O to N, and make O N equal to O M. Draw N K parallel to O P and join O K. Then, on the same scale on which O P represents E, the maximum magnitude of the impressed electromotive force, O K will represent the maximum magnitude of R I or the effective electromotive force, and O N will represent $LpI$ or the maximum magnitude of the inductive electromotive force. By the geometry of the figure we see that, if the angle P O K is called $\theta$, the projection $Ok$ of O K on O Y is equal to $OK \sin(pt-\theta)$, and also the projection $On$ of O N on O Y is equal to $ON \cos(pt-\theta)$. Hence $On = \frac{d}{dt}(Ok)$. Moreover, O K is the resultant of O P and O N, and O N is at right angles to O K; therefore O N and O K fulfil all the conditions requisite for being the representation of the maximum values of the inductive and effective electromotive forces. For O K is obviously the resultant of O P and O N. O N is in such a direction that its projection or instantaneous value is numerically determined by the rate of change of the projection or instantaneous value of O K; and we know, by fundamental principles, that the electromotive force of self-induction is determined by the rate of change of the current in this circuit; that is, by the rate of change of the effective electromotive force. By considering the relative positions of O P, O K, and O N, it will be seen that they are in the right directions to represent these three quantities. For, if the system of lines be supposed to revolve round O, then, when the projection of O K is above O X—that is, when the current in the circuit is increasing—the projection of O N is negative and is decreasing. This means that the electromotive force of self-induction is in such a direction as to oppose the current. Also, when the effective electromotive force or current is in the same direction, but decreasing, then the inductive electromotive force is positive, or in the same direction as the current, and is increasing. Accordingly, if the magnitude of O K is R I, where R is the resistance of the circuit and I is the maximum value of the current, and R I is therefore the maximum value of the effective electromotive force in the circuit, we see that the magnitude of O N must be L $p$ I, and that of O P must be $\sqrt{R^2+p^2L^2}\,I$, and this last, we know, is the value of the impressed electromotive force E. Accordingly, on whatever scale O P represents the impressed electromotive force E, then O K represents the effective electromotive force R I, and O N represents the inductive electromotive force L $p$ I. If we take one Rth part of O K, we have the value of the current in the circuit. The angle of lag $\theta$ by which the current is *behind* the impressed electromotive force in phase is an angle, such that—

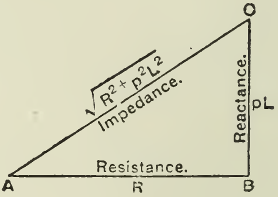

FIG. 59.

$$\cos\theta = \frac{\text{Resistance of circuit.}}{\text{Impedance of circuit.}} = \frac{R}{\sqrt{R^2+p^2L^2}}.$$

The diagram shows us, therefore, not only how to represent the current and impressed electromotive force in an inductive circuit properly as regards phase and magnitude, but tells us practically how the angle of lag should be measured.

The relation of the impedance and resistance of an inductive circuit may be represented geometrically as follows: Draw a right-angled triangle, A B C (Fig. 59), and take the base A B to represent the resistance of the circuit, and the hypotenuse, A C, to represent the impedance. Then the side B C will represent the magnitude of the quantity $p$ L. This has been called the *reactance* of the circuit, and since the angle C A B is an angle which has a cosine equal to $\frac{R}{\sqrt{R^2+p^2L^2}}$, we see that this angle, which we may call $\theta$, is the angle of lag of current behind electromotive force, and, moreover, that

$$\tan\theta = \frac{pL}{R} = \frac{\text{reactance of circuit}}{\text{resistance of circuit}}.$$

**§ 22. The Mean Value of the Power of a Periodic Current.** Having now seen how the fluctuation of current strength is related to that of the impressed E.M.F. in an inductive circuit under the conditions of a simple sine law of variation, we pass to the consideration of the measurement of the *power* taken up in or supplied to circuits traversed by periodic currents.

Let the thin line curve in Fig. 60 represent the curve of impressed electromotive force in an inductive circuit, and the thick line the corresponding curve of current. Then *at any instant* the rate at which energy is being expended on the circuit is equal to the product of the ordinates P M, Q M, which at any point M on the time line represent the electromotive force and current respectively. The mean rate of expenditure of energy, or the *mean power* being taken up in the circuit, is then the mean of all such products taken at equal and very near intervals of time during one complete period. This is not by any means identical with the product of the mean current and mean electromotive force. To arrive at an expression for this mean power, we must pave the way by a preliminary proposition on the mean product of two simple periodic quantities. An elegant geometrical method of

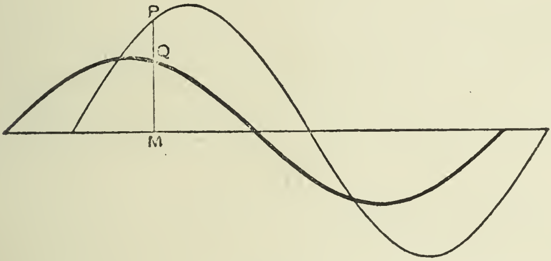

FIG. 60.

obtaining this has been given by Mr. Blakesley. We shall, however, give here an algebraical proof of this proposition. Let there be two radii O P, O Q (Fig. 61), which revolve in equal periodic times round a common centre O, separated by a fixed angle, P O Q. At equal small intervals of time corresponding to equal angular motions let the projections O $p$, O $q$ of these lines be taken on a vertical line through O. It is required to find the mean value of the product O $p$, O $q$ during one complete period.

Denote by X the length of O P, and by Y the length of O Q, and let the angle P O Q be $\beta$, and P O $p$ be $\alpha$. $\beta$ is the angle of phase difference, and X and Y are the maximum values of the periodic quantities O $p$, O $q$, which are the vertical projections of O P, O Q.

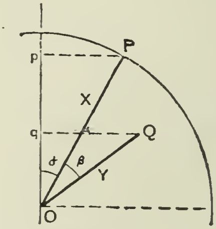

FIG. 61.

Let O$p$ be denoted by $p$, and O $q$ by $q$

Then $$p = X\cos\alpha,$$
and $$q = Y\cos(\alpha+\beta);$$
and therefore $$pq = XY\cos\alpha\cos(\alpha+\beta).$$

Let the pair of radii O P, O Q be supposed to turn round one complete revolution, proceeding by $n$ small steps, each step increasing the angle $\alpha$ by a very small amount, $\delta$, $\alpha$ and $n$ being a very large number. At each stage let the value of $p\,q$ be measured as above, then the mean value of the product $p\,q$ is one $n$th part of the sum of all the $n$ values so taken. Call this mean value of the product M. Then,

$$M = \frac{XY}{n}\left\{\begin{array}{l}\cos\beta + \cos\delta\alpha\cos(\delta\alpha+\beta) + \cos 2\,\delta\alpha\cos(2\,\delta\alpha+\beta) \\ \qquad \dots\dots + \cos\overline{n-1}\,\delta\alpha\cos(\overline{n-1}\,\delta\alpha+\beta)\end{array}\right\}$$

By trigonometry we have

$$\cos(\overline{n-1}\,\delta\alpha)\cos(\overline{n-1}\,\delta\alpha+\beta) = \tfrac{1}{2}\cos(\overline{2n-1}\,\delta\alpha+\beta) + \tfrac{1}{2}\cos\beta,$$
since $$\cos\overline{A+B} + \cos\overline{A-B} = 2\cos A\cos B.$$

Accordingly every term, except the first in the cosine series for M, splits up into the sum of two others, one of which is always $\frac{1}{2}\cos\beta$. Rearranging the terms, we get for the value of M as follows:—

$$M = \frac{XY}{n}\left[\tfrac{1}{2} n \cos\beta + \tfrac{1}{2}\Big(\cos\beta + \cos(2\,\delta\alpha + \beta) + \cos(4\,\delta\alpha + \beta) \right.$$
$$\left. \ldots\ldots + \cos(2\,\overline{n-1}\,\delta\alpha + \beta)\Big)\right]$$

The cosine series in the inner bracket consists of a series of cosines of angles in arithmetic progression taken all round the circle. Hence, since the cosine of any angle is numerically equal to that of the cosine of its supplement, but of opposite sign, these cosine terms will cancel each other out pair and pair, when $n$ becomes very great and $\delta\alpha$ very small, and $n\,\delta\alpha$ equal to $2\pi$. For when

$$n\,\delta\alpha = 1\pi,\ 2\,\overline{n-1}\,\delta\alpha + \beta = 4\pi + \beta, \text{ and } \cos(4\pi + \beta) = \cos\beta.$$

The first and last terms of the series are in this case identical, and for every term there will exist one of equal magnitude and opposite sign. The sum of the series of cosine terms in the inner bracket is accordingly zero.

The value of M reduces then to that of the first term, viz.:—

$$M = \frac{XY}{2}\cos\beta.$$

The mean value of the product of two simple harmonic or periodic functions of equal period but different amplitude and phase is equal to half the product of their maximum values, and the cosine of their difference of phase.

Returning to the consideration of the electrical problem, it is now clear that for simple periodic or sine variation the mean value of the product of the current at any instant and the simultaneous value of the impressed electromotive force in an inductive circuit is obtained by multiplying together half the product of their maximum values, and the cosine of the angle of lag. If $i$ be the current at any instant, and $e$ the impressed E.M.F., I and E being their maximum values, then the mean value of $e\,i$ during a complete period is

$$\frac{EI}{2}\cos\theta, \quad \ldots\ldots \quad (33)$$

and this is a measure of the mean rate of expenditure of energy on that circuit, or the mean power taken up. It is obvious, then, that if the lag is 90°, this mean product is zero, and that no work is done at all.

When $\theta$ has intermediate values between 0° and 90°, the real rate of dissipation or transformation of energy in the circuit will be intermediate between $\frac{E\,I}{2}$ and zero. In order to understand how this can be, and how it is that a circuit may be traversed by a current and yet take up no power, we must examine a little more closely the nature of the phenomena.

§ **23. Power Curves.**—Let the periodic curve in Fig. 62 represent a sinusoidal variation of electromotive force acting on a circuit which we shall for the moment assume has no sensible inductance. Let the curve in Fig. 63 represent the corresponding current. The length of each ordinate of the second curve is equal in magnitude to that of the corresponding ordinate of the first curve divided by the value of the resistance of the circuit. Let the lengths of corresponding ordinates of these two curves be multiplied together, and the product

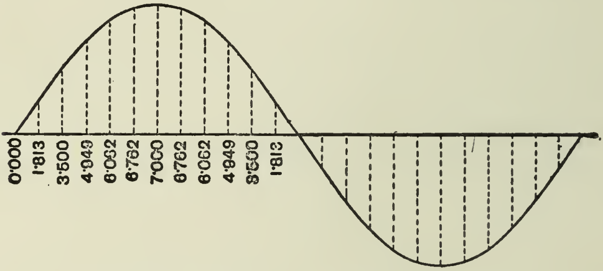

Fig. 62.—Electromotive Force Curve.

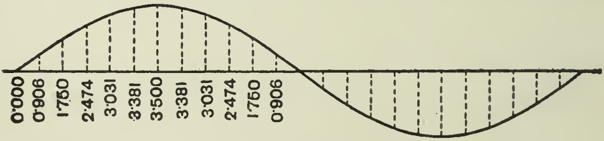

Fig. 63.—Current Curve.

set off as the ordinates of a new curve represented by the dotted line in Fig. 64. This dotted curve is, then, the curve of *power* or *activity*, and represents the variation of the product of the current and the electromotive force taken at every instant.

In multiplying together the ordinates of the first and second curves, we must pay attention to the algebraic sign of each ordinate. Ordinates of each curve drawn above the horizontal datum line of the curve must be reckoned *plus*, and ordinates drawn below must be reckoned *minus*, and in taking the product the algebraic law of signs must be regarded. It will be seen that the dotted line curve consists of a wavy line of two loops lying wholly above the mean datum line. If the area of the two hummocks enclosed by the dotted curve and the horizontal line is *indicated* or integrated, say, by an Amsler's planimeter,

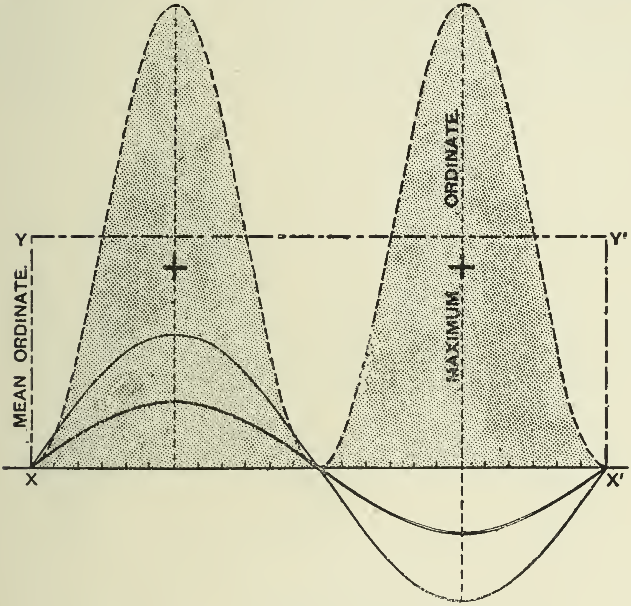

FIG. 64.

| | | | | | | |
|---|---|---|---|---|---|---|
| Ordinates | 1·81 | 3·5 | 4·95 | 6·06 | 6·76 | 7·0 |
| | ·906 | 1·75 | 2·47 | 3·03 | 3·38 | 3·5 |
| Products .. | 1·64 | 6·12 | 12·25 | 18·37 | 22·86 | 25·4 |

the area represented by the shaded part so obtained is a measure of the total work done in one complete period of the current oscillation, and, since this area lies wholly above the datum line, it must be reckoned as *positive*, or as work done by the electromotive force; in other words, it represents the total energy transformed from electrical energy into heat in one complete period.

Next let us suppose that the same periodic electromotive force acts upon a circuit having inductance as well as resistance, and that therefore, as already shown, the current is retarded in phase behind the electromotive force. Let the thin curve in Fig. 65 represent the periodic impressed electromotive force, and the thick curve the current retarded by 45° in phase behind the other. Proceed as before to obtain the power curve by multiplying the heights of corresponding ordinates, the multiplication of the ordinates being shown below the figure. We find that the power curve representing the variation of the activity is a wavy curve, consisting of four sections, two large hummocks above the datum line, which are positive areas, and two small ones below, which are

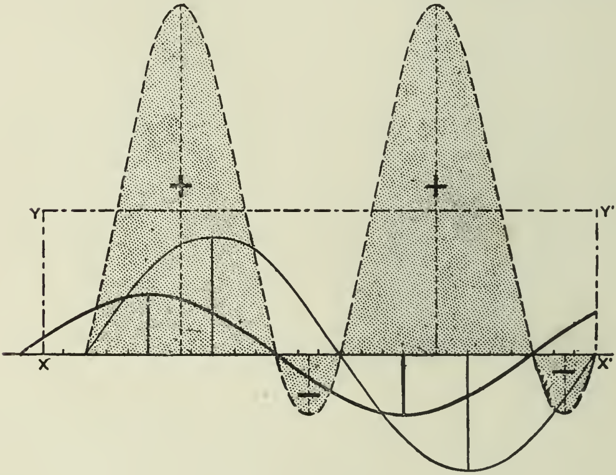

Fig. 65. Lag 45°.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ordinates | 2·47 | 3·03 | 2·38 | 3·5 | 3·38 | 3·03 | 2·47 | 1·75 | ·91 | 0 | ·91 | 1·75 | 2·47 |
| | 0 | 1·81 | 3.5 | 4·95 | 6·06 | 6·76 | 7 | 6·76 | 6·06 | 4·95 | 3·5 | 1·81 | 0 |
| Products .. | 0 | 5·5 | 11·83 | 17·3 | 20·48 | 20·48 | 17·3 | 11·83 | 5·5 | 0 | 3·18 | 3·18 | 0 |

negative areas. The algebraic sum taken with regard to sign of all these four areas, represented by the shaded parts, is a measure of the total work done in one complete period by the electromotive force. Going one step more, we may imagine that the current lag is 90°, and in Fig. 66 we have drawn the power curve in this case, obeying the same instructions. We see that the power curve consists of four loops, two positive and two negative, and that the area of these hummocks are equal. Hence, the total area or *indicated* value in this last case is zero, and the work done in one complete cycle is zero; hence, the rate of doing work, or the power, is zero. Returning to Fig. 64, the first case, it is easy to see that the rate of doing work, or the power, is measured by the *mean ordinate* of the shaded work areas considered as indicator diagrams. Since the dotted curves are perfectly symmetrical, if we draw a line Y Y′ at half the height of the maximum ordinate of the dotted curve, it will cut the two hummocks into two parts, and the area of the upper part, or mountain above the line Y Y′, would just fill up the valley between the bottom parts of the hummocks. Since the rate of doing work is equal to the work done in one complete cycle divided by the

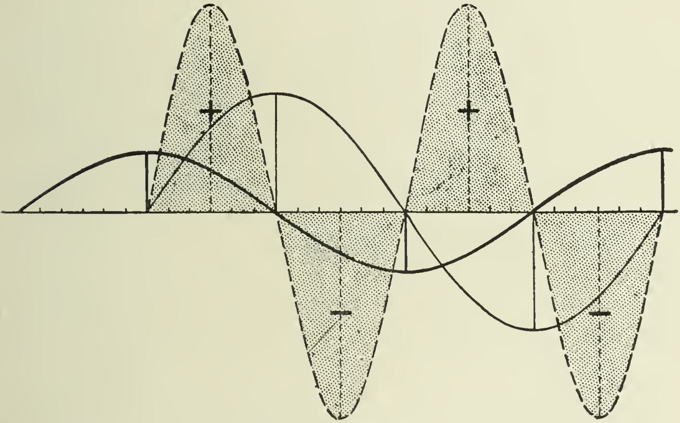

Fig. 66. Lag 90°.

| | | | | | | |
|---|---|---|---|---|---|---|
| Ordinates | 3·5 | 3·38 | 3·03 | 2·47 | 1 75 | ·91 |
| | 0 | 1·81 | 3·5 | 4·95 | 6-06 | 6 76 |
| Products | ..0 | 6·13 | 10 61 | 12 23 | 10 61 | 6·13 |

time of duration of that cycle, it obviously follows that this mean ordinate X Y measures the power or rate of doing work and is equal in magnitude to $\frac{E\,I}{2}$; hence this product is a measure of the rate at which the electromotive force does work. Referring to the second case, we see that the mean ordinate X Y is no longer equal to half the maximum ordinate of the positive or upper loop, but is equal to half the difference between the magnitudes of maximum ordinates of the positive and negative loops of the power curve, and is therefore less than $\frac{E\,I}{2}$. From the proof given previously we have seen that it is equal to $\frac{E\,I}{2}$ (cosine of lag).

In the third case considered, of a lag of 90 degrees, it is easy to see why the resultant rate of doing work is zero. In the first *quarter of a stroke* the electromotive force propels the current, and this last is in the direction of the E.M.F., but in the second quarter of a stroke the current is negative or opposite to the electromotive force; in other words, the current is moving against the force and does work against the E.M.F., and the same push and re-push is repeated in the second half of the period. Hence, on the whole, though there is an impressed electromotive force and a current flowing, no resultant work is done and no energy dissipated.

§ 24. **The Experimental Measurement of Periodic Currents and Electromotive Forces.**—At this stage it will be an advantage to direct attention to the practical means of measuring the $\sqrt{\text{mean square}}$ value of periodic currents and electromotive forces, and also the mean value of the power given to an inductive circuit of any kind. There are, amongst others, two instruments especially useful for measuring periodic currents. One of these, the Cardew voltmeter, depends upon the principle that when a wire traversed by a current, either steady and unidirectional or steadily periodic, is placed in an enclosure the walls of which are approximately at a constant temperature, the wire will itself, after a short time, attain a constant temperature. This constant temperature is reached when there is a state of equilibrium between the rate at which heat is radiated by the wire and the rate at which the walls of the enclosure radiate heat back to it.

The wire has a definite length corresponding to each temperature, and means are provided for measuring this elongation with great accuracy. The total amount of heat generated in the wire per second is dependent upon the rate of generation at each instant. The instantaneous rate of heat development is, by Joule's law, equal in mechanical units to the product of the resistance of the wire and the square of the value of the instantaneous current flowing in it.

If the wire is traversed by a simple periodic current, and we construct from the current curve diagram another curve whose ordinates are equal to the square of the corresponding current ordinates, we have a curve every ordinate of which is proportional to the instantaneous rate of generation of heat in a wire traversed by the periodic current. Since the horizontal line measures time, it is obvious that the whole area of the outer curve, or heat curve, represents the total work done per semi-period by the current in producing heat, and that the same total work would be done by a steady current which had a value equal to the square root of the mean of the squares of all the ordinates of the periodic current curve.

This square root of the mean of the squares of all the ordinates of a simple periodic curve has, however, been shown in § 9 to be numerically equal to the value of the maximum ordinate of the periodic curve divided by $\sqrt{2}$. It follows that the total heat generated per second in the wire is a numerical measure of the $\sqrt{\text{mean square}}$ value of the current or of half the square of the maximum value of a simple periodic current. A fine wire stretched out in the manner of a Cardew voltmeter wire has a very small inductance, and, when acted upon by a simple periodic electromotive force, the current produced in it is very nearly proportional to this impressed electromotive force. It follows, then, that, when a Cardew voltmeter is subjected to a simple periodic electromotive force, the needle takes a definite position, corresponding to a definite expansion of the wire, which is that which it would take if the wire were subjected to a steady electromotive force equal to $\frac{1}{\sqrt{2}}$ of the maximum value of the periodic electromotive force.

The Cardew voltmeter is not adapted to measure any but very small currents. The instrument generally employed to measure periodic currents of moderate and large magnitude is some modification of Weber's electro-dynamometer. In the best-known practical form of Siemens there are two coils of wires in series, one fixed and the other movable, and so placed that the currents in the movable coil circuit are traversed at right angles by the lines of force due to those in the fixed coil. When a simple periodic current traverses the coils in series, a force is brought into existence due to the electro-dynamic action, which is proportional to the instantaneous value of the square of the current strength. From instant to instant, however, the current strength varies. If the time of free vibration of the movable coil is very large compared with that of a complete period of the electrical vibrations, and if the movable coil is brought back by a restoring force due to a spring or bifilar suspension or gravity, &c., into a fixed normal position, then, during one complete electrical period, we may consider that the movable portion receives a number of small impulses which are in magnitude represented by the square of the ordinates of the current wave. Hence, the total impulse on the movable coil is equal to the magnitude of the integrated area of a sine curve whose ordinates are respectively the squares of those of the current curve, and the mean force on the movable coil will obviously be proportional to the mean ordinate of this force curve. If the movable coil is so heavy that its time of free vibration is very long compared with the time in which the periodic forces on it run through a complete cycle, it will experience a displacement exactly that due to the mean of the forces acting upon it—that is, to the square root of the mean of the squares of these instantaneous currents—or to $\frac{I}{\sqrt{2}}$, where I is the maximum value of the current during the period. The periodic force on the movable coil is equivalent to a steady force when this periodic force runs through all its values in a time very short compared with the time of free vibration of the coil. Hence, if a simple periodic current has a maximum value I, when it is sent through an electro-dynamometer it will cause a deflection equal to that which would be caused by a steady current equal to $\frac{I}{\sqrt{2}}$.

Let us suppose a coil of constant inductance L and resistance R to be traversed by a simple periodic current of frequency $n$ (where $2\pi n = p$). Let an electro-dynamometer be inserted in series with it, and let a Cardew voltmeter be connected to the extremities of the inductive circuit.

We have before seen that if E and I are the maximum values during the period of the impressed E.M.F. and current in an inductive circuit, then the *power* taken up in that circuit is equal to $\frac{\text{E I}}{2} \cos \theta$, where $\theta$ = angle of lag of current behind the E.M.F. But the reading of the Cardew voltmeter when connected to the ends of an inductive circuit is very nearly proportional to $\frac{\text{E}}{\sqrt{2}}$, and the dynamometer reading in that circuit is proportional to $\frac{\text{I}}{\sqrt{2}}$; therefore, the product of these readings is proportional to $\frac{\text{E I}}{2}$, and takes no account of the difference of phase. The product of the $\sqrt{\text{mean square}}$ values of the current and of the electromotive force in an inductive circuit is generally called the *apparent power* or *apparent watts* given to that circuit, but it is not a measure of the true power given to the circuit. For this reason we can derive no information from the use, in this manner, of these instruments. The observed readings, and hence their product, does not take into account the difference of phase between the current and impressed E.M.F. in the inductive circuit. A very small error, in practice negligible, is also introduced by disregarding the inductance of the wire of the Cardew instrument. On this account, strictly speaking, currents in the wire cannot be taken as accurately proportional to potential differences at the extremities, but this is in ordinary usage a negligible error.

**§ 25. Method of Measuring the True Value of the Power given to an Inductive Circuit. Theory of the Wattmeter.**— If a current traversing an inductive circuit under a periodic impressed electromotive force is made to pass through another circuit which acts electro-dynamically upon a movable circuit conveying another current proportional in strength to, and agreeing in phase with, the periodic variation of potential difference at the terminals of the inductive circuit, such an arrangement will, if it can be realised, afford a means for obtaining a true numerical measure of the power taken up in the inductive circuit.

An electro-dynamometer having its fixed coil composed of thick wire and its movable coil of fine wire, each circuit being independent, is most usually called a *wattmeter*. The examination of the circumstances under which the wattmeter can and cannot be used to measure the *power* expended in a circuit subject to simple periodic electromotive force, leads to some interesting considerations.

If the thick and thin wire coils of a wattmeter are traversed by two independent steady unidirectional currents, the force on the movable coil is at any instant proportional to the product of the strength of these two currents. If each of these currents are simple periodic currents the force varies with the product of the instantaneous values, and the compound curve formed by taking as ordinates the products of the corresponding values of these separate current strengths at each instant is itself a simple periodic curve, provided that the two component currents have constant amplitudes, equal period, and fixed difference of phase. Let a wattmeter be supposed to be joined up to an inductive circuit (Fig. 67); let R and L be the resistance and inductance of this inductive circuit between the points Q Q′; let the thick wire coil Th of the wattmeter be joined in series with this inductive resistance, and let the

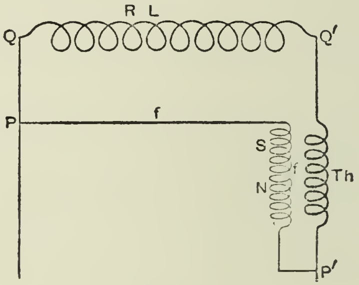

Fig. 67.

fine wire coil *f* of the wattmeter of resistance S and inductance N be joined to the points P P′; let the thick wire coil be of negligible resistance and inductance in comparison with the circuit Q Q′. If a simple periodic electromotive force operates on the double circuit between the points P and P′, we shall have a current flowing in R and S. It is required to calculate at any instant the currents in R and S respectively. Consider simply a divided circuit (Fig. 68) in which R and S are the branches. Let $x$ be the current at any instant in R, and $y$ that in S, and let $i$ be the strength of the current in that part of the circuit just before it divides; in other words, $i$ is the main current, which is divided into $x$ in the inductive resistance and $y$ in the fine wire coil of the wattmeter. Let $e$ be the potential difference between the points P P′ at the same instant, and let X, Y, I, and E be the maximum values

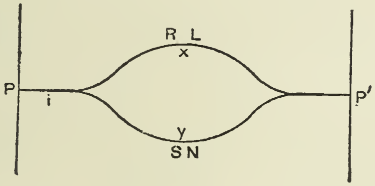

FIG. 68.

of all these quantities respectively. We assume that $i$ is a simple periodic function of I, and we then write $i = \mathrm{I} \sin pt$, where $p = 2\pi n$, $n$ being the frequency. Applying the fundamental equation of § 15 (p. 126) to each circuit, we see that

$$\mathrm{L}\frac{dx}{dt} + \mathrm{R}x = e;$$

also

$$\mathrm{N}\frac{dy}{dt} + \mathrm{S}y = e.$$

Accordingly,

$$\mathrm{L}\frac{dx}{dt} + \mathrm{R}x = \mathrm{N}\frac{dy}{dt} + \mathrm{S}y;$$

but, by the principle of continuity,

$$i = x + y$$

always, since there can be no accumulation of electricity at P or P′;

hence

$$x = i - y,$$

and, hence, $$\mathrm{L}\frac{d(i-y)}{dt}+\mathrm{R}(i-y)=\mathrm{N}\frac{dy}{dt}+\mathrm{S}\,y,$$

or $$\mathrm{L}\frac{di}{dt}+\mathrm{R}\,i=(\mathrm{L}+\mathrm{N})\frac{dy}{dt}+(\mathrm{R}+\mathrm{S})\,y.$$

But $$i=\mathrm{I}\sin p\,t.$$

and $$\frac{di}{dt}=\mathrm{I}\,p\cos p\,t,$$

$$\therefore\qquad(\mathrm{L}+\mathrm{N})\frac{dy}{dt}+(\mathrm{R}+\mathrm{S})\,y=\mathrm{I\,L}\,p\cos p\,t+\mathrm{I\,R}\sin p\,t,$$

or $$\frac{dy}{dt}+\frac{\mathrm{R}+\mathrm{S}}{\mathrm{L}+\mathrm{N}}y=\frac{\mathrm{L}\,p\,\mathrm{I}}{\mathrm{L}+\mathrm{N}}\cos p\,t+\frac{\mathrm{R\,I}}{\mathrm{L}+\mathrm{N}}\sin p\,t.$$

This differential equation is of the type

$$\frac{dy}{dt}+\mathrm{P}\,y=\mathrm{Q},$$

where Q is a function of $t$. The solution of this will be found in "Boole's Differential Equations," p. 38, and it is

$$y=e^{-\mathrm{P}t}\left(\int\mathrm{Q}\,e^{\mathrm{P}t}\,dt+\text{const}\right),$$

$e$ being here the base of Nap. logs. and not impressed E.M.F.

In the case before us $$\mathrm{P}=\frac{\mathrm{R}+\mathrm{S}}{\mathrm{L}+\mathrm{N}},$$

and $$\mathrm{Q}=\frac{\mathrm{R\,I}}{\mathrm{L}+\mathrm{N}}\sin p\,t+\frac{\mathrm{L}\,p\,\mathrm{I}}{\mathrm{L}+\mathrm{N}}\cos p\,t.$$

The integrals of $e^{\mathrm{P}t}\sin p\,t\,d\,t$ and $e^{\mathrm{P}t}\cos p\,t\,d\,t$ are required. They are as follows:—

$$\int e^{\mathrm{P}}\sin p\,t\,d\,t=\frac{e^{\mathrm{P}t}(\mathrm{P}\sin p\,t-p\cos p\,t)}{\mathrm{P}^2+p^2},$$

and $$\int e^{\mathrm{P}t}\cos p\,t\,d\,t=\frac{e^{\mathrm{P}t}(\mathrm{P}\cos p\,t+p\sin p\,t)}{\mathrm{P}^2+p^2};$$

hence it follows that

$$\int e^{\mathrm{P}t}\mathrm{Q}\,d\,t=\int e^{\mathrm{P}}\frac{\mathrm{R\,I}}{\mathrm{L}+\mathrm{N}}\sin p\,t\,d\,t+\int e^{\mathrm{P}t}\frac{\mathrm{L}\,p\,\mathrm{I}}{\mathrm{L}+\mathrm{N}}\cos p\,t\,d\,t$$

$$=\frac{\mathrm{R\,I}}{\mathrm{L}+\mathrm{N}}e^{\mathrm{P}}\frac{(\mathrm{P}\sin p\,t-p\cos p\,t)}{\mathrm{P}^2+p^2}+\frac{\mathrm{L}\,p\,\mathrm{I}}{\mathrm{L}+\mathrm{N}}\frac{e^{\mathrm{P}t}(\mathrm{P}\cos p\,t+p\sin p\,t)}{\mathrm{P}^2+p^2}$$

and $$\mathrm{P}^2+p^2=\frac{(\mathrm{R}+\mathrm{S})^2+(\mathrm{L}+\mathrm{N})^2p^2}{(\mathrm{L}+\mathrm{N})^2}.$$

Therefore we have by substitution

$$y = \frac{I}{(R+S)^2 + (L+N)^2 p^2} \left\{ \begin{aligned} &[R(R+S) + L(L+N)p^2] \sin pt \\ &+ [(R+S)Lp - R(L+N)p] \cos pt \end{aligned} \right\} \quad (34)$$

or

$$y = \frac{I}{(R+S)^2 + (L+N)^2 p^2} \left\{ \begin{aligned} &[R^2 + p^2 L^2 + R\,S + L\,N\,p^2] \sin pt \\ &\qquad + [S\,L\,p - R\,N\,p] \cos pt \end{aligned} \right\}.$$

Since the original equations are symmetrical in $x$ and $y$, R and S, L and N, the value for $x$ is given by changing R to S and L to N in the equation for $y$.

This equation for $y$ gives us the strength of the current in the fine wire coil, and it shows us that the phase of the currents $x$ and $y$ in the branch circuits differs from that of the main current $i$ by an amount which depends on L, N, R and S. In order to exhibit this in a simple form we may direct attention to a simple trigonometrical transformation.

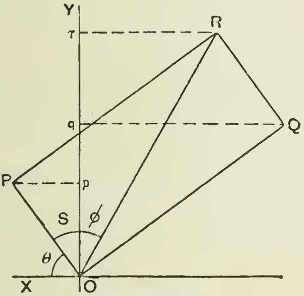

Fig. 69.

*Trigonometrical Lemma.*—The function A sin $\theta$ + B cos $\theta$, where A and B are constants, may otherwise be written

$$\sqrt{A^2 + B^2} \sin(\theta + \phi),$$

where

$$\tan \phi = \frac{B}{A}.$$

Draw any rectangle (Fig. 69) O P, O Q, and draw a pair of rectangular axes, OX, OY, through O. Project the points Q, R, P on O Y. Then, by geometry, if P O X = $\theta$ and P O R = $\phi$,

$$\begin{aligned} O\,r &= O\,p + O\,q, \\ &= O\,P \sin\theta + O\,Q \cos\theta, \\ &= O\,R \sin R\,O\,X = O\,R \sin(\theta + \phi); \end{aligned}$$

hence
$$\mathrm{O\,P}\sin\theta+\mathrm{O\,Q}\cos\theta=\mathrm{O\,R}\sin(\theta+\phi)$$
$$=\sqrt{\mathrm{O\,P}^2+\mathrm{O\,Q}^2}\sin(\theta+\phi).$$

But
$$\tan\phi=\frac{\mathrm{O\,Q}}{\mathrm{O\,P}};$$

hence
$$\mathrm{A}\sin\theta+\mathrm{B}\cos\theta=\sqrt{\mathrm{A}^2+\mathrm{B}^2}\sin(\theta+\phi), \quad . \quad (35)$$

where
$$\tan\phi=\frac{\mathrm{B}}{\mathrm{A}}.$$

Returning then to the equation for $y$, the coefficients of $\sin p\,t$ and $\cos p\,t$ in the equation are respectively $\mathrm{R}\,(\mathrm{R}+\mathrm{S})+\mathrm{L}\,(\mathrm{L}+\mathrm{N})\,p^2$, which represents the A, and $(\mathrm{R}+\mathrm{S})\,\mathrm{L}\,p-\mathrm{R}\,(\mathrm{L}+\mathrm{N})\,p$, which represents the B, in the above. Squaring each of these expressions, and adding the results, we obtain as a result
$$\{(\mathrm{R}+\mathrm{S})^2+(\mathrm{L}+\mathrm{N})^2\,p^2\}\,(\mathrm{R}^2+p^2\,\mathrm{L}^2);$$
hence we finally arrive by substitution at the equation for $y$
$$y=\frac{\mathrm{I}\sqrt{\mathrm{R}^2+p^2\,\mathrm{L}^2}}{\sqrt{(\mathrm{R}+\mathrm{S})^2+p^2\,(\mathrm{L}+\mathrm{N})^2}}\sin(p\,t+\theta), \quad . \quad (36)$$

where
$$\tan\theta=\frac{\mathrm{B}}{\mathrm{A}}=\frac{(\mathrm{R}+\mathrm{S})\,\mathrm{L}\,p-\mathrm{R}\,(\mathrm{L}+\mathrm{N})\,p}{\mathrm{R}\,(\mathrm{R}+\mathrm{S})+\mathrm{L}\,(\mathrm{L}+\mathrm{N})\,p^2}, \quad (37)$$

or
$$\tan\theta=\frac{(\mathrm{S\,L}-\mathrm{R{\cdot}N})\,p}{\mathrm{R}\,(\mathrm{R}+\mathrm{S})+\mathrm{L}\,(\mathrm{L}+\mathrm{N})\,p^2}. \quad . \quad . \quad (38)$$

In this form the equation for $y$ shows us that the phase of $y$ is *ahead* of that of $i$, or that the main current lags behind the current in the branch S, provided that S L is greater than R N; and, since the expression for the current $x$ is perfectly symmetrical, we can write it down at once, and it is
$$x=\frac{\mathrm{I}\sqrt{\mathrm{S}^2+p^2\,\mathrm{N}^2}}{\sqrt{(\mathrm{R}+\mathrm{S})^2+p^2\,(\mathrm{L}+\mathrm{N})^2}}\sin(p\,t+\theta') \quad . \quad (39)$$

where
$$\tan\theta'=\frac{(\mathrm{R\,N}-\mathrm{S\,L})\,p}{\mathrm{S}\,(\mathrm{R}+\mathrm{S})+\mathrm{N}\,(\mathrm{L}+\mathrm{N})\,p^2}; \quad . \quad . \quad (40)$$

and it is obvious that, if S L is greater than R N, $\tan\theta$ is positive, and $\tan\theta'$ is negative. If S L = R N, then there is no lag, and the branch currents $x$ and $y$ agreein phase with the main current $i$.

The general result is, therefore, this—When an impressed electromotive force acts on a circuit which branches into two, having each self but no mutual induction, there is a difference of phase between the currents in the main line and branches; that is, they do not come to their maximum values at the same instant. The main current lags behind the impressed electromotive force in phase, and the two branch currents respectively *lag behind* and are *pressed ahead* of the phase of the main current.

The question then arises, under what circumstances does the branch current which is in advance in phase of the main current get so much ahead that it comes into consonance with the phase of the impressed electromotive force?

To settle this question we shall have to discuss briefly the question of the *compound impedance* of branch circuits.

§ 26. **Impedance of Branched Circuits.**—Lord Rayleigh has treated the problem of the impedance of branched circuits under the assumption that any number of circuits are connected in parallel, posessing each self-induction, but having no mutual induction.*

The problem is: Given the resistance and inductance of each branch, to find the compound resistance and inductance, or equivalent resistance and inductance, of the system for simple periodic currents of given frequency.

Let R and L be the resistance and inductance of any branch, and $p$ the pulsation $= 2\pi n$. Let R′ and L′ be the compound or equivalent resistance and inductance of the system of parallel conductors.

The solution of the problem given, for which we refer the reader to the original paper, is

$$R' = \frac{A}{A^2 + p^2 B^2} \text{ and } L' = \frac{B}{A^2 + p^2 B^2},$$

where $$A = \Sigma \left( \frac{R}{R^2 + p^2 L^2} \right)$$

and $$B = \Sigma \left( \frac{L}{R^2 + p^2 L^2} \right).$$

If we take, as usual, $\tan \theta = \frac{pL}{R}$, and write (Im) for impedance, where $(\mathrm{Im})^2 = \mathrm{R}^2 + p^2\,\mathrm{L}^2$, we can write the above relations

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* See Lord Rayleigh "On Forced Harmonic Oscillations of Various Periods" (*Phil. Mag.*, May, 1886, p. 379).

$$\mathrm{A} = \Sigma \frac{\mathrm{R}}{(\mathrm{Im})^2}$$

$$\mathrm{B} = \Sigma \frac{\mathrm{L}}{(\mathrm{Im})^2}.$$

Let $\mathrm{R}'^2 + p^2\,\mathrm{L}'^2$ be written $(\mathrm{IM})^2$. This is the compound or equivalent impedance of the system of parallel conductors. It is obvious that

$$(\mathrm{IM})^2 = \mathrm{R}'^2 + p^2\,\mathrm{L}'^2 = \frac{1}{\mathrm{A}^2 + p^2\,\mathrm{B}^2},$$

where $$\mathrm{A} = \Sigma \frac{\mathrm{R}}{(\mathrm{Im})^2} \text{ and } \mathrm{B} = \Sigma \frac{\mathrm{L}}{(\mathrm{Im})^2};$$

hence $$(\mathrm{IM})^2 = \frac{1}{\left(\Sigma \frac{\mathrm{R}}{(\mathrm{Im})^2}\right)^2 + \left(\Sigma \frac{\mathrm{L}}{(\mathrm{Im}^2)}\right)^2 p^2}.$$

Consider the case of a pair of conductors in parallel (Fig. 70), having resistances R and S and inductances L and N, but no mutual inductance.

Let $$\sqrt{\mathrm{R}^2 + p^2\,\mathrm{L}^2} = (\mathrm{Im}_1),$$

and $$\sqrt{\mathrm{S}^2 + p^2\,\mathrm{N}^2} = (\mathrm{Im}_2),$$

and $$\sqrt{\mathrm{R}'^2 + p^2\,\mathrm{L}'^2} = (\mathrm{IM});$$

then $$(\mathrm{IM})^2 = \frac{1}{\left(\frac{\mathrm{R}}{(\mathrm{Im}_1)^2} + \frac{\mathrm{S}}{(\mathrm{Im}_2)^2}\right)^2 + \left(\frac{\mathrm{L}}{(\mathrm{Im}_1)^2} + \frac{\mathrm{N}}{(\mathrm{Im}_2)^2}\right)^2 p^2};$$

or $$(\mathrm{IM}) = \frac{(\mathrm{Im}_1)\,(\mathrm{Im}_2)}{\sqrt{(\mathrm{Im}_1)^2 + (\mathrm{Im}_2)^2 + 2\,(\mathrm{R}\,\mathrm{S} + p^2\,\mathrm{L}\,\mathrm{N})}};$$

or $$(\mathrm{IM}) = \frac{\sqrt{\mathrm{R}^2 + p^2\,\mathrm{L}^2}\ \sqrt{\mathrm{S}^2 + p^2\,\mathrm{N}^2}}{\sqrt{(\mathrm{R} + \mathrm{S})^2 + p^2\,(\mathrm{L} + \mathrm{N})^2}}.$$

The lag $\epsilon$ of the main current just before branching, considered with respect to the impressed electromotive force, will be given by the equation

$$\tan \epsilon = \frac{p\,\mathrm{L}'}{\mathrm{R}'};$$

hence $$\tan \epsilon = \frac{p\,\mathrm{B}}{\mathrm{A}} = \frac{\Sigma \frac{p\,\mathrm{L}}{(\mathrm{Im})^2}}{\Sigma \frac{\mathrm{R}}{(\mathrm{Im})^2}}$$

generally, and in the case considered will be

$$\tan \epsilon = \frac{\dfrac{pL}{R^2+p^2L^2} + \dfrac{pN}{S^2+p^2N^2}}{\dfrac{R}{R^2+p^2L^2} + \dfrac{S}{S^2+p^2N^2}};$$

hence after reduction

$$\tan \epsilon = \frac{(S^2+p^2N^2)pL + (R^2+p^2L^2)pN}{(S^2+p^2N^2)R + (R^2+p^2L^2)S}.$$

This is the equation which determines the *lag of phase* of the current $i$ behind the impressed electromotive force in the main branch before dividing into the branch currents $x$ and $y$ in R and S respectively.

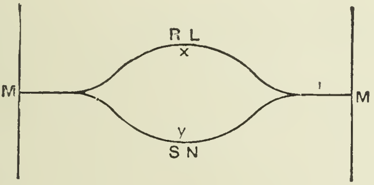

FIG. 70.

Compare this equation with that which determines the angle by which the phase of the branch current $y$ in S is *ahead* of the main current $i$. It is, as we have seen,

$$\tan \theta = \frac{(SL-RN)p}{R(R+S)+L(L+N)p^2}.$$

In the expressions for $\tan \epsilon$ and $\tan \theta$ put $N=0$, and they *both* become equal to

$$\frac{SLp}{R(R+S)+p^2L^2}.$$

This shows that, when $N=0$, the current $y$ in the branch S is as much ahead of the main current $i$ as $i$ is behind the impressed electromotive force, and hence that $y$ agrees in phase with the impressed E.M.F. acting on the double circuit; in other words, the current in the branch S is entirely unaffected by being joined in parallel with an inductive circuit R; but if N is *not* quite zero, then the current in branch S *is* affected, as regards its lag, by the fact of being joined in parallel with an inductive circuit. The nature of this affection will be dependent on whether $SL-RN$ is positive or negative—that is, whether $\frac{L}{R}$ or $\frac{N}{S}$ is the greater—that is, whether the time constant of the R circuit or the S circuit is greater. If $\frac{L}{R}$ is greater than $\frac{N}{S}$, then the current $y$ in S is *ahead* of the main current $i$, but lags behind the impressed electromotive force. If $\frac{L}{R}$ is less than $\frac{N}{S}$, then the current $y$ in S lags behind the main current $i$ in phase, and, *à fortiori*, behind the impressed electromotive force.

§ **27. Wattmeter Measurement of Periodic Power.**—Returning to the wattmeter problem, let one of these divided circuits, viz., the one of resistance R, be a circuit in which it is desired to measure the *electrical power*. In the ordinary way of using the wattmeter, the fine-wire coil, which we will assume has a resistance S, is placed in parallel with the inductive circuit, the thick-wire coil united in series with the inductive circuit. The main current $i$ is thus divided between the inductive circuit R and the wattmeter fine-wire circuit S. The electro-dynamic action in the wattmeter is then one between a current in S, which we have called $y$, and one in the thick-wire circuit, which is the same as that in the inductive circuit R, which we have called $x$.

We have above arrived at expressions for the values of $x$ and $y$. The question then arises how far the indications given by the instrument, and which are due to the electro-dynamic action of the currents $x$ and $y$, and proportional to their numerical product, are proportional to the real power taken up in the circuit R.

The current $x$ is the same as the current in R; hence the error, if any, will result from the current $y$ in S differing in phase or in proportionality from the potential difference between the ends of the circuit R.

In the ordinary mode of calibrating the wattmeter the instrument would be applied to measure a power in a non-inductive circuit traversed by a known current, and having a known potential difference at its ends.

From this the real watts taken up in the circuit are known, and, since the force required to bring back the movable coil to its initial position is proportional to the product of the numerical values of the currents in the fixed and movable coils, we have at once the desired constant of the instrument.

If a wattmeter so calibrated is applied to measure power in an inductive circuit, there are two causes of error which may or may not neutralise each other, and which may cause the measured watts as determined by the instrument to be greater than, equal to, or less than, the real watts or power taken up in the circuit.

The first of these causes of error is due to the fact that the fine-wire circuit of the wattmeter always has a sensible inductance—that is, N is not zero. It may be made very small by arranging the chief part of the wire resistance of the fine-wire circuit as a non-inductive resistance in series with the small inductive resistance which forms the movable coil. It follows that, if E be the maximum potential difference during the period between those points to which the fine-wire circuit is attached, the mean-square ($\sqrt{\text{mean}^2}$) value of the current in the fine-wire circuit is equal to $\frac{1}{\sqrt{2}} \frac{E}{\sqrt{S^2+p^2N^2}}$ when subjected to a simple periodic E.M.F. of angular velocity $p$. This quantity is not proportional merely to E, but depends also on the value of $p$. One effect of the impedance of the fine-wire circuit is to make the mean-square current in it under periodic E.M.F. *less* than it would be if produced by a steady E.M.F. equal to the mean-square value of the periodic E.M.F. But, in addition, the impedance causes a lag in phase of the current in the fine-wire circuit behind the phase of the potential difference between its ends. This is the second cause of error, and the effect of this lag is dependent upon the nature, whether inductive or non-inductive, of the circuit R.

To dissect its action, first let us suppose the circuit R is non-inductive—that is, let L be zero. The current $x$ in it will, therefore, coincide in phase with that of the potential difference at the points of junction. The current in S, viz., $y$, will, however, lag in phase behind that of the potential difference at the junction. The effect of this lag in S will be to increase the phase difference between $x$ and $y$, and to diminish the cosine of this angle of phase difference. Hence, the effect is to diminish the product $\frac{XY}{2}\cos\delta$, which measures the true mean product of $x$ and $y$, X and Y being their maximum values and $\delta$ their difference of phase. Since by assumption X agrees in phase with E, any reduction of the above product reduces the instrumental reading, and makes it less than the true-power reading. If, however, we have to deal with a circuit possessing inductance, and in which, therefore, there is a current $x$, of which the phase lags behind that of the potential difference of the junctions, then the lag in the current $y$ in the circuit S, so far from increasing the difference of phase of $x$ and $y$, may operate to bring them nearer into accordance, and to increase the instrumental reading, and more than make up for the decrease due to the first-named cause of error.

§ 28. **Correcting Factor of a Wattmeter.**—The action of these two causes of error may be illustrated and explained best by the graphic method by a construction which at the same time shows us how to obtain geometrically the value of the compound resistance and impedance of a branched circuit.

Describe a circle with centre O (Fig. 71), and take any line O A to represent the maximum value of the potential difference between the two points M M′ of the divided circuit, of which R is the resistance of the inductive circuit consisting of the thick wire of the wattmeter in series with the circuit in which the *power* is being measured, and S that of the fine wire of the wattmeter. Then, as before, the vertical projection of O A as it revolves represents the periodic variation of this potential difference. On O A describe a semi-circle, and set off on O A, as a base, two right-angled triangles O C A, O B A, of which the sides O B, B A, and O C, C A are in the ratio respectively of the resistance to the reactance of these circuits. Otherwise the angle A O B is one whose tangent is $p$ times the time-constant of the S circuit, and A O C is one whose tangent is $p$ times the time-constant of the R circuit. Take one $S^{th}$ portion of O B, and set off O Y equal to it, then, as in § 21 (p. 144), O Y represents the maximum value Y of the current in the S circuit.

Similarly, set off O X equal to one R$^{\text{th}}$ part of O C, and O X represents the maximum current X in R. On O X, O Y describe a parallelogram O Y I X, and draw the diagonal O I, and produce it to O D. Then O I represents the maximum value of the main current I just before division. Join A D; A D and O D will represent the product of the current I and the equivalent reactance and resistance of the two circuits R and S in parallel respectively.

To prove this last proposition, we must refer again to the paper by Lord Rayleigh on "Forced Harmonic Oscillations of Various Periods" (*Phil. Mag.*, 1886).

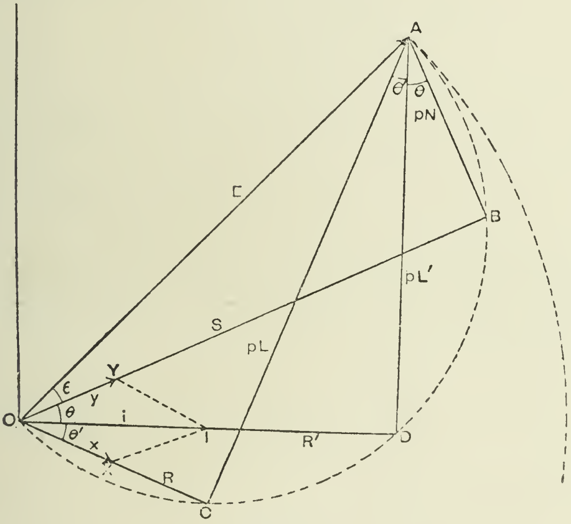

Fig. 71.

If R′ represents the equivalent resistance of a number of resistances joined in parallel between two points, and L′ represents the equivalent inductance of the system, then it is shown in Lord Rayleigh's paper that

$$R' = \frac{A}{A^2 + p^2 B^2} \text{ and } L' = \frac{B}{A^2 + p^2 B^2},$$

where
$$A = \Sigma \frac{R}{R^2 + p^2 L^2},$$

and
$$B = \Sigma \frac{L}{R^2 + p^2 L^2},$$

R and L being the resistance and inductance of any branch, and the mutual inductance being zero.

Apply this theorem to the case under consideration, viz, the two inductive resistances (R, L) (S, N) in parallel, and we have

$$A = \frac{R}{R^2 + p^2 L^2} + \frac{S}{S^2 + p^2 N^2},$$

$$B = \frac{L}{R^2 + p^2 L^2} + \frac{N}{S^2 + p^2 N^2}.$$

Effecting the multiplication we have

$$A = \frac{R\,(S^2 + p^2 N^2) + S\,(R^2 + p^2 L^2)}{(S^2 + p^2 N^2)\,(R^2 + p^2 L^2)},$$

$$B = \frac{L\,(S^2 + p^2 N^2) + N\,(R^2 + p^2 L^2)}{(S^2 + p^2 N^2)\,(R^2 + p^2 L^2)},$$

and
$$R' = \frac{A}{A^2 + p^2 B^2} = \frac{R\,(S^2 + p^2 N^2) + S\,(R^2 + p^2 L^2)}{(R + S)^2 + p^2\,(L + N)^2},$$

$$L' = \frac{B}{A^2 + p^2 B^2} = \frac{L\,(S^2 + p^2 N^2) + N\,(R^2 + p^2 L^2)}{(R + S)^2 + p^2\,(L + N)^2}.$$

Turning back to Fig. 71, we see from the geometry of the figure that, if the angle B O D is as before called $\theta$, B O D = D A B.

We have then $$\frac{O\,D}{\cos\theta} = O\,B - A\,B \tan\theta.$$

But since A B = $p$ N Y and O B = S Y by construction,

therefore $$O\,D = S\,Y \cos\theta - p\,N\,Y \sin\theta.$$

In § 25 we have found the value of $\tan\theta$ to be

$$\tan\theta = \frac{(S\,L - R\,N)\,p}{R\,(R + S) + L\,(L + N)\,p^2};$$

hence, eliminating the sin and cos terms, and substituting for Y the value obtained from equation (36), page 162, we get

$$O\,D = \frac{R\,S\,(R + S) + p^2\,(S\,L^2 + R\,N^2)}{(R + S)^2 + (L + N)^2\,p^2}\,I,$$

where I is the maximum value of the main current, and Y that of the current in the S circuit.

On comparing this value for O D with the value above calculated for R′ we see that O D = R′ I.

So that, on the same scale on which O B and O C represent S Y and R X, O D represents R′ I. Similarly, it may be shown that A D $=p$ L′ I. For the angle C O D $=\theta'=$ angle C A D, and

$$\frac{\mathrm{A\,D}}{\cos\theta'}=\mathrm{A\,C}-\mathrm{O\,C}\tan\theta',$$

or

$$\mathrm{A\,D}=p\,\mathrm{L\,X}\cos\theta'-\mathrm{R\,X}\sin\theta';$$

and, since

$$\tan\theta'=\frac{(\mathrm{R\,N}-\mathrm{S\,L})\,p}{\mathrm{S}\,(\mathrm{R}+\mathrm{S})+\mathrm{N}\,(\mathrm{L}+\mathrm{N})\,p^2},$$

a similar substitution, with help of equation (39), page 162, enables us to see that

$$\mathrm{A\,D}=\frac{p\,\mathrm{L}\,(\mathrm{S}^2+p^2\,\mathrm{N}^2)+p\,\mathrm{N}\,(\mathrm{R}^2+p^2\,\mathrm{L}^2)}{(\mathrm{R}+\mathrm{S})^2+(\mathrm{L}+\mathrm{N})^2p^2}\,\mathrm{I},$$

and this is equal to the value found by analysis for $p$ L′ I.

This diagram shows us, then, what is the effect of the inductance of the wattmeter fine-wire circuit, and what must be the correction applied to the readings to get the real power expended in the inductive circuit.

The actual reading of the wattmeter is proportional to the true mean value of the product of $x$ the current in the inductive circuit R and $y$ the current in the fine-wire circuit S; and this, as previously shown, is equal to half the product of their maximum values, and the cosine of the difference of phase.

From Fig. 71 this mean value is therefore

$$\frac{\mathrm{O\,X}\,.\,\mathrm{O\,Y}}{2}\ \text{cosine B O C}.$$

This, however, is *not* the measure of the power expended in the R circuit. The true watts are proportional to the mean product of $x$ and a current equal to one S^th^ part of $e$, having a phase difference equal to the angle C O A, viz., that of the angle of lag of the current in R and the potential difference O A of its ends. Hence the real power or watts are proportional

to

$$\tfrac{1}{2}\frac{\mathrm{E}}{\mathrm{S}}\,.\,\mathrm{O\,X}\,.\ \text{cosine C O A},$$

since E is the maximum of $e$, viz., the instantaneous potential difference between the extremities of the branch circuits.

Now O Y is taken as one S$^{\text{th}}$ part of the effective electromotive force in the S circuit; and on the same scale on which

O A represents the impressed E.M.F. O B represents the effective E.M.F. in that circuit. Hence, in taking the reading of the wattmeter, which is proportional to the quantity

$$\frac{\mathrm{O\,X} \,.\, \mathrm{O\,Y}}{2} \text{ cosine } \mathrm{B\,O\,C},$$

as the watts, we are making an error; the quantity really required is the value of

$$\frac{1}{2}\frac{\mathrm{E}}{\mathrm{S}} \mathrm{O\,X} \text{ cosine } \mathrm{A\,O\,C},$$

which is numerically equal to the real power. We see that two errors come in—one due to the maximum current in the fine-wire circuit being O Y or $\frac{\mathrm{O\,B}}{\mathrm{S}}$ instead of $\frac{\mathrm{E}}{\mathrm{S}}$ or $\frac{\mathrm{O\,A}}{\mathrm{S}}$, and the other due to the phase difference being taken as the angle C O B instead of C O A.

To correct the instrumental reading or observed watts to true value or real watts, we have to multiply the observed readings by two factors.

*First*, the ratio of $\quad \frac{\mathrm{O\,A}}{\mathrm{O\,B}}$ or $\frac{\sqrt{\mathrm{S}^2+p^2\,\mathrm{N}^2}}{\mathrm{S}}$,

which is the correction due to the self-induction of the fine-wire circuit or to the potential part of the wattmeter having a sensible inductance. The *second* is the ratio of the cosines of the angles C O A and C O B, or

$$\frac{\cos \mathrm{C\,O\,A}}{\cos \mathrm{C\,O\,B}} = \frac{\cos \mathrm{C\,O\,A}}{\cos\,(\mathrm{C\,O\,A} - \mathrm{B\,O\,A})} = k.$$

But from the diagram

$$\text{cosine } \mathrm{C\,O\,A} = \frac{\mathrm{R}}{\sqrt{\mathrm{R}^2+p^2\,\mathrm{L}^2}},$$

and

$$\text{cosine } \mathrm{B\,O\,A} = \frac{\mathrm{S}}{\sqrt{\mathrm{S}^2+p^2\,\mathrm{N}^2}}$$

$$\therefore k = \frac{\dfrac{\mathrm{R}}{\sqrt{\mathrm{R}^2+p^2\,\mathrm{L}^2}}}{\dfrac{\mathrm{R}}{\sqrt{\mathrm{R}^2+p^2\,\mathrm{L}^2}} \cdot \dfrac{\mathrm{S}}{\sqrt{\mathrm{S}^2+p^2\,\mathrm{N}^2}} + \dfrac{p\,\mathrm{L}}{\sqrt{\mathrm{R}^2+p^2\,\mathrm{L}^2}} \cdot \dfrac{p\,\mathrm{N}}{\sqrt{\mathrm{S}^2+p^2\,\mathrm{N}^2}}}$$

Combining these two corrections into a single product, we get as the full correcting factor :—

$$\frac{\sqrt{S^2+p^2N^2}}{S} \cdot \frac{R\sqrt{S^2+p^2N^2}}{RS+p^2LN},$$

or

$$\frac{RS^2+p^2N^2R}{RS^2+p^2LNS}=F.$$

If we put $T_S=\frac{N}{S}$, where $T_S$ is the time-constant of the S, or fine-wire circuit, and $T_R=\frac{L}{R}$, where $T_R$ is the time-constant of the R circuit, we have

$$F=\frac{1+p^2T_S^2}{1+p^2T_RT_S}, \quad . \quad . \quad . \quad . \quad . \quad (41)$$

and the real watts or power taken up in the circuit R is obtained by multiplying the observed watts by F. F becomes unity for two cases when L and N are both zero, and also when $T_S=T_R$.

Hence, the ordinary wattmeter, applied as usual to measure the *electrical power* in a circuit traversed by a simple periodic current, gives absolutely correct readings only in two cases. First, when the fine-wire circuit and the circuit being measured have no inductance; second, when the fine-wire circuit and the circuit being measured have equal time-constants.

But if $T_R$ is greater than $T_S$, then F is a proper fraction. The wattmeter reads too high, and the real watts are less than the observed. If $T_R$ is less than $T_S$, then the observed readings are too low. If $T_R=T_S$, then the observed readings are correct. Hence the wattmeter may read too high, too low, or correct. Generally speaking, it reads too high, since the time-constant of the measured circuit will most often be in excess of that of the fine-wire circuit.

**§ 29. Mutual Induction of Two Circuits of Constant Inductance.**—As an illustration of the above principles, it is useful to consider the case of the mutual induction of two circuits in one of which a simple periodic electromotive force operates. We suppose two circuits to be so placed relatively to each other that when a change of current occurs in one, which is called the Primary (Pr.), a change of magnetic induction takes place through the other, called the Secondary (Sec.). We have, then, to regard the primary and secondary as linked together by loops of induction, and the closed lines of induction, together with the two circuits, must be considered as forming three links of a chain. We shall suppose the self and mutual inductance to be known and to be constant, as also the resistance of each circuit. The primary inductance and resistance will be denoted by L and R, and those of the secondary by N and S, and the mutual inductance by M. The primary circuit is to be subjected to a simple periodic electromotive force of which the maximum value is E, and the result is to generate in the primary circuit a primary current, which, as we have seen, is also a simple periodic quantity, and is to be denoted by its maximum value, $I_1$. The change of induction through the secondary follows the change of current, and gives rise to an impressed electromotive force in the secondary circuit, which, being represented by the rate of change of the simple periodic induction, is also a simple periodic quantity, and gives rise to a simple periodic current in the secondary, to be denoted by its maximum value $I_2$.

The general description of the phenomena produced in such a system of primary and secondary circuits connected by an air magnetic circuit is a follows :—

1. The application of a simple periodic impressed electromotive force, E, produces a simple periodic current, $I_1$, moving under an effective electromotive force, $R\,I_1$, and brings into existence a counter electromotive force of self-induction, which causes the primary current $I_1$ to lag behind E by an angle called the primary lag $\theta_1$. If $n$ is the frequency of the vibrations and $2\pi n = p$, as before; then, as we have before seen, $\tan\theta_1 = \frac{Lp}{R}$, and this counter electromotive force of self-induction is a periodic quantity of which the maximum is $Lp\,I_1$, and of which the phase is 90° behind that of the effective electromotive force or current, or is in quadrature with it.

2. The field round the primary, and therefore the induction through the secondary, is in consonance with the primary current $I_1$; but, since it is also a simple periodic quantity, its

time-rate of change, and therefore the impressed electromotive force in the secondary, is in quadrature with the primary current. Since the induction through the secondary, due to a current $I_1$ in the primary, is $M I_1$, by the definition of M the maximum value of the rate of change of this induction for a pulsation $p$ is $M p I_1$.

It is useful to note that in all dealings with simple periodic quantities, if X is the maximum value of a simple periodic quantity which runs through its cycle $n$ times in a second, the maximum value of its time-rate of change is denoted by $p X$, where $p = 2\pi n$.

If, then, as usual, simple periodic quantities are denoted by the letter signifying their maximum values, prefixing $p$ to any one gives us the value of the maximum of its first differential coefficient with regard to time, or $p$ is here equivalent in notation to $\frac{d}{dt}$.

3. This secondary impressed electromotive force gives rise to a secondary current, $I_2$, moving under an effective secondary electromotive force, $S I_2$, and creating a counter electromotive force of self-induction in the secondary, represented by $N p I_2$.

The secondary current lags behind the secondary impressed electromotive force by an angle $\theta_2$ such that

$$\tan \theta_2 = \frac{N p}{S}.$$

4. This secondary current, $I_2$, reacts, in its turn, on the primary, and it creates what is called a back electromotive force, or reacting inductive electromotive force, on the primary circuit. The phase of this must be in consonance with that of the electromotive force of self-induction in the secondary, and it is represented by the quantity $M I_2 p$. This is obviously in quadrature with the phase of the secondary current or secondary effective electromotive force.

5. There is, then, a phase difference between the primary and secondary currents, and also between the primary impressed electromotive force and the primary current.

The general problem is, then: Given the value of the inductances L, M, N, and the resistances R, S, and that of the impressed electromotive force E and the frequency $n$, find from these seven quantities other four, viz., the primary current $I_1$, the secondary $I_2$, and the difference of phase between E, $I_1$, and $I_2$.

We shall attack the problem geometrically, as this method exhibits far better than the algebraic method the relation between the various quantities involved. The method adopted is to construct an electromotive force diagram, in which all lines represent on any scale volts; and moreover, as each of the quantities considered is a periodic quantity, the lines all represent the maximum value of each quantity, and the value at any instant can be obtained by taking the projections of all lines on any straight line through the centre of the diagram suitably placed.

Let O (Fig. 72) be taken as a centre; draw any line O Q′, and on it set off any length, O T, which we *assume* as the magnitude of the maximum of the primary current. All other lines will be in proper proportion to this. Produce O T to O Q so that $OQ = RI_1$. O Q is then the effective electromotive force in the primary circuit. From Q draw Q P at right angles to O Q, and set off Q P equal to $Lp$ times O T or to $LpI_1$; Q P represents the electromotive force of self-induction in the primary circuit. Join O P.

From O draw O C at right angles to O Q, and set off O C equal to $Mp$ times O T. O C is then equal to $MpI_1$, and O C represents the impressed electromotive force in the secondary circuit. On O C describe a semi-circle, and set off O B, making an angle C O B with O C such that $\tan COB = \frac{Np}{S}$, or tan C O B = the ratio of the inductive to the ohmic resistance for the secondary circuit. Join B C. On the same scale on which O C represents the impressed electromotive force in the secondary circuit, viz., $MpI_1$, O B will then represent the effective electromotive force in the secondary, or will represent $SI_2$, and hence, if O D is taken equal to one $S^{th}$ part of O B, O D will represent $I_2$, or the secondary current. Next draw a line O K perpendicular to O B, and therefore parallel to B C, and on it set off a length, O K, equal to $Mp$ times O D or to $MpI_2$. O K represents then the back inductive electromotive force of the secondary on the primary.

The impressed electromotive force which has to be applied to the primary to produce in it the primary current O T and to induce in the secondary the secondary current O D has therefore to be equal and opposite to the resultant of three electromotive forces, or to equilibrate three electromotive forces, viz., the effective electromotive force of the primary

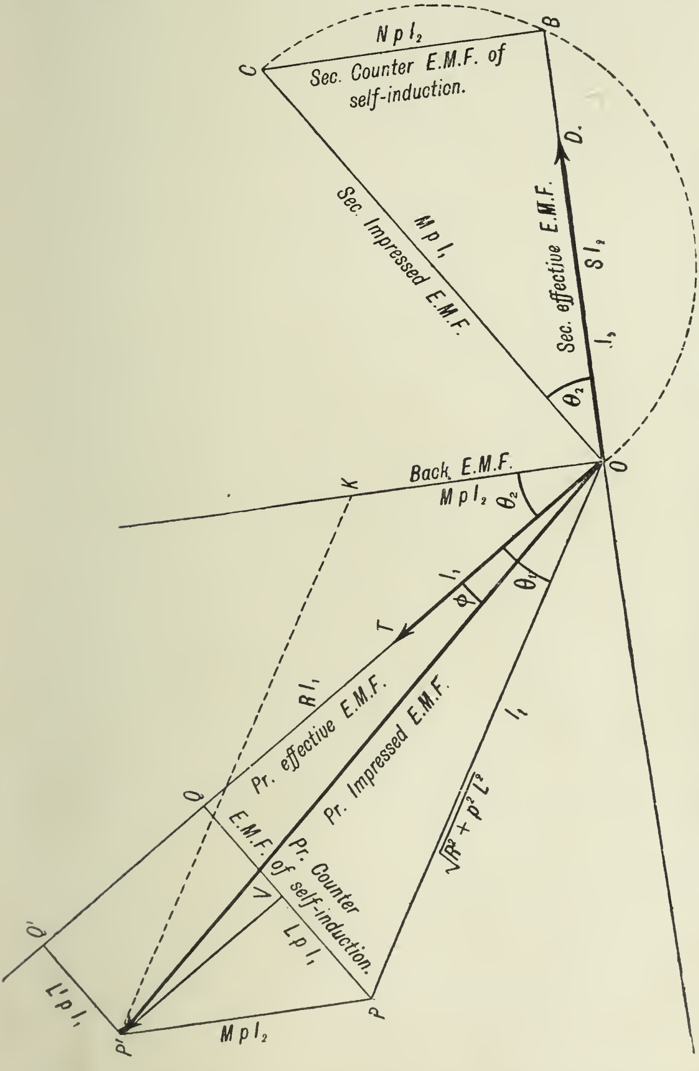

FIG. 72.—Electromotive Force Diagram for a Pair of Mutually Inductive Circuits.

O Q, the electromotive force of self-induction in the primary P Q, and the back electromotive force in the primary due to the inductive effect of the secondary on the primary, viz., O K.

The resultant of O Q and Q P is O P. If, then, we draw P P′ from P, and make it equal and parallel to O K, and join O P′, O P′ will be the resultant of O Q, Q P, and O K; and hence O P′ will represent E, or the impressed electromotive force required to be applied to the primary to maintain the currents $I_1$ and $I_2$.

It is to be understood that in this diagram a unit of length stands for a volt, or unit of electromotive force; and hence, on that assumption, E represents in volts the impressed E.M.F.—that is, the maximum of the simple periodic E.M.F. required to maintain the currents $I_1 I_2$, of which O T, O D represent the maximum values. The relative phases are indicated by the positions of these lines. To obtain the actual values of the E.M.F. and currents at any instant we have only to take the projections of O P′, O T, and O D on any line drawn through O suitably placed, and the magnitudes of these projections will give the required quantities. We must then suppose the whole diagram to be enlarged or diminished without distortion until the length of O P′ is numerically equal to the maximum value in volts of the impressed E.M.F. E, and then O T and O D, will represent the currents $I_1$ and $I_2$ in magnitude. We may consider the two right-angled triangles O Q P, O B C as pivoted together at O, and revolving round O; the fluctuations of the projections of O P′, O T, O D on any line will give us the cyclic values of E, $I_1$, and $I_2$. We can next obtain some useful relations between these quantities from the geometry of the figure. In the triangle O B C,

$$OC^2 = OB^2 + BC^2.$$

Hence

$$I_1^2 M^2 p^2 = S^2 I_2^2 + N^2 p^2 I_2^2;$$

or

$$\frac{I_1}{I_2} = \frac{\sqrt{S^2 + p^2 N^2}}{M p};$$

or

$$\frac{\text{primary current}}{\text{secondary current}} = \frac{\text{impedance of secondary}}{M p}.$$

$Mp$ might by analogy be called the mutual reactance.

To obtain the value of $I_1$ in terms of E and the inductances and resistances, we project the lines O P′ and O P on the vertical line O K, and express the fact that O P′ or E is in all cases the resultant of O K and O P. Let the angle P′ O Q′ be called $\phi$. $\phi$ is the angle by which the primary current lags behind the total impressed electromotive force. Then C O B is $\theta_2$, and T O K = C O B = $\theta_2$, since Q O C and K O B are both right angles.

Hence we have by resolution on O K

$$\mathrm{E}\cos(\phi+\theta_2)=\mathrm{M}\,p\,\mathrm{I}_2+\sqrt{\mathrm{R}^2+p^2\,\mathrm{L}^2}\,\mathrm{I}_1\cos(\theta_1+\theta_2)\,;$$

but, since
$$\mathrm{I}_2=\mathrm{I}_1\,\frac{\mathrm{M}\,p}{\sqrt{\mathrm{S}^2+p^2\,\mathrm{N}^2}},$$

we have by substitution

$$\mathrm{E}\cos(\phi+\theta_2)=\left\{\frac{\mathrm{M}^2p^2}{\sqrt{\mathrm{S}^2+p^2\,\mathrm{N}^2}}+\sqrt{\mathrm{R}^2+p^2\,\mathrm{L}^2}\cos(\theta_1+\theta_2)\right\}\mathrm{I}_1,$$

and therefore a relation established between E and $\mathrm{I}_1$ which is known when $\phi$ is known.

Since
$$\tan\theta_1=\frac{p\,\mathrm{L}}{\mathrm{R}},\text{ and }\tan\theta_2=\frac{p\,\mathrm{N}}{\mathrm{S}},$$

it follows by an easy transformation that

$$\cos(\theta_1+\theta_2)=\frac{\mathrm{R}\,\mathrm{S}-p^2\,\mathrm{L}\,\mathrm{N}}{\sqrt{\mathrm{R}^2+p^2\,\mathrm{L}^2}\,\sqrt{\mathrm{S}^2+p^2\,\mathrm{N}^2}}\,;$$

hence
$$\mathrm{E}=\frac{\mathrm{M}^2p^2+\mathrm{R}\,\mathrm{S}-p^2\,\mathrm{L}\,\mathrm{N}}{\sqrt{\mathrm{S}^2+p^2\,\mathrm{N}^2}}\;\frac{\mathrm{I}_1}{\cos(\phi+\theta_2)}\,.$$

To find the value of $\phi$, suppose that whilst E and $\mathrm{I}_1$ remain the same, the secondary circuit is suppressed. We should then only have an impressed electromotive force, E, creating a current, $\mathrm{I}_1$, and from the diagram and from what has been before explained it is obvious that the effective and self-inductive electromotive forces in the circuit would then be represented by O Q′ and Q′ P′. If we denote these by the symbols $\mathrm{R}'\mathrm{I}_1$ and $\mathrm{L}'p\,\mathrm{I}_1$ we may properly call R′ and L′ the equivalent resistance and inductance; that is to say, these quantities are the resistance and inductance which the primary circuit should have in order that, when there is no secondary circuit, the primary impressed electromotive force may generate in it the same current which it does when the secondary circuit is present and the primary has its natural resistance R and inductance L. We see, then, that the effect of bringing up the secondary and allowing it to be acted upon and react upon the primary is to increase the effective resistance and diminish the effective inductance of the primary; in other words, the equivalent resistance of the primary circuit is greater and the equivalent inductance is less by reason of the presence of the secondary circuit.

We have then to find the value of $\cos(\phi + \theta_2)$.

$$\operatorname{Cos}(\phi + \theta_2) = \cos\phi \cos\theta_2 - \sin\phi \sin\theta_2 ;$$

but $$\cos\phi = \frac{R'}{\sqrt{R'^2 + p^2 L'^2}}, \qquad \sin\phi = \frac{L'p}{\sqrt{R'^2 + p^2 L'^2}},$$

and $$\cos\theta_2 = \frac{S}{\sqrt{S^2 + p^2 N^2}}, \qquad \sin\theta_2 = \frac{Np}{\sqrt{S^2 + p^2 N^2}};$$

hence $$\cos(\phi + \theta_2) = \frac{R' S - L' N p^2}{\sqrt{S^2 + p^2 N^2} \ \sqrt{R'^2 + p^2 L'^2}}.$$

Substituting this in the equation connecting E and $I_1$, we arrive at

$$E = I_1 \frac{(M^2 p^2 + R S - p^2 L N) \sqrt{R'^2 + p^2 L'^2}}{R' S - L' N p^2}.$$

Returning to Fig. 72, we see from it that $P' Q'$ is parallel to $P Q$, and hence, if we draw $P'V$ parallel to $Q' Q$, we have

$$\begin{aligned} P' Q' &= P Q - P V \\ &= P Q - P P' \cos P' P V \\ &= P Q - P P' \sin\theta_2, \end{aligned}$$

or $$L' p I_1 L p = I_1 - M p I_2 \frac{N p I_2}{M p I_1};$$

or, since $$\frac{I_2^2}{I_1^2} = \frac{M^2 p^2}{S^2 + p^2 N^2},$$

we have $$L' = L - \frac{N M^2 p^2}{S^2 + p^2 N^2}. \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Also, again, $O Q$ represents $R I_1$ and $O Q'$ on the same scale $R' I_1$, and

$$\begin{aligned} O Q' &= O Q + Q Q' \\ &= O Q + P' V \\ &= O Q + P P' \sin P' P V \\ &= O Q + P P' \cos\theta_2 ; \end{aligned}$$

hence $$R' I_1 = R I_1 + M p I_2 \frac{S I_2}{M p I_1};$$

and, therefore, $$R' = R + \frac{M^2 p^2 S}{S^2 + p^2 N^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

These formulæ (42) and (43) give us the effective inductance and resistance of the primary circuit as affected by the secondary. They were first given by Clerk Maxwell in a paper in the *Philosophical Transactions* of the Royal Society in 1865, entitled "A Dynamical Theory of the Electromagnetic Field" (*Phil. Trans.*, 1865, p. 475).*

If we form from (42) and (43) the function $R' S - L' N p$, we find it to be $M^2 p^2 + R S - p^2 L N$; and hence, by substitution in the expression already given connecting E and $I_1$, we arrive finally at the result

$$E = I_1 \sqrt{R'^2 + p^2 L'^2},$$

or

$$I_1 = \frac{E}{\sqrt{R'^2 + p^2 L'^2}}.$$

Following the usual nomenclature, we may call the expression $\sqrt{R'^2+p^2 L'^2}$ the *equivalent impedance* of the primary circuit, and we have as the final result for the induction coil of constant inductance

$$\text{Primary current strength} = \frac{\text{impressed electromotive force}}{\text{equivalent impedance of primary circuit}},$$

$$\text{Secondary current strength} = M p \times \frac{\text{primary current}}{\text{impedance of secondary circuit}}.$$

The angle of lag of primary current behind impressed E.M.F. $= \phi$, where $\tan \phi = \frac{p L'}{R'}$, and the angle of lag of the secondary current behind the primary is seen to be $90° + \theta_2$ and $\tan \theta_2 = \frac{N p}{S}$; hence we have the values and relative phases of the currents and the impressed electromotive force.

In the above equations we are to understand current strengths and electromotive forces to be the maximum values during the period. If $i_1$ and $i_2$ be the actual values at any time $t$, reckoning time from the instant of the zero value of the electromotive force, then, from the principles previously explained in this chapter, it is obvious that

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* *See also* Lord Rayleigh on "Forced Harmonic Oscillations of Various Periods," *Phil. Mag.*, May, 1886, p. 375.

$$i_1 = \mathrm{I}_1 \sin(p\,t - \phi),$$

or
$$i_1 = \frac{\mathrm{E}}{\sqrt{\mathrm{R}'^2 + p^2\,\mathrm{L}'^2}} \sin(p\,t - \phi);$$

and
$$\mathrm{I}_2 = \frac{\mathrm{M}\,p\,\mathrm{I}_1}{\sqrt{\mathrm{S}^2 + p^2\,\mathrm{N}}},$$

or
$$i_2 = \frac{\mathrm{M}\,p}{\sqrt{\mathrm{S}^2 + p^2\,\mathrm{N}^2}} \frac{\mathrm{E}}{\sqrt{\mathrm{R}'^2 + p^2\,\mathrm{L}^2}} \sin(p\,t - \phi - \theta_2 - 90^\circ).$$

The student will find the above expressions for the primary and secondary currents can be deduced by analytical processes from the simultaneous equations.

$$\mathrm{L}\frac{d\,i_1}{d\,t} + \mathrm{M}\frac{d\,i_2}{d\,t} + \mathrm{R}\,i_1 = \mathrm{E}\sin p\,t, \quad . \quad . \quad (44)$$

$$\mathrm{N}\frac{d\,i_2}{d\,t} + \mathrm{M}\frac{d\,i_1}{d\,t} + \mathrm{S}\,i_2 = 0, \quad . \quad . \quad . \quad . \quad (45)$$

which equations can be established for two circuits by analogous methods to that by which in § 18 a current equation was arrived at for one circuit, subject to a simple periodic electromotive force.* It is easily seen that if $n$ is very great, or the alternations extremely rapid, then

$$\frac{\mathrm{I}_1}{\mathrm{I}_2} = \frac{\mathrm{N}}{\mathrm{M}}.$$

If the primary and secondary circuits consist of two equal circuits, so interwound that for these circuits $\mathrm{L} = \mathrm{M} = \mathrm{N}$, then for very rapid alternations we see that the secondary current $\mathrm{I}_2$ is equal in magnitude, and exactly opposite in phase, to the primary current $\mathrm{I}_1$, and the magnetic fields due to these currents respectively are also equal and opposite in direction at every instant.

§ 30. **The Flow of Simple Periodic Currents into a Condenser.**—The electrical capacity of a conductor of any kind is measured by the quantity of electricity required to charge it to unit potential. Two conducting surfaces so arranged as to have constant capacity are generally called a condenser. The most simple and familiar form of this appliance is the Leyden jar, in which two surfaces of tin foil are separated by a sheet of glass. Condensers for practical purposes are generally formed of sheets of tinfoil, separated by some dielectric, such as paraffined paper, mica, gutta-percha, tissue, or ebonite. If a quantity of electricity, Q, measured in microcoulombs, is given to one of the plates or sets of plates of a condenser, and if the other set are kept at zero potential, the charged plates will be raised to a certain potential, say V volts, above that of the earth. The capacity of the condenser C in microfarads is then such that by definition

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* *See* Mascart and Joubert's "Electricity," Vol. I., p. 521.

$$\mathrm{C\,V}=\mathrm{Q}.$$

If the potential difference of the condenser terminals at any instant is $v$, and if it is changing, and if $q$ is the charge or quantity of electricity in the condenser at that same instant, then

$$\mathrm{C}\,v=q,$$

or

$$\mathrm{C}\frac{d\,v}{d\,t}=\frac{d\,q}{d\,t};$$

but $\frac{d\,q}{d\,t}$ is the time rate at which the charge is changing or is the value of the current $i$ at that instant flowing into or out of the condenser. Hence

$$\frac{d\,q}{d\,t}=i,$$

and

$$\mathrm{C}\frac{d\,v}{d\,t}=i.$$

Suppose the condenser is being charged through a circuit whose resistance is R, then $i$ is the current which is flowing through this resistance at the instant considered; and the fall of potential down the resistance is R $i$ volts. If a constant potential, V, is being applied to the outer extremity of the resistance, so that the total difference of potential between one plate of the condenser and the outer end of the resistance attached to the other plate is V volts, we have at any instant the equation

$$v+\mathrm{R}\,i=\mathrm{V},$$

or

$$v+\mathrm{C\,R}\frac{d\,v}{d\,t}=\mathrm{V},$$

or

$$\mathrm{C\,R}\frac{d\,v}{d\,t}+v=\mathrm{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

This is the differential equation for the potential $v$ of the condenser at any instant, $t$, when being charged through a resistance, R, by a constant potential, V. The equation (46) is easily integrated as follows:—Multiply all through by $e^{\frac{1}{\mathrm{RC}}t}$, where $e$ is the base of the Napierian logarithms, viz., 2·71828. Then we have, as a result, the equation

$$e^{\frac{t}{\mathrm{RC}}}\frac{dv}{dt}+v\frac{1}{\mathrm{RC}}e^{\frac{t}{\mathrm{RC}}}=\frac{\mathrm{V}}{\mathrm{RC}}e^{\frac{t}{\mathrm{RC}}},$$

or

$$\frac{d}{dt}\left(v\,e^{\frac{t}{\mathrm{RC}}}\right)=\mathrm{V}\frac{d}{dt}\left(e^{\frac{t}{\mathrm{RC}}}\right).$$

Both sides of this last equation can then be integrated. Hence, integrating with respect to time, we obtain the equation

$$v\,e^{\frac{t}{\mathrm{RC}}}=\mathrm{V}\,e^{\frac{t}{\mathrm{RC}}}+a\ constant,$$

or

$$v=\mathrm{V}+\mathrm{C}\,e^{\frac{t}{\mathrm{RC}}}.$$

The constant C is determined by the condition that when the time $t$ is zero then $v$ is also zero. Hence C = – V, and, therefore,

$$v=\mathrm{V}\left(1-e^{-\frac{}{\mathrm{RC}}}\right). \quad . \quad . \quad . \quad . \quad (47)$$

The student should compare equations (46) and (47) for the value of the instantaneous potential of a condenser charged by a constant voltage with the equations (26), page 127, and (27), page 132, for the instantaneous value of a current in an inductive circuit acted on by a constant pressure, and it will be seen that they are similarly constructed. The quantity C R is called the *time-constant* of the condenser just as $\frac{\mathrm{L}}{\mathrm{R}}$ is called the *time-constant* of the inductive coil. The greater the condenser time-constant the larger will be the time taken by the condenser potential to arrive at a given fraction of the steady impressed voltage applied to charge it. Practically, the condenser is fully charged in a time equal to eight or ten times its true constant. The reader must note that if the capacity C of the condenser is measured in microfarads, then the resistance R must be measured in megohms to obtain the correct measure of the time-constant C R in seconds.

Suppose, for example, that a condenser of one microfarad is being charged through a resistance of one megohm by an impressed voltage of 100 volts. It is required to find the potential difference at the terminals of the condenser at the end of the 1st, 2nd, 3rd, 4th, $n$th second. Let $v_1$, $v_2$, $v_3$, $v_n$ be these potential differences.

Then $\quad v_1 = 100\,(1 - e^{-1}) = 63$ volts nearly,

where $\quad e = 2{\cdot}71828$,

and $\quad v_2 = 100\,(1 - e^{-2}) = 86$ volts nearly,

or $\quad v_n = 100\,(1 - e^{-n})$.

Since $e^{-10}$ is an exceedingly small number, in ten seconds the condenser potential is practically equal to 100 volts. We see, therefore, that in charging condensers through resistances sufficient time must always be allowed for the charge, depending on the value of the quantity C R, and the charge is not practically complete unless contact with the source of impressed voltage endures at least for a time equal to 5 C R, or, better still, 10 C R.

Supposing, in the next place, that the impressed voltage acting on the condenser is periodic in character or alternating, and that the condenser, instead of being charged through a resistance, has its terminals shunted by a resistance. Let the capacity of the condenser be, as before, C microfarads, and let the resistance of the shunting conductor be $\frac{1}{\mathrm{K}}$ ohms; that is, let K be the shunt conductance. Let the impressed voltage be periodic, and let its frequency be $n$, and write $p$ for $2\pi n$, as usual. Then, as above, the current flowing into the condenser at the instant when its terminal potential difference is $v$ is $\mathrm{C}\frac{dv}{dt}$. Also the current flowing through the shunt is $\mathrm{K}v$. We have, therefore, as the equation for the total current flowing into the shunted condenser, the expression

$$\mathrm{C}\frac{dv}{dt} + \mathrm{K}v = i, \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where $i$ is the total current flowing into condenser and through the shunt at the instant when the terminal potential difference of the condenser is $v$. Let time be measured from the instant when the total current denoted by $i$ is zero, then, if $i$ is a simple periodic current, it may be expressed in terms of its maximum value by the equation

$$i = \mathrm{I} \sin p t.$$

Hence
$$\mathrm{C}\frac{d v}{d t} + \mathrm{K} v = \mathrm{I} \sin p t. \quad . \quad . \quad . \quad . \qquad (49)$$

On comparing these equations, (48) and (49), for the instantaneous value of the condenser potential difference when a periodic current is flowing through it with the equations (29), page 133, and (31), page 135, it will be seen that they are structurally the same, and we can at once write down the solution of equation (49) by imitating the solution of (29), writing C instead of L, K instead of R, $v$ instead of $i$, and I instead of E. Hence the solution of

$$\mathrm{C}\frac{d v}{d t} \ \mathrm{K} v = \mathrm{I} \sin p t$$

is
$$v = \frac{\mathrm{I}}{\sqrt{\mathrm{K}^2 + \mathrm{C}^2 p^2}} \sin (p t - \theta) + \text{a constant}, \quad . \qquad (50)$$

where $\theta$ is an angle such that $\tan \theta = \frac{\mathrm{C} p}{\mathrm{K}}$.

The quantity $\sqrt{\mathrm{K}^2 + \mathrm{C}^2 p^2}$ has not yet received an acknowledged name, but it is analogous to the impedance in the case of the current flow in inductive circuits. It has been suggested that this quantity should be called the *admittance* of the condenser.

From equation (50) we see that the terminal voltage of the condenser lags behind the charging current in phase; or otherwise that the current is in *advance* of the potential difference. If the shunt wire is removed this is equivalent to making K equal to zero, and under these circumstances we have

$$v = \frac{\mathrm{I}}{\mathrm{C} p} \sin (p t - 90),$$

or
$$v = \frac{\mathrm{I}}{\mathrm{C} p} \cos p t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (51)$$

This shows that in the case of a condenser having a dielectric with no true conducting power, the charging current is exactly 90deg. in advance in phase of the terminal potential difference of the condenser.

It is clear, therefore, that as regards producing lag or lead of current *capacity* acts in the opposite direction to *inductance*, and may be considered to be equivalent to a *negative inductance*.

It is also evident that if a shunted condenser is placed in series with an inductive circuit a proper relation may be formed between the inductance, resistance, capacity, and conductance, such that the combination of inductive resistance and shunted condenser acts to an impressed simple periodic electromotive force just as if it were totally non-inductive. This annulment of inductance has important applications in practice in telegraphy, and it is therefore desirable to define these conditions carefully.

**§ 31. A Shunted Condenser in Series with an Inductive Resistance.**—Let a condenser of capacity C have its terminals closed by a circuit whose conductance is K, and hence its resistance $K^{-1}$. Let this shunted condenser be placed in series with an inductive resistance whose inductance is L and resistance R. At any instant let $v_1$ be the potential difference of the terminals of the condenser and $v_2$ those of the inductive resistance when a periodic current having at this same instant a value $i$ is flowing through the system. Let $v$ be the value of the over-all potential difference. Then suppose that $v$ is a simple periodic potential difference, or that

$$v = V \sin p\,t.$$

Then the current $i$ will differ in phase from $v$, and may be represented by the expression

$$i = I \sin (p\,t - \phi).$$

Capital letters represent maximum values as usual. We can then form the fundamental equation for the shunted condenser in series with an inductive circuit as follows:—We have for the shunted condenser terminal potential $v_1$ the equation

$$C\frac{d\,v_1}{d\,t} + K\,v_1 = i, \quad . \quad . \quad . \quad . \quad . \quad (52)$$

and for the inductive resistance the equation

$$L\frac{d\,i}{d\,t} + R\,i = v_2, \quad . \quad . \quad . \quad . \quad . \quad (53)$$

and also

$$v_1 + v_2 = v, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

since the over-all potential $v$ is the sum of the separate potential steps $v_1$ and $v_2$. Eliminate $v_1$ and $v_2$ from these equations, and we get

$$\mathrm{C\,L}\frac{d^2 i}{d t^2}+(\mathrm{C\,R}+\mathrm{K\,L})\frac{d i}{d t}+(\mathrm{K\,R}+1)\,i=\mathrm{C}\frac{d v}{d t}+\mathrm{K}\,v, \quad . \quad (55)$$

also, since $v=\mathrm{V}\sin p\,t$,

we have $\mathrm{C}\dfrac{d v}{d t}=\mathrm{C}\,p\,\mathrm{V}\cos p\,t,$

and $\mathrm{K}\,v=\mathrm{K\,V}\sin p\,t.$

Hence we have

$$\mathrm{C}\frac{d v}{d t}+\mathrm{K}\,v=\sqrt{\mathrm{K}^2+\mathrm{C}^2 p^2}\;\mathrm{V}\sin(p\,t-\theta). \quad . \quad (56)$$

This last follows at once from the lemma on page 161.

Then, since $i=\mathrm{I}\sin(p\,t-\phi),$

we obtain $\dfrac{d i}{d t}=\mathrm{I}\,p\cos(p\,t-\phi),$

and $\dfrac{d^2 i}{d t^2}=-\mathrm{I}\,p^2\sin(p\,t-\phi),$

and by substitution of these last values in equation (55), we arrive at the equation

$$(\mathrm{K\,R}+1-\mathrm{C\,L}p^2)\,\mathrm{I}\sin(p\,t-\phi)+(\mathrm{C\,R}p+\mathrm{K\,L}p)\,\mathrm{I}\cos(p\,t-\phi)$$
$$=\sqrt{\mathrm{K}^2+\mathrm{C}^2p^2}\;\mathrm{V}\sin(p\,t-\theta) \quad . \quad . \quad (57)$$

or

$$(\mathrm{C\,R}+\mathrm{K\,L})\frac{d i}{d t}+(\mathrm{K\,R}+1-\mathrm{C\,L}p^2)\,i$$
$$=\sqrt{\mathrm{K}^2+\mathrm{C}^2p^2}\;\mathrm{V}\sin(p\,t-\theta). \quad . \quad . \quad (58)$$

Since it is shown on page 161 that any function of the form $\mathrm{A}\sin\theta+\mathrm{B}\cos\theta$ can always be expressed in the form

$$\sqrt{\mathrm{A}^2+\mathrm{B}^2}\,\sin(\theta+\alpha),\ \text{where}\ \tan\alpha=\frac{\mathrm{B}}{\mathrm{A}},$$

it follows at once that the maximum value of the current **I** is given by the equation

$$\mathrm{I}^2=\frac{\mathrm{V}^2(\mathrm{K}^2+\mathrm{C}^2p^2)}{(\mathrm{K\,R}+1-\mathrm{C\,L}\,p^2)+(\mathrm{C\,R}\,p+\mathrm{K\,L}\,p)^2}, \quad . \quad . \quad (59)$$

and it is not difficult to show that this last equation (59) may be written in the form

$$\mathrm{I}^2=\frac{\mathrm{V}^2}{\left\{\mathrm{R}+\dfrac{\mathrm{K}}{\mathrm{K}^2+\mathrm{C}^2p^2}\right\}^2+\left\{\mathrm{L}-\dfrac{\mathrm{C}}{\mathrm{K}^2+\mathrm{C}^2p^2}\right\}^2p^2}.$$

Let $R^1$ stand for $\dfrac{K}{K^2+C^2p^2}$,

and $L^1$ for $\dfrac{C}{K^2+C^2p^2}$.

If the condenser and shunt are removed it is clear that the maximum value of the current I will be given by

$$I^2=\frac{V^2}{R^2+p^2L^2}.$$

Hence the effect of adding the shunted condenser is as if the resistance of the inductive circuit were *increased* by an amount equal to $\dfrac{K}{K^2+C^2p^2}$, and the inductance of the inductive circuit *diminished* by an amount equal to $\dfrac{C}{K^2+C^2p^2}$.

Hence, if the quantity $\dfrac{C}{K^2+C^2p^2}$ is equal to L, which is the inductance of the circuit, then the total effective inductance of the circuit will vanish, and the inductive resistance and shunted condenser together will act as if there were no resultant inductance at all. This condition is fulfilled when we have

$$C=K^2L+C^2p^2L,$$

which equation gives us the required magnitude of the capacity to annul the inductance. Uniting $\dfrac{1}{r}$ instead of K, so that $r$ is the resistance of the shunt, we have

$$r^2=\frac{L}{C(1-CLp^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

as the final equation, which tells us what must be the resistance of the shunt to be put across the terminals of a condenser of capacity C in order that the combination may just annul the inductance of a resistance in series with it whose resistance is R and inductance L. If L is measured in *henries*, C must be in *farads*, and R and $r$ be given in ohms.

It will be seen that the annulment is only exact for one particular frequency, and that change of frequency means a change in the value of $r$ requisite to neutralise the inductance. In telegraphic work it is usual to employ a shunted condenser to annul more or less completely the self-induction of relay magnets.

§ 32. **Representation of Periodic Currents by Polar Diagrams.** Two methods of graphically representing simple periodic currents or electromotive forces, viz., clock diagrams and wave diagrams, have been hitherto used. Each of these methods has some peculiar advantages of its own. There is, however, a third method which has sometimes especial utility, and this is by a polar diagram.

Let a straight line O P (Fig. 73) revolve round one of its extremities O, and let the angle of inclination $\theta$ which the revolving line makes with another fixed straight line O A passing through this centre of revolution be called the angle of displacement of the revolving line. Let any point P be taken on the revolving line, and let the distance of this point from the centre be denoted by $r$. The length $r$ is called the radius vector. Let $r$ increase or diminish as O P revolves, but so that the length $OP = r$ varies according to any law connecting it with the angle of displacement A O P. The extremity P of the line O P will then trace out a curve called a polar diagram.

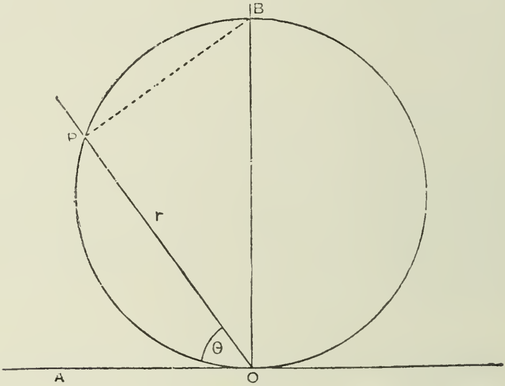

Fig. 73.

Suppose that O P runs through a cycle of values beginning with a zero value when $\theta = 0$, and, after reaching a maximum value, becoming zero again after the line O P has completed one half revolution, or when $\theta=\pi$, the polar diagram will be a closed curve passing through the origin. Since the radius O P runs through a cycle of values from zero to a maximum, and returns to zero again during a revolution of the radius through two right angles, or an angle $\pi$, the radius $r$ may suitably represent the value of any periodic quantity which completes a cycle of values in the time represented by one complete half revolution of the radius vector. Suppose, then, that O P varies as sin $\theta$, where $\theta$ is the displacement phase angle—that is, let

$$r=\mathrm{R}\sin\theta.$$

It is then clear that in this case the polar curve is a circle. For if we draw the line O B through O perpendicular to O A, and draw P B at right angles to O P, since A O P$=\theta$ and O P$=r$, and since by supposition $r=\mathrm{R}\sin\theta$, it follows that the angle P B O is always equal to $\theta$, and that, therefore, the length O B is a constant length equal to R. In other words, the *locus* of the point P is a circle passing through O, P, and B. Hence the circle is the curve which represents in a polar diagram a simple periodic quantity, and if the length O P is taken to represent the magnitude of a simple periodic current or electromotive force at any instant corresponding to a phase angle A O P, the circle passing through O will be the polar curve representing this simple periodic current or electromotive force.

If the current or electromotive force is periodic, but not simply periodic, then the polar curve representing it will be a unicursal curve passing through the origin O, but will not be a circle. The polar diagram of an alternating current or electromotive force enables us very easily to determine the square root of the mean-square value of the periodic quantity represented by it. It is more difficult to do this with the ordinary wave-curve diagram.

Suppose, for example, that in Fig. 74 we have a wave diagram drawn representing a half wave of an alternating electromotive force which is not a simple sine curve. The ordinates represented by the dotted lines are proportional to the instantaneous values of the periodic quantity taken every 20deg. If we wish to find the $\sqrt{\text{mean}^2}$ value of the ordinates of this curve there is no other way of doing this than by drawing a number of numerous equidistant vertical ordinates, measuring their lengths, squaring these magnitudes, and taking the square root of the mean of these squares. This is a troublesome arithmetical process, and in proportion as the wave curve is more irregular or complex, so much the more numerous must be the ordinates to obtain a correct result for the mean-square value. It is, however, a much easier process if the periodic quantity is represented by a polar curve. Let Fig. 75 represent the same periodic function as in Fig. 74, drawn in a polar form. The dotted radii are proportional to the instantaneous values of the periodic quantity, and are placed at angular intervals of 10deg. Let any radius O P (*see* Figs. 75 and 76) be denoted by $r$, and let the corresponding phase angle P O A be denoted by the letter $\theta$. If we consider the radius to move forward by a small angle $d\,\theta$ and to increase in length by a small amount $d\,r$, then it is obvious that the small increment of area $d$ A swept out by the radius $r$ is equal to $\frac{1}{2}\,r^2\,d\,\theta$. To obtain the whole area, and included by the polar curve O P Q R, we have to integrate this quantity $\frac{1}{2}\,r^2\,d\,\theta$ from 0 to $\pi$, or to obtain the integral

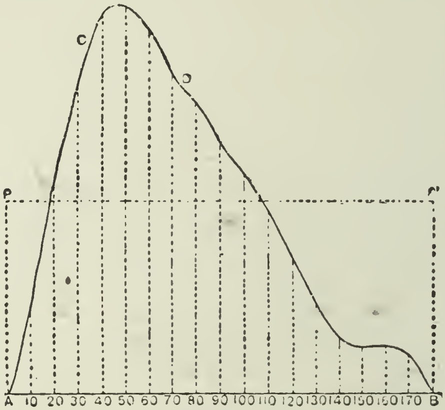

Fig. 74.

$$A = \frac{1}{2}\int_0^{\pi} r^2\,d\,\theta.$$

On the base line drawn through the polar centre O let a semicircle A C B (*see* Fig. 75) be described equal in area to the area included by the polar curve O P Q. Let $OA = R$ be the radius of this semicircle. Then

$$\frac{\pi R^2}{2} = A = \tfrac{1}{2}\int_0^\pi r^2\,d\theta,$$

or

$$R = \sqrt{\frac{1}{\pi}\int_0^\pi r^2\,d\theta}.$$

The expression on the right hand side of this last equation is obviously the square root of the mean of the squares of the instantaneous values of the periodic quantity $r$. Hence, we can obtain at once the $\sqrt{\text{mean}^2}$ value of a periodic quantity current or electromotive force as follows. On any straight line describe a semicircle and draw radii of this semicircle at angular intervals equal to those phase intervals for which the instantaneous values of the current or electromotive force are observed. Set off on these radii lengths equal to the respective values of these instantaneous quantities and join the extremities of all these lines so set off by a curve. This curve is the polar diagram, and it is a closed curve. Take the area included by the polar diagram with an Amsler's or other planimeter, and from a table of areas of circles find the circle whose area is double that of the polar curve. Then the radius of this circle is the square root of the mean of the squares of the periodic quantity. If, for instance, we have a periodic current

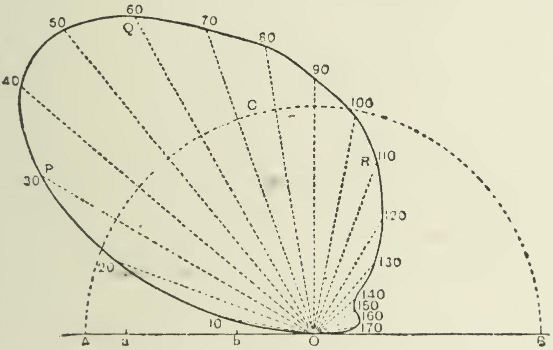

Fig. 75.

curve plotted down in polar form, this operation will give us the dynamometer value of the current, or the value which would be read on a dynamometer.

The polar curve, therefore, lends itself very easily to the determination of the mean-square value of a periodic current or electromotive force, of which the instantaneous values are known throughout a semi-period, whether those values are taken at equidistant intervals of time or not. The reader will, therefore, note that if we plot down a periodic quantity in rectangular co-ordinates (Fig. 74) or *wave form*, and find the rectangle A P P′ B of equivalent area to the half wave, the altitude A P of this rectangle gives us the *true mean ordinate* of the periodic curve; but if we plot down the periodic quantity to a *polar diagram* (Fig. 75) and find the semicircle A C B of equivalent area, the radius O A of this semicircle gives us the *square root of the mean-square value* of the periodic quantity represented by the radii of the polar curve.

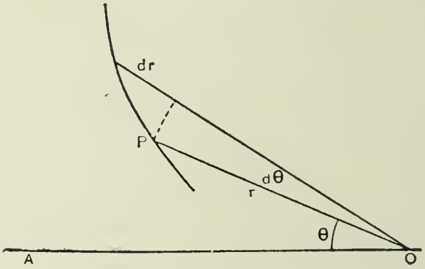

FIG. 76.

§ 33. **Initial Conditions on starting Current Flow in a Circuit having Resistance and Inductance.**—It has been shown in the foregoing sections that if an impressed electromotive force of simple periodic kind acts upon a circuit having inductance, the resulting current is a simple periodic current, but lags behind the impressed electromotive force in phase. These, however, are the conditions when the resulting current has become steady. At the instant of closing the circuit there are peculiar conditions of augmentation of the current which are called initial conditions, and which have very important consequences in practice. It will be advisable, therefore, to examine a little more closely how these are produced, and what results may be expected at the instant of starting or stopping the current in such a circuit. To do this we will, in the first place, examine more carefully the solution of the fundamental equations for current creation in an inductive circuit. It has been shown that the differential equation for current at any instant in the circuit of constant inductance L and resistance R under an impressed simple periodic electromotive force $v = \mathrm{V} \sin p\,t$ is

$$\mathrm{L}\frac{d\,i}{d\,t} + \mathrm{R}\,i = v = \mathrm{V} \sin p\,t. \quad . \quad . \quad . \quad (61)$$

To solve this equation completely, we differentiate it twice, and eliminate thereby the term $\mathrm{V} \sin p\,t$, thus obtaining the equation

$$\mathrm{L}\frac{d^3 i}{d\,t^3} + \mathrm{R}\frac{d^2 i}{d\,t^2} = -p^2\left(\mathrm{L}\frac{d\,i}{d\,t} + \mathrm{R}\,i\right),$$

or

$$\mathrm{L}\frac{d^3 i}{d\,t^3} + \mathrm{R}\frac{d^2 i}{d\,t^2} + p^2\,\mathrm{L}\frac{d\,i}{d\,t} + p^2\,\mathrm{R}\,i = 0. \quad . \quad (62)$$

A differential equation of this type is called a linear differential equation of the third order. It is shown in treatises on differential equations that its solution depends on the solution of a cubic equation called the auxiliary equation. The auxiliary equation in this case is

$$\mathrm{L}\,m^3 + \mathrm{R}\,m^2 + p^2\,\mathrm{L}\,m + p^2\,\mathrm{R} = 0. \quad . \quad . \quad . \quad (63)$$

This cubic equation can be split up into two factors and be written

$$(m^2 + p^2)\,(\mathrm{L}\,m + \mathrm{R}) = 0,$$

and hence the roots of the cubic equation (63) are

$$m = \pm\sqrt{-1}\,p$$

$$m = -\frac{\mathrm{R}}{\mathrm{L}}.$$

For a linear equation of the type of (62), the roots of the auxiliary cubic being $\alpha$ and $\pm\sqrt{-1}\,\beta$, the solution is known to be of the form

$$i = \mathrm{A}\,e^{\alpha t} + \mathrm{B} \sin \beta\,t + \mathrm{B}' \cos \beta\,t.$$

The solution of the differential equation (62) is, then, given by

$$i = \mathrm{A}\,e^{-\frac{\mathrm{R}}{\mathrm{L}}t} + \mathrm{B} \sin p\,t + \mathrm{B}' \cos p\,t. \quad . \quad . \quad (64)$$

By the Trigonometrical Lemma on page 161 we can write, instead of the second and third terms on the right hand side of (64), the single term $\sqrt{B^2+B'^2}\sin(pt-\phi)$, where $\sqrt{B^2+B'^2}$ is obviously the maximum value of the current—call it I—when the initial state is over, and $\phi$ is the angle by which the current, when steady, lags behind the electromotive force in phase. Hence (64) may be written,

$$i = A\,e^{-\frac{R}{L}t} + I\sin(pt-\phi). \quad . \quad . \quad . \quad (65)$$

To find the constant quantity A, we note that at the instant when the circuit is closed the current has necessarily a value zero. Let this closing of the current happen at a time $t'$ reckoned from the instant when the electromotive force is zero. Then at the instant of closing the circuit we have

$$0 = A\,e^{-\frac{R}{L}t'} + I\sin(pt'-\phi),$$

or

$$A = -I\sin(pt'-\phi)\,e^{+\frac{R}{L}t}.$$

Hence, substituting this value of A in equation (65), we obtain

$$i = I\sin(pt-\phi) + I\sin(pt'-\phi)\,e^{-\frac{R}{L}(t-t')}, \quad . \quad . \quad (66)$$

and this is the complete solution of the differential equation (61) for the current in the circuit.

We note that the expression for the current at any instant in the inductive circuit is made up of two terms; the first term, $I\sin(pt-\phi)$, is a simple sine function, and represents by itself a simple periodic current having a maximum value I. The second term, $I\sin(pt'-\phi)\,e^{-\frac{R}{L}(t-t')}$, is an exponential function having a maximum value $I\sin(pt'-\phi)$, when $t'=t$, and this term represents, therefore, a logarithmic curve beginning with the value $I\sin(pt'-\phi)$ and dying away gradually to zero.

Hence the resulting current curve consists of these two curves superimposed upon one another, a periodic curve and a logarithmic or diminishing curve.

In Fig. 77 are shown two such curves; curve 1 being the sine curve, curve 2 the logarithmic curve, and the resultant curve 3, represented by the dotted line, which is obtained by adding together the ordinates of the sine curve and the logarithmic curve. It will be seen that the effect of the superposition is to make the resultant curve lopsided with respect to the curve axis for a certain period, but beyond that time it is sensibly symmetrically situated with respect to the time axis. Hence, during this initial period, the maximum values of the current in opposite directions are not the same. At the instant when $t = t'$ the value of the current is zero. It is obvious that, when $p t' = 90 + \phi$, $\sin (p t' - \phi) = 1$, and that then the logarithmic curve begins with its greatest value; but that, when $p t' = \phi$, then the logarithmic curve has no existence at all. When $p t' = 90 + \phi$—that is if the circuit is closed at an instant $t = \frac{90 + \phi}{p}$ reckoning from the instant when the impressed electromotive force is zero—the disturbance of the uniformity of the periodic curve of current is the greatest possible. At the same time the maximum value of the current in the negative direction can never be greater than 2 I where I is the maximum value of the steadily periodic current. For the maximum value of the logarithmic curve at the instant of closing the circuit is $-$I, and at that instant $i = 0$; hence the value at which the periodic component of the current must begin will be +I. At the time when the periodic part has reached a maximum of $-$I, which happens after half a period, the ordinate of the logarithmic curve will have fallen to something less than I by an amount depending on the rate of decrease, which in turn depends upon the value of $\frac{R}{L}$ or the reciprocal of the time-constant of the circuit.

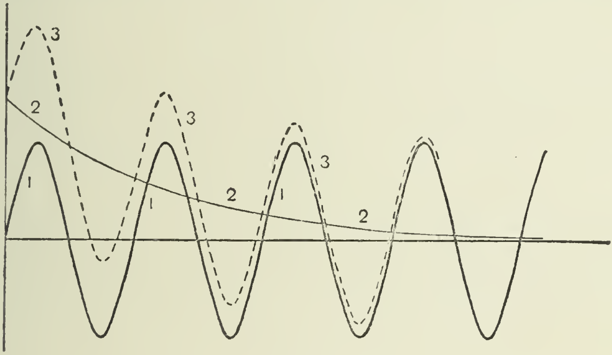

Fig. 77.

Consider the case when the circuit is closed at a time $t'$ reckoned from the zero of electromotive force such that $t' = \frac{90+\phi}{p}$, then $p\,t' - \phi = 90$.

If T is the complete periodic time, then at a time $\frac{T}{2}$ after the instant of closing the value of the periodic part in the expression for the current is

$$\begin{aligned} & I \sin \left\{ p \left( t' + \frac{T}{2} \right) - \phi \right\} \\ = & I \sin (p\,t' - \phi + \pi) \\ = & -I \end{aligned}$$

since $p\,t' - \phi = 90$.

The value for the exponential part at this instant $t' + \frac{T}{2}$ is

$$-I \sin (p\,t' - \phi)\, e^{-\frac{R}{L}\frac{T}{2}}$$

and hence the total value of that current at that instant is

$$\begin{aligned} & -I \left\{ 1 + \sin (p\,t' - \phi)\, e^{-\frac{R}{L}\frac{T}{2}} \right\} \\ = & -I \left\{ 1 + e^{-\frac{R}{L}\frac{T}{2}} \right\}, \end{aligned}$$

since $p\,t' - \phi = 90$.

The greatest value which the time constant $\frac{L}{R}$ can have is infinity. Hence, when $\frac{R}{L} = 0$, or approximately zero, $e^{-\frac{R}{L}\frac{T}{2}} = 1$, and the quantity in the bracket in the last equation may approach to 2 but can never exceed it. This shows us that, with an inductive circuit of very large time-constant, the value of the current after half a period from the instant of closing the circuit may be something a little less than *twice* the value of its periodic steady maximum, provided that the circuit is closed at the instant when the electromotive force has a value $e$ such that

$$e = E \sin (90 + \phi),$$

or

$$e = E \cos \phi,$$

where $\tan \phi = \frac{Lp}{R}$.

This amounts to saying that the circuit must be closed at the instant of zero electromotive force.

Hence we see that during the initial period there may be a greatly increased mean-square value of the current, and thus the production of a *current-rush* on closing the circuit.

When therefore a circuit of constant inductance is switched on to a source of steadily periodic electromotive force at the instant when the electromotive force has a value corresponding to the maximum value of the current, when the variable state is over, we find that, before the current settles down into its steady swing, lagging behind the electromotive force in phase, there is a period of disturbance during which the current has greater maximum values in one direction than in the other, and the current virtually consists of a rapidly evanescent unidirectional current superposed upon the normal periodic current which ultimately survives. If however the circuit is closed at the instant when the electromotive force is passing through a value which corresponds to that at which the current has its zero value, when the variable period is passed, then there is no period of disturbance, but the current begins at once in its normal manner lagging behind the electromotive force and having constant maximum values $+\text{I}$ and $-\text{I}$ alternately. This phenomenon of current rushes into inductive circuits such as transformers will be treated at length in a later chapter, and the attention of the reader is merely at this point directed to the general nature of the effects taking place in a circuit of constant inductance when suddenly switched on to a source of simple periodic electromotive force.

**§ 34. Initial Conditions in Circuits having Capacity, Inductance and Resistance.**—It is somewhat more difficult to discuss completely the conditions which arise at the instant of connecting to a source of periodic electromotive force a circuit having not only inductance and resistance but also capacity. Generally speaking, they may be described as consisting of a variable period and a succeeding steady period. We shall in outline indicate how these conditions respectively arise. Suppose, in the first place, that a condenser of capacity C is connected through an inductive resistance of inductance L and resistance R to a source of periodic electromotive force of which the value at any instant is represented by

$$v' = \text{V}' \sin p\,t.$$

Let $i$ be the instantaneous value of the current flowing into the circuit, and let $v$ be the value at the same instant of the potential difference between the terminals of the condenser, and $v_1$ the fall of potential down the inductive resistance. Then we have the following fundamental equation connecting the current and potential :—

For the resistance $$L\frac{d\,i}{d\,t}+R\,i=v_1, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (67)$$

for the condenser $$C\frac{d\,v}{d\,t}=i, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (68)$$

and for the total fall of potential

$$v'=v+v_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (69)$$

Hence, from (67) and (68),

$$L\frac{d\,i}{d\,t}+R\,i+\frac{1}{c}\int i\,d\,t=v'=V'\sin p\,t,$$

and eliminating $i$ by the help of (68), we get

$$L\,C\frac{d^2 v}{d\,t^2}+R\,C\frac{d\,v}{d\,t}+v=V'\sin p\,t. \quad . \quad . \quad . \qquad (70)$$

This is the differential equation defining the value of the condenser terminal potential in terms of the constants and the time. To solve (70) we must eliminate $V'\sin p\,t$. This is done by differentiating (70) twice and eliminating $V'\sin p\,t$ between the original equation (70) and the twice differentiated equation. As a result we reach the equation

$$L\,C\frac{d^4 v}{d\,t^4}+C\,R\frac{d^3 v}{d\,t^3}+(1+C\,L\,p^2)\frac{d\,v^2}{d\,t^2}+C\,R\,p^2\frac{d\,v}{d\,t}+p^2 v=0. \quad (71)$$

The solution of this linear differential equation of the fourth order depends on the solution of the biquadratic

$$L\,C\,m^4+C\,R\,m^3+(1+C\,L\,p^2)\,m^2+C\,R\,p^2\,m+p^2=0,$$

and this last splits up into two factors

$$(m^2+p^2)\;(C\,L\,m^2+C\,R\,m+1)=0.$$

Hence the roots of this biquadratic are,

$$m=\pm\sqrt{-1}\,p$$

$$m=-\frac{R}{2\,L}\pm\sqrt{-1}\,\sqrt{\frac{4\,L-C\,R^2}{4\,C\,L^2}}.$$

It is shown in treatises on differential equations that the solution of equation (71) is then

$$v = \mathrm{A}\sin p\,t + \mathrm{B}\cos p\,t + \mathrm{A}'\,e^{-\frac{\mathrm{R}}{2\mathrm{L}}t}\sin q\,t + \mathrm{B}'\,e^{-\frac{\mathrm{R}}{2\mathrm{L}}t}\cos q\,t, \quad . \qquad (72)$$

where A, B, A′, B′ are constants and

$$q = \sqrt{\frac{4\,\mathrm{L} - \mathrm{C}\,\mathrm{R}^2}{4\,\mathrm{C}\,\mathrm{L}^2}}.$$

The solution for $v$ may obviously be written

$$v = \sqrt{\mathrm{A}^2 + \mathrm{B}^2}\sin(p\,t - \phi) + \sqrt{\mathrm{A}'^2 + \mathrm{B}'^2}\,e^{-\frac{\mathrm{R}}{2\mathrm{L}}t}\sin(q\,t - \phi'). \quad (73)$$

This equation for the value of the condenser potential $v$ shows us that the variation of $v$ is made up of two parts. First a simple periodic part $\sqrt{\mathrm{A}^2 + \mathrm{B}^2}\sin(p\,t - \phi)$, which may be written $\mathrm{V}\sin(p\,t - \phi)$, and which indicates a simple periodic variation of $v$ differing in phase from the impressed electromotive force $v'$ by an angle $\phi$. The other term of the solution indicates a superposed periodic variation, gradually decreasing in amplitude as time increases, and dying out as the exponent $\frac{\mathrm{R}}{2\mathrm{L}}t$ increases with time. If, for the sake of brevity, we write C′ for $\sqrt{\mathrm{A}'^2 + \mathrm{B}'^2}$, we can put the solution for $v$ in the form

$$v = \mathrm{V}\sin(p\,t - \phi) + \mathrm{C}'\,e^{-\frac{\mathrm{R}}{2\mathrm{L}}t}\sin(q\,t - \phi'). \quad . \quad (74)$$

Two constants have therefore to be determined, viz., C′ and $\phi'$, in order that we may completely solve the problem.

Since the quantity of electricity in the condenser at any instant is numerically equal to the product of the capacity and potential, if we multiply (74) all through by C, the condenser capacity, we have an expression for the charge $x$ in the condenser at any instant. Since this charge is null at the instant of closing the circuit, if the circuit is closed at the instant $t'$, reckoning time from the instant when the impressed electromotive force is zero, we have

$$0 = \mathrm{V}\sin(p\,t' - \phi) + \mathrm{C}'\,e^{-\frac{\mathrm{R}}{2\mathrm{L}}t'}\sin(q\,t' - \phi')$$

as an equation to determine the constant C′.

Therefore
$$\mathrm{C}' = -\frac{\mathrm{V}\sin(p\,t' - \phi)\,e^{\frac{\mathrm{R}}{2\mathrm{L}}t'}}{\sin(q\,t' - \phi')}. \quad . \quad . \quad . \quad (75)$$

No sufficient advantage is to be obtained by working out the rather complicated algebraical expressions in terms of C, L, R, and $p$, for the constant $\phi'$, but we can indicate generally what the equations teach. They show us that if such a condenser in series with an inductive circuit is switched on to a source of periodic impressed electromotive force, before the oscillations of the condenser potential settle down into a regular state, there is a variable period in which a second set of oscillations of gradually diminishing amplitude are superimposed on the steady set, and the second set of oscillations have a quite different frequency and initial amplitude to the steady set which ultimately survive. In the initial period the superposition of the two sets of oscillations may increase the instantaneous value of the condenser terminal potential difference to a value greater, and perhaps much greater, than it would have if there were no such additional oscillations. If the circuit is closed at an instant $t'$ such that $p\,t' = \phi$, viz., at an interval after the impressed electromotive force has passed its zero value equal to the final permanent difference of phase of condenser and impressed electromotive force, then, since this value makes the constant C′ zero, we see that there are no superposed vibrations at all. On the other hand, if $t'$ is so chosen that $p\,t' = 90^\circ + \phi$, then the disturbing effect is greatest. It is clear, therefore, that, in switching on a condenser to an alternating current circuit through an inductive resistance, initial effects of abnormal rise of condenser voltage may result. We shall see later on that these are practically very important matters in dealing with alternating current systems of supply, and that caution must always be used in switching on a condenser to such a circuit.

§ **35. Complex Periodic Functions.**—Before leaving the subject of periodic currents and electromotive forces it is desirable to explain some properties of the trigonometrical expressions or series by which such functions can be represented as the sum of a series of simple periodic constituents or terms. It has already been explained that the value of an ordinate $y$ of any single valued function can be expressed by Fourier's method as follows:

$$y = Y_0 + Y_1 \sin (p\,t + \phi_1) + Y_2 \sin (2\,p\,t + \phi_2) + , \text{\&c.}; \quad (76)$$

or, taking advantage of the trigonometrical equality

$$A \sin \theta + B \cos \theta = \sqrt{A^2 + B^2} \sin (\theta + \phi), \text{ where } \theta \tan \phi = \frac{B}{A},$$

we can write the above expression for $y$

$$y = Y_0 + y_1 \sin pt + z_1 \cos pt + y_2 \sin 2pt + z_2 \cos 2pt + \&c. \quad (77)$$

The coefficients $Y_1$, $Y_2$, &c., in the series in (76) are the amplitudes of the simple sine curves or harmonic constitutents whose added ordinates together build up the function $y$.

If the periodic quantity represented is a wave curve symmetrical with respect to the axis of time, and repeating the same form continually, then the constant term $Y_0$ is absent.

We can therefore express the instantaneous value $e$ of any periodic electromotive force by the expression

$$e = E_1 \sin pt + F_1 \cos pt + E_2 \sin 2pt + F_2 \cos 2pt + , \&c \quad (78)$$

and the instantaneous value $i$ of any periodic current by the series

$$i = I_1 \sin pt + J_1 \cos pt + I_2 \sin 2pt + J_2 \cos 2pt + , \&c. \quad (79)$$

In the series (78) the amptitude of the first harmonic is $\sqrt{E_1^2 + F_1^2}$ and that of the second $\sqrt{E_2^2 + F_2^2}$, and so on.

We have already seen that if we take the mean or average value of $\sin^2 \theta$ over one half-period, or from 0 to $\pi$, the value of the mean is $\frac{\pi}{2}$; but that the mean value of such a product as $\sin \theta \cos \theta$ or $\sin \theta \sin 2\theta$ over half a period is zero.

Accordingly, if we square the series in (78) or (79) we find that the values for $e^2$ and for $i^2$ partly involve terms like $\sin^2 pt$, $\sin^2 2pt$, &c., and partly terms like $\sin pt \sin 2pt$. Hence, if we integrate the value of $e^2 dt$ or $i^2 dt$ over half a period and divide the result by $\pi$, or take the definite integrals $\frac{1}{\pi}\int_0^\pi e^2 dt$, $\frac{1}{\pi}\int_0^\pi i^2 dt$, which is equivalent to finding the mean-square values of $e$ and $i$, we find by the above theorem that

$$\frac{1}{\pi}\int_0^\pi e^2 dt = \frac{E_1^2}{2} + \frac{F_1^2}{2} + \frac{E_2^2}{2} + \frac{F_2^2}{2} + , \&c. \quad . \quad (80)$$

$$\frac{1}{\pi}\int_0^\pi i^2 dt = \frac{I_1^2}{2} + \frac{J_1^2}{2} + \frac{I_2^2}{2} + \frac{J_2^2}{2} + , \&c. \quad . \quad . \quad (81)$$

and, similarly, by multiplying (78) and (79) and taking the mean value of the products from $p\,t=0$ to $p\,t=\pi$, we obtain

$$\frac{1}{\pi}\int_0^\pi e\,i\,dt=\frac{E_1 I_1}{2}+\frac{F_1 J_1}{2}+\frac{E_2 I_2}{2}+\frac{F_2 J_2}{2}+, \text{\&c.} \quad (82)$$

The ordinary alternate current ammeter or dynamometer measures the $\sqrt{\text{mean}^2}$ value of the current $i$, or the quantity $\sqrt{\frac{1}{\pi}\int_0^\pi i^2\,d\,t}$; and the ordinary alternating voltmeter, such as an electrostatic voltmeter, measures the $\sqrt{\text{mean}^2}$ value of the electromotive force $e$, or the quantity $\sqrt{\frac{1}{\pi}\int_0^\pi e^2\,d\,t}$, whatever be the force of the curves of $e$ and $i$. A wattmeter in proper form reads the true mean product of $e$, $i$ or $\frac{1}{\pi}\int_0^\pi e\,i\,d\,t$ when currents respectively proportional to $e$ and $i$ traverse its two coils. For shortness, let us write $e'$ for $\sqrt{\frac{1}{\pi}\int_0^\pi e^2\,d\,t}$, and $i'$ for the similar function of $i^2$, and $(e'\,i')$ for $\frac{1}{\pi}\int_0^\pi e\,i\,d\,t$; then

$$2\,e'=E_1^2+F_1^2+, \text{\&c.},$$
$$2\,i'=I_1^2+J_1^2+, \text{\&c.},$$
$$2\,(e'\,i')=E_1\,I_1+F_1\,J_1+, \text{\&c.}$$

We see therefore that *twice* the $\sqrt{\text{mean}^2}$ value of $e$ or $i$ is equal to the sum of the squares of the coefficients of the sine and cosine terms, or to the sum of the squares of the amplitude of the harmonic constituents. Also that the *twice* mean product of two periodic functions taken over a half period at similar instants is equal to the sum of the products of the coefficients of similar sine or cosine terms taken in pairs from each expansion. Suppose that $e$ and $i$ represent the instantaneous values of the impressed electromotive force and current of any circuit, we may ask whether the product of the $\sqrt{\text{mean}^2}$ value of $e$ and the $\sqrt{\text{mean}^2}$ value of $i$ is equal to, greater, or less than the mean of the product of $e$ and $i$,

or is $$\sqrt{\frac{1}{\pi}\int_0^\pi e^2\,d\,t}\times\sqrt{\frac{1}{\pi}\int_0^\pi i^2\,d\,t}\ \overset{=}{\underset{<}{\overset{>}{\text{or}}}}\ \frac{1}{\pi}\int_0^\pi e\,i\,d\,t.$$

We see that the square of the expression for 2 $(e' i')$ is made up of terms like $E_1^2 I_1^2$, and $F_1^2 J_1^2$, and also of terms like $2\,E_1 I_1 F_1 J_1$.

The product of the expressions for $e'$ and $i'$ consists obviously of terms like $E_1^2 I_1^2$, $F_1^2 J_1^2$, and also of terms like $F_1^2 I_1^2$ and $E_1^2 J_1^2$.

Hence, the question whether the product of $e'$ and $i'$ or $e' \times i'$ is or is not greater than $(e' i')$ will depend upon the relative collected magnitude of terms like $F_1^2 I_1^2 + E_1^2 J_1^2$ in the one series, and of terms like $2\,E_1 I_1 F_1 J_1$. If $F_1 : E_1 :: J_1 : I_1$, or if $\frac{F_1}{E_1} = \frac{J_1}{I_1}$, then $E_1^2 J_1^2 + F_1^2 I^2 = 2\,E_1 F_1 I_1 J_1$, and if the same proportion holds good for the coefficients of the terms in $\sin 2pt$, &c., then we see that $e' \times i' = (e' i')$, or the product of the $\sqrt{\text{mean}^2}$ values $e$ and $i$ is equal to the mean product of $e$ and $i$. If the above proportionality does not hold good, then, since generally $a^2 + b^2$ is greater than $2\,a\,b$, it is not difficult to see that $e' \times i'$ is greater than $(e' i')$ or the product of

$$\sqrt{\frac{1}{\pi}\int_0^\pi e^2\,dt} \times \sqrt{\frac{1}{\pi}\int_0^\pi i^2\,dt}$$

is greater than the mean value $\frac{1}{\pi}\int_0^\pi e\,i\,dt$.

Hence it follows that, in this last case, the product of the amperes and volts as read on alternating current instruments is greater than the true value of the power as read on a wattmeter. The mean value $\frac{1}{\pi}\int_0^\pi e\,i\,dt$ is called generally the *true watts or power* given to the circuit, and the product $\sqrt{\frac{1}{\pi}\int_0^\pi e^2\,dt} \times \sqrt{\frac{1}{\pi}\int_0^\pi i^2\,dt}$ is called the *apparent watts* or power given to the circuit. The apparent watts are equal under some conditions to the true watts. This is the case when the circuit is non-inductive, and when the different harmonic constituents of the current and electromotive force are in step or in synchronism with each other. Under these conditions the angle of lag of the electromotive force harmonics is equal to the lag of the current harmonics of the same degree, and this is expressed by relations of the form $F_1 : E_1 :: J_1 : I_1$ holding good. Under other conditions, the apparent watts are greater than the true watts. The ratio of the true power or watts taken up in any circuit to the apparent power or watts is called the *power factor* of the circuit, and the power factor (P.F.) is therefore given for any alternating current circuit by the ratio of the wattmeter reading to the product of the ammeter reading and voltmeter reading for that circuit.

# CHAPTER IV.

## MUTUAL AND SELF INDUCTION.

§1. **Researches of Prof. Joseph Henry.**—We have already, in the first chapter, made a brief allusion to the share taken by Joseph Henry in the fundamental discovery of the induction of electric currents. A full account of his labours in this field is to be found in the collected "Scientific Writings of Joseph Henry," republished by the Smithsonian Institution. It will be of advantage to consider at this stage some of his chief investigations.

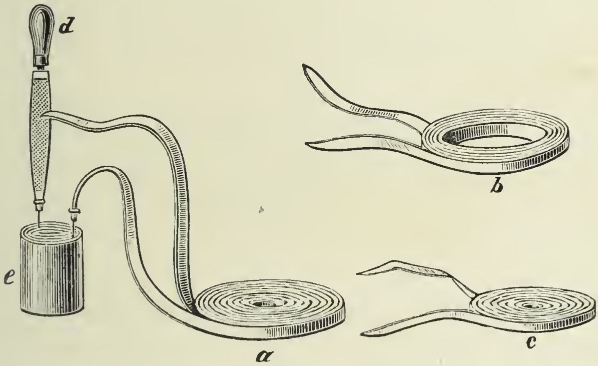

FIG. 78.

The principal pieces of apparatus used by Henry in his experiments on the induction of electric currents consisted of

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[*] See also the *Philosophical Magazine*, Vol. XVI., 3rd Ser., 1840, and *Transactions* of the American Philosophical Society, Vol. VI., 1838, pp. 303-337.

a number of flat coils of copper strip or band, which were designated by the names *Coil No. 1*, *Coil No. 2*, &c., also several long bobbins of wire, and these, to distinguish them from the ribands, were called *Helix No. 1*, *Helix No. 2*, &c.

His description of these coils and helices is as follows: Coil No. 1 was formed of thirteen pounds of copper strip one inch and a-half wide and ninety-three feet long; it was well covered with two coatings of silk, and was generally used in the form represented in Fig. 78, which is that of a flat spiral sixteen inches in diameter. It was, however, sometimes formed into a ring of larger diameter, as is shown in Fig. 79.

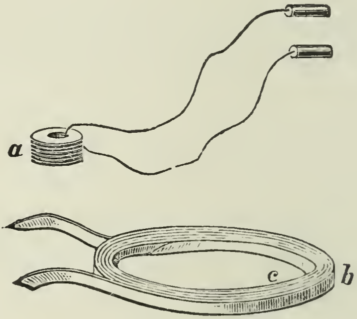

Fig. 79.

Coil No. 2 was also formed of copper strip of the same width and thickness as coil No. 1. It was, however, only sixty feet long. Its form is shown at *b* in Fig. 78. The opening at the centre was sufficient to admit helix No. 1. Coils No. 3, 4, 5, 6, were all about sixty feet long, and of copper strip of the same thickness, but of half the width of coil No. 1.

Helix No. 1 consisted of sixteen hundred and sixty yards of copper wire $\frac{1}{49}$th of an inch in diameter; No. 2 of nine hundred and ninety yards, and No. 3 of three hundred and fifty yards of the same wire. These helices were wound on bobbins of such size as to fit into each other, thus forming one long helix of three thousand yards, or, by using them separately and in different combinations, seven helices of different lengths. The wire was covered with cotton thread saturated with bees' wax, and between each stratum of spires a coating of silk was interposed.

Helix No. 4, shown at *a*, Fig. 79, was formed of five hundred and forty-six yards of wire $\frac{1}{49}$th of an inch in diameter, the several spires of which were insulated by a coating of cement.

Helix No. 5 consisted of fifteen hundred yards of silvered copper wire, $\frac{1}{125}$th of an inch in diameter, covered with cotton, and of the form of helix No. 4.

In addition, a long spool of copper wire covered with cotton, $\frac{1}{16}$th of an inch in diameter and five miles long, was provided. It was wound on a small axis of iron, and formed a solid cylinder of wire eighteen inches long and thirteen in diameter.

For determining the direction of the induced currents Henry employed a magnetising spiral, which consisted of about thirty spires of copper wire in the form of a cylinder, and so small as just to admit a sewing needle into the axis.

Also a small iron horseshoe is frequently referred to, which was formed of a piece of soft iron about three inches long and $\frac{2}{5}$ths of an inch thick; each leg was surrounded with about five feet of copper bell wire. This length was so small that only a current of considerable strength could develop sensible magnetism in the iron. This horseshoe was used for indicating the existence of such a current. The battery which was used was a simple copper-zinc cylinder battery, having about $1\frac{3}{4}$ square feet of zinc surface. In some experiments a series of cells was used, but most experiments were performed with one or two cells of the above kind. For interrupting the circuit of the conductor Henry employed the simple device of scraping one end of the conductor along a rasp held in contact with the battery terminal.

Provided with this apparatus, Henry entered on a preliminary series of experiments on the induction of electric currents, and in 1838 published an account of his investigations on the phenomenon which had been previously named by Faraday electro-dynamic induction. The fact which seems to have chiefly attracted the attention of the numerous investigators who rapidly entered the region of research opened out by Faraday's discovery of the mutual induction of electric circuits and the production of electric currents in conducting circuits by the variation of the magnetic induction linked with them, and by Henry's discovery of the self-induction of electric circuits, seems to have been the possibility of obtaining from a single cell of a galvanic battery effects such as spark and shock. These effects connected what was then known as voltaic electricity with the then more familiar effects of electrification by friction.* Henry took up the train of investigation at this point, and proceeded to employ the above described helices and coils in an investigation of the facts of the self- and mutual-induction of electric circuits. His mode of operating was to close the battery circuit by dipping the ends of a coil or helix into two mercury cups connected with the terminal plates, and then to break the circuit by lifting out one end from its mercury cup, the hands being at the same time in contact with the battery terminal and the end of the conductor which is being raised. In this way the *extra current*, or electro-magnetic discharge of the coil, passed through the operator's body.

When the electromotive force was small, as in the case of a thermopile or a large single cell, and the circuit taken was the flat riband coil No. 1, ninety-three feet long, it was found to give brilliant snaps at the surface of the mercury when contact was broken, but the shocks were very feeble, and could only be felt in the fingers or through the tongue. The induced current in a short coil, which thus produced deflagration but not shocks, he called, for distinction, one of quantity.

When the length of the coil was increased, the battery being the same, the deflagrating power decreased, while the intensity of the shock continually increased. With five-riband coils in series, making an aggregate length of three hundred feet, and a small battery the deflagration was less than with coil No. 1, but the shocks were more intense.

There appeared to be, however, a limit to this increase of intensity of the shock, and this took place when the increased resistance or diminished conduction of the lengthened coil began to counteract the influence of the increasing length of the current. The following experiment illustrated this fact.

A coil of copper wire $\frac{1}{16}$th of an inch in diameter was increased in length by successive additions of about thirty-two feet at a time. After the first two lengths, or sixty-four feet, the brilliancy of the spark began to decline, but the shocks continually increased in intensity until a length of five hundred and seventy feet was obtained, when the shocks also began to decline. This was, then, the proper length to produce the maximum effect with a single battery and a wire of the above diameter. With a battery of sixty cells (Cruickshank's trough), having plates four inches square, scarcely any shock could be obtained when the coil formed a part of the circuit. If the length of the coil was increased, then the inductive effect became very apparent.

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* For a more extended description of the historical order of discoveries in connection with the induction coil the reader is referred to the First Chapter in the Second Volume of this treatise.

When the current from ten cells of the above-mentioned trough was passed through the large spool of copper wire, the induced shock was too severe to be taken through the body. Again, when a small battery of twenty-five cells having plates one inch square, which alone would give but a very feeble shock, was used with helix No. 1, an intense shock was received from the induction when the contact was broken. Also a slight shock in this arrangement was given when the contact was formed, but it was very feeble in comparison with the other. The spark, however, with the long wire and compound battery was not as brilliant as with the single battery and short riband coil.

When the shock was produced from a long wire, as in the last experiments, the size of the plates of the battery might be very much reduced without a corresponding reduction in the intensity of the shock. A small battery was made, formed of six pieces of copper bell wire one inch and a-half long and an equal number of pieces of zinc of the same size. When the current from this was passed through a coil consisting of five miles of wire, the shock was given at once to twenty-six persons joining hands.

With the same coil, and the single battery used in the former experiments, no shock, or at most a very feeble one, could be obtained.

The induced current in these last experiments he called one of *considerable intensity* and *small quantity*.

§ 2. **Mutual Induction.**—Henry then passed on to consider the mutual induction of two circuits. Coil No. 1 (see *c*, Fig. 80) was arranged to receive the current from a small battery of a single cell, and coil No. 2, *b*, was placed over it with a plate of glass between to secure perfect insulation. As often as the current in No. 1 circuit was interrupted, a powerful secondary current was induced in No. 2. When the ends of the secondary were joined to a magnetising spiral, the enclosed needle became strongly magnetic. Also when the ends of the second coil were attached to a small water decomposing apparatus, a stream of gas was given off at each pole; and when the secondary current was passed through the wires of the iron horse-shoe, magnetism was developed. The shock, however, from the secondary coil was very feeble, and scarcely felt above the fingers. This secondary current had, therefore, the properties of one of moderate intensity but considerable quantity (to use the terms then employed) when developed by the current in one flat riband coil acting on another flat riband coil.

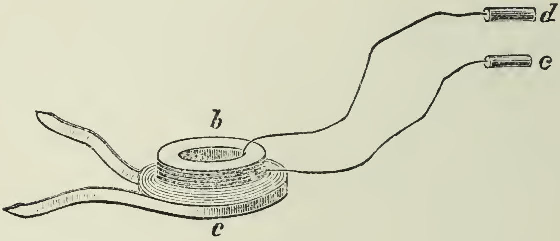

Fig. 80.

Coil No. 1, remaining as before a longer coil, formed by uniting Nos. 3, 4, and 5, was substituted for No. 2. With this arrangement as a secondary circuit the magnetising power of the current and the brilliancy of the spark at breaking contact was less than before, but the shocks were more powerful—in other words, the intensity of the secondary induced current was increased, whilst its quantity was decreased.

A compound helix, formed by uniting Nos. 1 and 2 helices, and therefore containing two thousand six hundred and fifty yards of wire, was next placed on coil No. 1. The weight of this helix happened to be precisely the same as that of coil

No. 2, and hence the different effects of the same quantity of metal (as secondary circuit) in the two forms of a long and short conductor could be compared. With this arrangement the magnetising effects with the apparatus above-mentioned disappeared. The sparks were much smaller and the decomposition less than with the short coil, but the shock was almost too intense to be received with impunity except through the fingers of one hand. The secondary current in this case was one of small quantity but of great intensity.

The following experiment is important in establishing the fact of a limit to the increase of the intensity of the shock as well as to the power of decomposition with a wire of given diameter.

Helix No. 5, consisting of a wire $\frac{1}{125}$th of an inch in diameter, was placed on coil No. 2, and its length increased to about seven hundred yards. With this extent of wire neither decomposition nor magnetism could be obtained, but shocks were given of a peculiarly pungent nature. The wire of the helix was further increased to about fifteen hundred yards; the shock was now found to be scarcely perceptible in the fingers.

As a counterpart to the last experiment, coil No. 1 was formed into a ring of sufficient internal diameter to admit the great spool of wire, and, with the whole length of this (five miles), the shock was found so intense as to be felt at the shoulder when passed only through the forefinger and thumb. Sparks and decomposition were also produced, and needles rendered magnetic. The wire of this spool was $\frac{1}{16}$th of an inch in diameter; and Henry noted therefore from this experiment that, by increasing the diameter of the wire, its length might also be increased with increased effect of shock.

The previous experiments were repeated, using a battery of sixty cells (Cruickshank's trough). When the current from this was passed through the riband coil No. 1, no indication, or a very feeble one, was given of a secondary current in any of the coils or helices arranged as in the preceding experiments; but when the long helix No. 1 was placed as a primary, instead of coil No. 1, a powerful inductive action was produced on each of the circuits used as before.

First, helices Nos. 2 and 3 were united into one coil and placed within helix No. 1, which still conducted the battery current.

With this disposition a secondary current was produced, which gave intense shocks but feeble decomposition and no magnetism in the soft iron horse-shoe. It was therefore one of intensity, and was produced by a battery current also of intensity. Instead of the helix used in the last experiment for receiving the induction (secondary), one of the coils, No. 3 (copper riband), was now placed on helix No. 1, the battery remaining as before. With this arrangement the induced current gave no shocks, but it magnetised the small horse-shoe, and when the ends of the coil were rubbed together produced bright sparks. It had, therefore, the properties of a current of quantity, and it was produced by the induction of a current from a battery of intensity.*

This experiment was considered of so much importance that it was varied and repeated many times, but always with the same result; and it therefore established the fact *that an intensity current could induce one of quantity*; and by the preceding experiments the converse has also been shown, *that a quantity current could induce one of intensity.*

§ 3. **Induction at a Distance.**—In the experiments on mutual induction detailed above, the primary circuit was separated from the secondary only by a pane of glass, but the action was so energetic that an obvious experiment was to investigate the effect of distance on the mutual induction. For this purpose coil No. 1 was formed into a ring of about two feet in diameter (*see* Fig. 79), and helix No. 4 placed as shown. When the helix was at the distance of about sixteen inches from the middle of the plane of the ring, shocks could be perceived through the tongue, and these rapidly increased in intensity as the helix was lowered, and when it reached the plane of the ring they were quite severe. The effect, however, was still greater when the helix was moved from the centre to the inner circumference, as at *c*, but when it was placed without the ring, in contact with the outer circumference at *b*, the shocks were very slight, and when placed within, but with its axis at right angles to that of the ring, not the least effect could be observed. Coil No. 1 remaining as before (the primary) helix No. 1, which was nine inches in diameter, was substituted for the small helix in the last experiment, and with this the effect at a distance was much increased. When coil No. 2 was added to coil No. 1, and the currents from two small batteries sent through, these shocks were distinctly perceptible through the tongue when the distance of the planes of the coil and the three helices united as one was increased to thirty-six inches. The action at a distance was still further increased by coiling the long wire of the large spool into the form of a ring of four feet in diameter, and placing parallel to this another ring formed of the four ribands of coils Nos. 1, 2, 3, 4. When a current from a single cell having thirty-five feet of zinc surface, was passed through the riband conductor, shocks through the tongue were felt when the rings were separated to a distance of four feet. In another experiment, to illustrate induction across a distance, Henry (*Phil. Mag.*, Vol. XVIII., 1841, 3rd Series, p. 592) joined all his coils, so as to form a single conductor of about 400 feet in length, and this was rolled into a ring of five and a-half feet in diameter and suspended vertically against a door. On the other side of the door, and opposite to the coil, was placed a helix formed of upwards of a mile of copper wire one sixteenth of an inch in thickness and wound in a hoop of four feet in diameter. With this arrangement, and with a battery of one hundred and forty-seven square feet of zinc surface divided into eight elements, shocks were perceptible on the tongue when the two conductors were separated by a distance of seven feet, and at a distance of between three and four feet the shocks were quite severe.

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* This last experiment is very interesting, as showing that in 1838 Prof. Henry had already realised that which used to be called the reverse use of the induction coil. He had employed a current flowing in a fine wire of many turns and moving under a high electromotive force, to induce a current of greater strength in a secondary circuit, consisting of a lesser number of turns of copper riband, and moving under a less electromotive force. In other words, he had constructed what we should now call *a step-down transformer.* Note Henry's explicit statement in the following paragraph.

In the fifty years which have elapsed since Henry performed the classical experiments described above, the progress of knowledge has placed in our hands an appliance vastly more delicate than physiological shock for detecting induction at a distance, viz., the articulating telephone receiver. Aided by this, it has recently been found possible to find indications of the mutual induction between conductors separated by miles instead of feet.

Along the Gray's Inn-road, London, the English Post-office service placed a line of iron pipes buried underground carrying many telegraph wires. The United Telephone Company placed a line of open wires along the same route over the housetops, situated 80ft. from the underground wires. Considerable disturbances were experienced on the telephone circuit, and even Morse signals were read, which were said to be caused by the continuous and parallel telegraph circuits. A very careful series of experiments,* extending over some period, proved unmistakably that this was so, and that the well-known pattering disturbances due to induction are experienced at a much greater distance than was anticipated.

Experiments conducted on the Newcastle Town Moor extended the area of the disturbance to a distance of 3,000ft., while effects were detected on parallel lines of telegraph between Durham and Darlington at a distance of 10¼ miles. But the greatest distance experimented upon was between the east and west coast of the Border, when two lines of wire 40 miles apart were affected the one by the other, sounds produced at Newcastle on the Jedburgh line being distinctly heard at Gretna on a parallel line, though no wires connected the two places.

Distinct conversation has been held by telephone through the air without any wire through a distance of one quarter of a mile, and this distance can probably be much exceeded.

Effects are not confined to the air, for submarine cables half a mile apart in the sea disturb each other. It may well be doubted whether the inductive effects above described as taking place over very large distances above mentioned are not complicated by current leakage, but it has been abundantly established that inductive effects can be produced and detected between circuits separated by great distances.

Practical application of current induction across large air spaces has been made in the methods of carrying on telegraphic communication with railway trains when in motion. There are two methods by which this has been accomplished. (1) The magneto-induction method, which was devised by

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* Mr. W. H. Preece on "Induction between Wires and Wires" (*The Electrician*, Vol. XVII., 1886, p. 410 ; *British Association Report*, Birmingham, 1886).

Mr. L. J. Phelps and was tried about the year 1885 on a line about 15 miles long between Haarlem River and New Rochelle Junction, in the United States. In the other system (2) the principle involved is that of electrostatic induction, and, after having been suggested in a more or less imperfect form by Mr. W. Wiley Smith in 1881 (U.S. Patent No. 247,127), has been worked out in great detail by Messrs. Edison and Gilliland.

In the magneto-induction system a telegraphic car attached to the train carries a great circuit of wire wound on a frame extending the whole length of the car, and so placed that one side of the windings is as near the track as possible and one side as high above as the height of the car will permit. Between the rails is laid down a fixed insulator conductor, and the fluctuations of a current in this last induce currents in the lower side of the large coil carried on the car. The secondary current so induced is detected by a telephone and by suitable interruptions. A Morse code of audible signals can be transmitted from the fixed conductor to the moving train. The signals are thus made to jump over the air space, and continuous communications can be kept up between a station or stations in connection with the fixed conductor and a person in the moving telegraph car.

Mr. Phelps used a conductor of No. 12 (A.G.) insulated wire, which was placed in a kind of small wooden trough mounted on blocks attached to the sleepers. The car containing the telegraphing instruments carried beneath its floors, about 7in. above the rail level, a 2in. iron pipe, in which was a rubber tube holding about 90 convolutions of No. 14 (A.G.) copper wire, so connected as to form a continuous circuit about a mile and a-half long, and presenting something like three-quarters of a mile of wire parallel to the primary line wire mounted between the metals. The instrument, consisted of a delicate polarised relay as a receiving instrument, which acted as a sounder, and a "buzzer," or rapid current-breaker, for transmitting signals by means of the Morse key, which were received at the station in a telephone. This arrangement was so far a practical success that Mr. Phelps was encouraged to proceed; but meantime it was discovered that the patent above referred to had already been issued, while Edison and Gilliland had also been working on similar lines. In Wiley Smith's specification no mention is made of a "buzzer," which turns out to be an important feature in the invention; but the practical success of the experiments made is due to a combination of the devices of Phelps, Edison, and Gilliland. The latest system is an improvement on that of Phelps, briefly described above, in that it dispenses with the insulated line wire laid between the metals, and uses ordinary telegraph wire strung on what are known as short poles alongside the permanent way. The line wire is, in fact, stretched on poles about 16ft. high and at an average distance of 8ft. from the rails. On the Lehigh Valley Railroad, U.S., from Perth Junction to Easton, interesting experiments of this kind were made in 1887. As a rule the roof of the car, usually sheathed with metal, is available for securing the necessary electrical condition, but, where a metal covering is absent, all that is necessary is to attach a wire or rod to the roof and another to some portion of the metallic or rolling part of the coach in order to obtain "earth." The instruments are inserted in this circuit, and comprise a 12-cell chromic acid battery (the cells being 2in. wide by 4in. deep), which is closed on an induction coil having a primary of about 3 ohms and a secondary of about 500 ohms, and provided with an ordinary vibrating make-and-break. The messages are sent by means of a Morse key placed in the secondary circuit, this key being of the double-pointed kind with extra contact. The receiving telephone has a resistance of about 1,000 ohms; but Mr. Phelps states that, even when wound so as to have a resistance of 10,000 ohms, the sound is quite clear, so high is the electromotive force of the induction on the roof. The car-roofs are frequently of metal—usually painted tin plates, sheet zinc, or galvanised iron, and these answer admirably as inductive receivers; but where the roofs are of wood, covered with painted canvas, an iron or brass rod or tube, ½in. in diameter, is carried along under the eaves throughout the length of the train. The metallic roof or the rod is connected by a wire to the secondary of an induction coil, while the primary of the coil is connected to the front contact of the double-pointed key, and through that with the battery. Opposite the core of the coil is the "buzzer," which transmits a series of impulses to the line whenever the key is worked. The extra contact, which is placed on the upper surface of the front contact of the key, closes the secondary circuit, and allows the charges to be sent into the roof, while, when the key is on the back contact, the secondary and primary coils are cut out, and the charge from the roof then passes direct to the key and through the telephone to earth, which, as a rule, is made by connecting wires from the coil and the telephone to one of the axle-boxes. The coil and the key, with suitable connections, are mounted on a board which is large enough to contain a telegraph form besides, and the telephone is attached by flexible connections, and is, when in use, strapped to the operator's head. The battery is put up in a case with a handle, so that the whole apparatus can be carried from one end of a train to the other. The arrangements at the terminal and other stations on the line, so far as induction telegraphy is concerned, are practically identical with those in the railway coach; but, in addition, they have a duplex Morse equipment, by which ordinary messages can be sent by the dot-and-dash system.

Of late years interesting experiments have been made under the direction of Mr. W. H. Preece in carrying on telegraphic communication across considerable distances by means of induction between parallel circuits.

§ **4. Induced Currents of Higher Orders.**—In 1888 Henry made a further remarkable discovery, viz., that secondary currents, though only of momentary duration, could in their turn induce other induced currents in neighbouring conductors; and these he called tertiary and currents of higher orders.

A primary current was passed through coil No. 1, while coil No. 2 was placed over it to receive the secondary current, and the ends of this last coil joined to a third coil, No. 3. By this disposition the secondary current passed through No. 3, and since this was at a distance (*see* Fig. 81), and beyond the influence of the primary, its separate induction could be rendered manifest by the effects on helix No. 1, arranged as a secondary circuit to this third coil. When the handles *a b* of the last helix *d* were grasped a powerful shock was received, proving the induction of a *tertiary current* in the last coil. By a similar more extended arrangement of inducing coils (shown in Fig. 82) shocks were received from currents of a *fourth* and *fifth* order; and with a more powerful primary current and additional coils a still greater number of successive inductions might be obtained. Henry thus established by decisive experiments that, in a properly placed series of connected coils, a primary current could give birth to secondary currents, and these last to tertiary currents, and so on, a whole family of induced currents arising from the starting or stopping of the primary current.

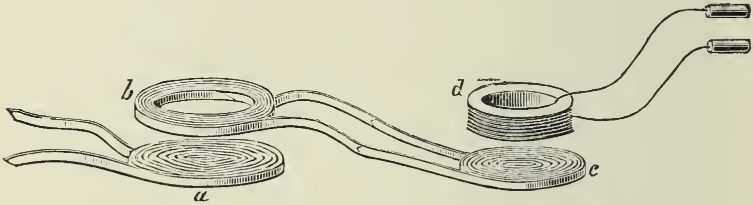

Fig. 81.

It was found that with a small battery a shock could be given from the current of the third order to 25 persons joining hands; also shocks perceptible in the arms were obtained from a current of the fifth order.

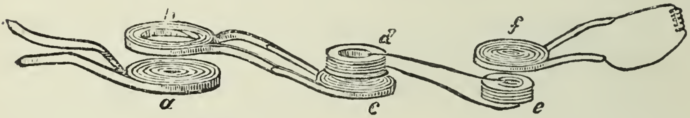

Fig. 82.

When the long helix is placed over a secondary current generated in a short coil, and which is one of *quantity*, a tertiary current of *intensity* was obtained capable of producing shocks. When the intensity current of the last experiment was passed through a second helix, and another flat riband coil placed over this (*see* Fig. 82a), a quantity current was again produced. Therefore, in the case of these currents of higher orders also *a quantity current could be induced from one of intensity, and the converse.*

The arrangement in Fig. 82 shows these different results produced at once. The induction from coil No. 3 to helix No. 1 produces an intensity current, and from helix No. 2 to coil No. 4 a quantity current.

The next stage in Henry's inquiry had reference to the direction of these induced currents. Knowing that a current on starting in a conductor induces an *inverse* or oppositely-directed induced current in a neighbouring secondary circuit, and a *direct* or like directed induced current on stopping, it was clear that each tertiary current must consist, in its simplest form, of two oppositely directed currents succeeding each other instantaneously; for at the "make" or "break" of the primary the secondary circuit is traversed by a brief secondary current in "opposite" or "like" direction. We shall speak of these as the *inverse* and *direct* secondary currents produced on closing or opening the primary circuit.

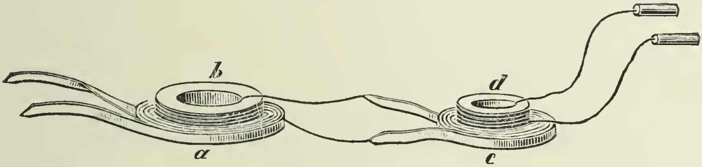

FIG. 82A.

Each of these secondary currents rises to a maximum and then sinks to zero again. If there is a tertiary circuit present, then during the rise of the secondary current to its maximum it is developing an *inverse* tertiary and during its decrease to zero a *direct* tertiary current. Since, as we shall see, the duration of the secondary current is a very small fraction of a second, these two component tertiary currents must succeed each other at an excessively short interval of time. Physiologically their separate effect is, so to speak, united, and they make themselves felt as one shock. Henry adopted the method of employing a magnetising spiral containing a sewing needle as a means of analysing the nature of these induced currents of higher order. By inserting such a spiral in the circuit of the successive conductors and noting the direction of the magnetisation of the steel needle he arrived at the conclusion that there exists an alternation in the direction of the currents of the several orders, and that the directions of the several induced currents could be expressed by saying that at the "make" of the primary we get an inverse secondary, a direct tertiary, and inverse quarternary current, and so on; or, symbolically:—

| *Primary current* | *started* | *stopped* |
|---|---|---|
| *Secondary* ,, | *inverse* − | *direct* − |
| *Tertiary* ,, | *direct* + | *inverse* + |
| *Quarternary* ,, | *inverse* − | *direct* − |
| | *&c.* | *&c.* |

The use of a magnetising spiral as a means of determining the direction of an induced current is, however, liable to lead to serious errors in drawing conclusions as to direction of currents, and the above experiment cannot be regarded as an exhaustive examination of the whole phenomena of induced currents of higher orders. Before entering into a more detailed discussion of the exact nature of the effects which here present themselves, it will be of assistance to gather together the principal observations on the induction of transient electric currents.

§ 5. **Inductive Effects Produced by Transient Electric Currents.**—It was an obvious inference, from all the foregoing facts, that Leyden jar discharges, or the transient currents formed by discharging charged condensers, should in like manner be able to give rise to a family of induced currents in suitably-placed circuits. Henry thus opened up a new field of research, which was diligently cultivated by Marianini, Abria, Matteucci, Reiss, Verdet, and many other physicists. Henry's first experiment was as follows: A hollow glass cylinder (*see* Fig. 83) of about six inches in diameter was prepared with a narrow riband of tinfoil about thirty feet long pasted spirally around the outside, and a riband of the same length pasted on the inside, so that the corresponding spires of the two were directly opposite each other. The ends of the inner spiral passed out of the cylinder through a glass tube to prevent direct communication between the two circuits. When the ends of the inner riband were joined by the magnetising spiral containing a sewing needle and a discharge from a half-gallon jar sent through the outer riband, the needle was strongly magnetised in such a manner as to indicate an induced current through the inner riband in the *same direction* as that of the current of the jar. If, instead of using the magnetising spiral, the ends of the inner riband were brought near together, a small spark was detected at the instant of sending a jar discharge through the outer conductor. Experiments were next made in reference to the production of induced currents of different orders by electric discharges. For this purpose a series of glass cylinders with tinfoil spirals pasted on them was prepared and joined up so that the inner spiral of one cylinder was in connection with the outer spiral of another. When a discharge was passed through the outer riband of the first cylinder it produced an induced secondary discharge circulating in the inner spiral of the first and the outer spiral of the second cylinder. This in turn generated a tertiary current, and so forth. Each of these discharges was a brief wave of current, and by the use of the magnetising spiral in each circuit an attempt was made to determine the direction of the discharge. Here, however, an anomaly presented itself. By the use of this magnetising spiral it appeared that the induced discharges were all in the *same direction*. Leyden jar discharges were then passed through the first member of the series of coils and helices used in the experiments on galvanic currents, and here the directions of the induced discharges in the several conductors were found to *alternate*. After various experiments Henry considered that he had found the solution of this anomaly in the different distances of the inducing and inductive circuits. As an experiment illustrating this he gives the following:—Two narrow strips of tinfoil about twelve feet long

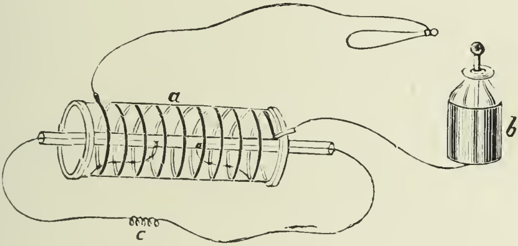

FIG. 83.

were stretched parallel to each other, and separated by thin plates of mica to the distance of about $\frac{1}{50}$th of an inch. When a discharge from a half-gallon jar was passed through one of these an induced current in the *same* direction was obtained from the other. When the ribands were separated to a distance of about $\frac{1}{8}$th of an inch, no induced current, as evidenced by the absence of effect in the magnetising spiral, could be obtained. When the circuits were still further separated the induced current reappeared, but in the *opposite* direction to the primary discharge. The distance at which the induced discharge changes direction appears, according to Henry, to be dependent on a number of circumstances, such as the capacity and charge of the jar and the length and thickness of the wires.

With a battery of eight half-gallon jars and parallel wires of about ten feet long the change in direction did not take place until the wires were separated by twelve or fifteen inches. The currents of all the higher orders were found to change sign with a change in the distance between the inducing and inductive circuit.

One interesting experiment was made by Henry to illustrate the inductive effect of jar discharges across considerable distances. In this case a primary circuit was formed consisting of an insulated wire eighty feet long. Around this, and separated from it by a distance of about twelve feet, was another circuit consisting of a wire one hundred and twenty feet long. When the discharge from thirty large Leyden jars was sent through the primary wire an induced discharge was obtained in the other sufficiently strong to magnetise to saturation a small needle placed in a magnetising spiral interpolated in the secondary circuit. We may, however, remark here, once for all, that all these experiments directed to determine the direction of induced discharges in which the magnetising power of the discharge is made use of for this purpose are difficult to interpret, and too much reliance must not be placed on the conclusions thus drawn. Leaving out of account for the moment all consideration of what are called electric oscillations, to which we shall allude subsequently, we may say that, if two discharges are passed through a magnetising spiral, the discharges being oppositely directed and of equal *quantity* but different *durations*, the resulting direction of magnetisation will be dependent upon several conflicting elements. Speaking generally, the intensity of magnetisation is determined by the relative magnitude of the maximum current strength during the discharge, and of two discharges having equal quantity the one lasting the shortest time would rise to the highest current strength during the period of the discharge, and exercise the greatest magnetising force. Even then it would not be safe to draw too pronounced a conclusion from the direction of magnetisation as to the relative magnitudes of the maxima of two alternate discharge currents rapidly succeeding each other, for, as Abria pointed out long ago,* the demagnetisation of a steel needle requires a less magnetising force than that necessary to magnetise it in the first instance, and hence the final results are complicated by the relative order of imposition as well as the relative maximum magnitude of the magnetising discharge currents. One fact which has to be borne in mind in attempting to interpret these results of Henry is that the magnetising current whose direction we are seeking to determine acts by induction also on the mass of the needle or iron in the testing magnetising coil, and generates in its mass induced currents circulating round its surface. Under the head of *Magnetic Screening* in a later section we shall examine the circumstances under which such currents induced in a metallic mass shield to a greater or less extent conducting circuits lying beyond them from inductive effects. Meanwhile we may say that the effect of a very *sudden* discharge in one direction in the magnetising coil is to induce eddy currents in the surface of the needle which shield the inner and deeper portions of the steel from the magnetising action, and the resulting magnetisation is chiefly superficial. If, however, the discharge is prolonged or dragged out whilst retaining the same electric quantity, the shielding action will not be so pronounced, and the magnetisation will penetrate deeper down into the mass of the steel. Accordingly two equal discharges, *i.e.*, discharges of equal quantity, may produce a greater or less magnetic moment in the steel, according as the duration of the same is greater or less, a *very* sudden discharge having much less magnetic-moment-producing power than the same quantity more dragged out. We may in general also say that the magnetising power of a discharge current is determined by the value of the maximum current strength during the discharge, and hence of two equal quantity discharges, the one which lasts the shorter time, and which has, therefore, the greatest maximum value, will, if the discharges are approximately equal in duration, produce the greatest magnetising effect.

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* Abria, *Ann. de Chem. et de Phys.*, [3] Vol. I., p. 429, 1844.

The tertiary currents, produced by ordinary galvanic currents, and the secondary currents, produced by Leyden jar discharges, consist, as we have seen, in their simplest form of a double discharge or flow, one part *inverse*, or oppositely directed to its inducing parent current, and the succeeding part *direct*, or similarly directed, the two component currents of the total discharge having equal quantity but different durations. In general the first portion, or the inverse current, is that which has the greatest maximum value and the shortest duration, the second half, or the direct current, being more dragged out in time; and, for a reason to be stated further on, the approximation of the induced and inducing circuits exaggerates this difference, or increases the maximum value of the inverse current at the expense of its duration. The explanation which may be offered, however, of the phenomena of the magnetisation of steel by tertiary currents, or by the secondary currents due to Leyden jar discharges, is as follows:—When the induced and inducing circuits are not very near to each other, and when the inducing current reaches its maximum not very suddenly, the two induced currents are not very different in duration, but the first or inverse current has, of the two, a rather greater maximum and less duration. It follows that a magnetisation is produced in the needle, which, on the whole, is in the direction produced by the inverse current, and the inference from the direction of magnetisation is that the induced and inducing currents are in the opposite direction. If, however, the inducing current reaches its maximum value very suddenly, as it does if the circuits are very close, then the first half, or the inversed induced current, is so brief in its duration that the magnetisation of the needle due to it is very superficial. On the other hand, the magnetisation due to the rather more prolonged direct current is more diffused through the needle, and the resultant magnetisation found on testing the needle is that apparently due to the *direct* current, and the inference from the resulting magnetism of the needle would be that the induced and inducing currents are in the *same* direction. By some such explanation as the above we may reconcile these experimental results of Henry with known facts, but it is evident, since the resulting magnetisation of the needle is an effect determined by the relative maximum values of the two portions of the total induced current, and by their duration, as well as by their order of superposition, that considerable caution is necessary in attempting to interpret the results of experiments made with a magnetising helix. Henry was followed in the same field of investigation by Abria, Marianini, Reiss, and Matteucci. Matteucci endeavoured to determine the direction of the induced discharges by employing a process founded upon the experiment of the pierced card, in which the hole made by the spark on a piece of paper or a card is always nearer to the negative electrode. By means of this process, combined with the employment of the galvanometer, Matteucci considered that the inductive discharges are determined by the following law:—If the inducing and induced circuit are both closed, the induced discharge is in the *opposite* direction to the inducing discharge. If, however, the induced circuit is interrupted at any point so that there is a spark, the induced discharge is in the *same* direction as the inducing. Abandoning these methods above described, M. Verdet* employed another, which depends upon the *polarisation of electrodes* in dilute sulphuric acid.

From more recent knowledge we may state the facts with regard to the action of alternate currents upon a dilute sulphuric acid voltameter as follows† :—

If a current of electricity consisting of alternate short fluxes of current of opposite sign is passed through a voltameter having platinum electrodes, and if these electrodes are large, there is no visible decomposition, but if the electrodes are reduced in size below a certain limit visible decomposition begins. For every current there is a certain size of electrode, above which gas is not visibly evolved, and for every given size of electrode there is a current below which gas is not apparently liberated. When the conditions are suitable for the liberation of gas, the gases collected at both electrodes have the same composition. If the quantities of electricity passing in each alternate and oppositely-directed flux are equal, then the electrodes are not sensibly polarised. If, however, the quantities are not equal, then there is, on the whole, a greater flow of current in one direction than in the other, and the electrodes exhibit the state known as polarisation, and yield a reverse current when connected with the galvanometer. Verdet, in his experiments, made use of flat spirals, the wires of which were insulated from each other with great care by silk and a layer of gum-lac varnish. The primary spiral was made of copper wire $\frac{3}{5}$ths of an inch in diameter and 92 feet in length, forming 24 spirals. The secondary circuit consisted of three spirals of wire $\frac{1}{50}$th of an inch in diameter and 157 feet in length, making 95 turns. The inducing discharge was supplied from a Leyden jar battery of nine large jars. The induced discharge was sent through a voltameter having small platinum electrodes, and which could be connected with a delicate galvanometer for detecting polarisation of the electrodes immediately after the discharge. Verdet's experiments led him to recognise that when the induced circuit is continuous, and not interrupted anywhere except by the insertion of the voltameter, no traces of polarisation are obtained except by very powerful discharges. This indicates that the induced discharge consists of a double current of two oppositely-directed and equal quantities of electricity. In the case of very powerful discharges there was a slight galvanometric deflection, indicating a preponderating secondary discharge in the *same* direction as the primary. If the induced or secondary circuit is interrupted at one point, so that the discharge has to pass as a spark at that place, then very perceptible polarisation of the electrodes presents itself, and the direction of this is such as to indicate a predominant induced current passing in the *same* direction as the primary.

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* Verdet, *Ann. de Chem. et de Phys.*, [3] Vol. XXIX., p. 501, 1850.

† *See* a Paper by MM. Maneuvrier and J. Chappuis, abstracted in *The Electrician*, June 29, 1888, Vol. XXI., p. 237.

To sum up. It follows from all the numerous researches on induced discharges that this is a very complex phenomenon, and is influenced by a large number of conditional circumstances, and also by the very mode employed for determining it. It may be, however, taken as proved that an induced discharge, produced either as a secondary discharge by a transitory primary, such as the discharge from a Leyden jar, or a tertiary current produced by induction by a secondary current of very brief duration, is, in its simplest form, a wave of electric current, consisting of two short fluxes or currents in opposite directions, and succeeding each other immediately. This Poggendorff* holds to be shown by the action of such tertiary or higher order currents on a galvanometer. If these currents are led through a galvanometer of which the arrangement is such that the magnetic axis of the needle is accurately at right angles to the direction of the magnetic axis of the coil, then no deflection of the needle is observed, or at most a very slight one. If, however, the needle makes an angle with the plane of the coils, then these induction currents cause a deviation of the needle. This effect (*die doppelsinnige Ablenkung*) arises from the fact that the magnetism of the needle is not rigid, and that the alternate twisting couples to which the needle is subjected are not equal, by reason of the fact that one of the halves of the complete induced current—say the direct half—increases the magnetic moment of the needle, and hence increases slightly the deflecting couple in the direction tending to increase the deviation of the needle; the other half—say the inverse part of the induced current—tends to reduce the moment of the needle, and hence to subject it to a smaller reverse couple. Hence it follows that, if discharges of equal quantity and opposite sign succeed each other through a galvanometer when the needle is accurately in the plane of the coils, little or no deviation is observed; but if the coils are turned so that the needle makes an angle with them, then these alternate currents will affect the needle and increase the angle of deflection.

This behaviour towards a galvanometer, and the action on a voltameter of liberating mixed gases of equal composition at each pole, prove that each induced current of the third and higher orders consists of two oppositely-directed discharges, produced by the operation of two successive electromotive impulses of opposite sign and very brief duration acting upon the circuit. The *quantities* of these discharges are equal; but the *durations* are different, and hence the maximum value of the current strength during the opposite discharges may be very different.

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* Poggendorff, *Pogg. Ann.*, Bd. XLV., 1838, p. 353.

This may be illustrated graphically thus:—

Let the curve *a* P *b* Q *c* (Fig. 84) be a current curve representing two waves of current of opposite sign succeeding each other. Let the horizontal line *a c* be a time line, and vertical ordinates represent instantaneous current strengths. Then the shaded areas will represent the *quantity* in each discharge. Let these shaded areas be equal, then the diagram represents two discharges of equal quantity succeeding each other in opposite directions, but having different maximum current strengths I and I′. The duration of the first discharge is represented by *a b*, and that of the second by *b c*. This diagram represents the conditions in the simplest case of tertiary current. If the instantaneous value of the current at any time is called $i$, then the whole quantity of the discharge will be represented by the shaded area and by the integral $\int i\,d\,t$ between proper limits.

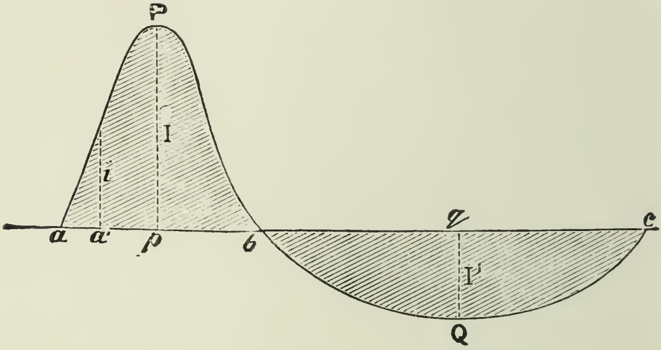

Fig. 84.

We may classify the effects of induced discharges or currents in the following way:—

(1) Those effects dependent upon $\int i\,d\,t$, or upon the whole quantity of the discharge. These are the *galvanometric* and the *electro-chemical* effects. If a discharge is passed through a

galvanometer, the duration of which is very small compared with the time of free oscillation of the needle, the galvanometer needle experiences a "throw" such that the sine of half the angle of deflection is proportional to the whole quantity of the discharge. Also in a voltameter, by Faraday's law, the whole quantity of the electrolyte broken up is proportional to the quantity of electricity which has passed through it.

(2) Those effects dependent upon $\int i^2 dt$, or upon the average of the square of the strength of the current at every instant during the discharge. These are the *heating* and the *electro-dynamic* effects. By Joule's law, at every instant the rate of dissipation of energy is proportional to the square of the current strength, and hence the whole heat generated by the discharge is proportional to the integral above. Similarly, if the discharge passes through a circuit, part of which is movable and can react upon a fixed part, so that attraction or repulsion may take place between them, the force is dependent at any instant on the square of the current strength, and hence the whole effect or average force upon the same integral.

(3) We have, lastly, effects dependent chiefly upon the maximum ordinate I, or upon the rate of change of the current—that is, upon the steepness of the slope of the current curve. These are the *physiological*, *telephonic*, *luminous*, and *magnetic* effects.

The physiological effect of a discharge in giving a shock appears to depend in great part upon the suddenness with which the maximum current strength is reached. Of two discharges which reached equal maxima, that which arrived at it in the shortest time would be the most effective in producing shocks. The value of the maximum current strength is also important. Two induced currents of equal quantity but different durations cause a greater shock in proportion to their lesser duration. The telephone in this respect resembles the animal body. It is affected more by the rate of change of the current strength than by the absolute current strength at any instant.

The *magnetic* effect depends, as has been shown by Lord Rayleigh,* upon the maximum current strength during the discharge, or upon the initial current strength, in those cases in which the current dies gradually away. In the two Papers referred to below it is shown by direct experiment that, since the time required for the permanent magnetisation of steel is small compared with the duration of induced currents generally, the amount of acquired magnetism depends essentially on the initial or maximum current strength during a transitory current, without regard to the time for which it lasts. It is, then, not difficult to understand that the effort to settle by experiment with a magnetising coil the direction of induced discharges may lead to very conflicting results, and, in any case, it is hardly competent to do more than indicate the direction in which the maximum current flow takes place during the discharge.

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* See *Phil. Mag.*, Ser. 4, Vol. XXXVIII., 1869, p. 8: The Hon. J. W Strutt (Lord Rayleigh), "On some Electromagnetic Phenomena." Also *Phil. Mag.*, Ser. 4, Vol. XXXIX., 1870, p. 431.

The *spark* effects are also included in this category. The air or other dielectric is broken down when the difference of potentials between the two discharging points reaches a certain magnitude, and in the case of a varying electric pressure the question whether a spark will pass or not is evidently determined by the maximum magnitude of that quantity.*

It is evident from the above considerations that the complete analysis of the effects and phenomena of induced currents of the higher orders, and of those of secondary currents due to discharges from condensers, requires a knowledge of the *form* of the current curve in each case. We proceed to consider the problem of the theory of induced currents in some of its simpler aspects.

**§ 6. Elementary Theory of the Mutual Induction of Two Circuits.**—Aiming rather at the elucidation of principles than very copious treatment, we shall consider in the next place the problem of the mutual induction of two circuits in its simplest form. Let there be two bobbins of wire in suitable positions for producing mutual induction and without iron cores. Let the constant inductance of the first or primary coil be denoted by L and its resistance R, and the similar quantities for the second or secondary coil be N and S, and let M be the coefficient of mutual induction.*

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* *See* Bertin, "Notes on Electrodynamic Induction," *Ann. de Chimie*, 4th Ser., Vol. XXII., April, 1871, p. 486.

Let there be a source of constant electromotive force, E, which can be applied or withdrawn from the primary circuit. We shall denote by $x$ the strength of the current in the primary at any time $t$ after closing the primary circuit by applying the battery to it. Also we shall denote by $y$ the current in the secondary circuit at any time reckoned from the same zero.

If, then, at any instant the currents are $x$ and $y$, the following state of things exists in the circuits.

The electromotive force E is the impressed force on the primary circuit.

That part of the impressed electromotive force producing the current $x$ is $\mathrm{R}\,x$. That part employed in overcoming the counter-electromotive force of self-induction is $\mathrm{L}\frac{dx}{dt}$, and the counter-electromotive force of mutual induction due to the current $y$ at that instant in the secondary circuit is $-\mathrm{M}\frac{dy}{dt}$.

Hence the relation which at any instant holds good between these quantities is

$$\mathrm{L}\frac{dx}{dt}+\mathrm{M}\frac{dy}{dt}+\mathrm{R}\,x=\mathrm{E}.$$

The above equation is an expression of the fact that the external impressed electromotive force at any instant is equal to the internal electromotive forces and the effective electromotive force driving the current.

Similarly, for the secondary circuit we have an induced electromotive force due to the induction of the primary on the secondary equal to $\mathrm{M}\frac{dx}{dt}$ and a counter-electromotive force of self-induction $\mathrm{N}\frac{dy}{dt}$.

Hence $$\mathrm{N}\frac{dy}{dt}+\mathrm{M}\frac{dx}{dt}+\mathrm{S}\,y=0,$$

since there is no external impressed electromotive force. The complete solution of the problem of finding the currents $x$

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* Continental writers often call L and N the potentials of the bobbins on themselves, and M the potential of one bobbin on the other.

and $y$ at any instant is obtained by the solution of these simultaneous differential equations—

$$L\frac{dx}{dt}+M\frac{dy}{dt}+Rx=E,$$

$$N\frac{dy}{dt}+M\frac{dx}{dt}+Sy=0.$$

As our object is to illustrate principles rather than mathematical methods, we shall simplify the problem by supposing that the two circuits are similar in every respect. This makes $R=S$ and $L=N$, and the equations become

$$L\frac{dx}{dt}+M\frac{dy}{dt}+Rx=E, \quad . \quad . \quad . \quad . \quad (83)$$

$$L\frac{dy}{dt}+M\frac{dx}{dt}+Ry=0. \quad . \quad . \quad . \quad . \quad (84)$$

Bearing in mind that the inductance L is, in ordinary parlance, the "number of lines of force" which are linked with the primary circuit when unit current flows in its own circuit, and that M signifies the number of lines of force which are common to both, or linked in with both circuits, when unit current flows in each, we see that M can never be greater than L, but that under all circumstances we must have

$$M < \text{or} = L,$$

also $$M < \text{or} = N;$$

hence $$M^2 < \text{or} = LN,$$

or $LN-M^2$ always a positive quantity, and the maximum value which the co-efficient of mutual inductance M can have is $\sqrt{LN}$, or the square root of the product of the self-inductances of the separate circuits.

In order to separate the differentials in (83) and (84) we differentiate each equation with respect to $t$, and obtain—

$$L\frac{d^2x}{dt^2}+M\frac{d^2y}{dt^2}+R\frac{dx}{dt}=0, \quad . \quad . \quad . \quad (85)$$

$$L\frac{d^2y}{dt^2}+M\frac{d^2x}{dt^2}+R\frac{dy}{dt}=0. \quad . \quad . \quad . \quad (86)$$

Multiply (85) by L, (86) by $-$M, and (83) by R, and then adding the three equations together we obtain—

$$\frac{d^2 x}{d t^2}+\frac{2\,\mathrm{L\,R}}{\mathrm{L}^2-\mathrm{M}^2}\frac{d x}{d t}+\frac{\mathrm{R}^2}{\mathrm{L}^2-\mathrm{M}^2}x=\frac{\mathrm{E\,R}}{\mathrm{L}^2-\mathrm{M}^2}, \quad . \quad (87)$$

and a similar elimination gives us

$$\frac{d^2 y}{d t^2}+\frac{2\,\mathrm{L\,R}}{\mathrm{L}^2-\mathrm{M}^2}\frac{d y}{d t}+\frac{\mathrm{R}^2}{\mathrm{L}^2-\mathrm{M}^2}y=0. \quad . \quad . \quad . \quad (88)$$

We have now separated the differentials in $x$ and $y$, and the solution of these equations depends, as is well known,* upon the solution of an auxiliary quadratic equation—

$$m^2+\frac{2\,\mathrm{R\,L}}{\mathrm{L}^2-\mathrm{M}^2}m+\frac{\mathrm{R}^2}{\mathrm{L}^2-\mathrm{M}^2}=0,$$

the solution of which is—

$$m=-\frac{\mathrm{R}}{\mathrm{L}+\mathrm{M}}, \text{ or } -\frac{\mathrm{R}}{\mathrm{L}-\mathrm{M}}.$$

Hence the general solution of (83) and (84) is—

$$x=\mathrm{A}\,e^{-\frac{\mathrm{R}t}{\mathrm{L+M}}}+\mathrm{B}\,e^{-\frac{\mathrm{R}t}{\mathrm{L-M}}}+\frac{\mathrm{E}}{\mathrm{R}}, \quad . \quad . \quad . \quad (89)$$

and

$$y=\mathrm{A}'e^{-\frac{\mathrm{R}t}{\mathrm{L+M}}}+\mathrm{B}'e^{-\frac{\mathrm{R}t}{\mathrm{L-M}}}, \quad . \quad . \quad . \quad . \quad . \quad (90)$$

where A, B, A′, B′ are constants of integration to be determined from the circumstances of the flow. To do this, however, a preliminary discussion is necessary. Let us suppose that the primary current is fully established, and has a steady value I, and hence that M I lines of induction penetrate through the secondary circuit. This quantity is then the electromagnetic momentum of the secondary circuit, because when the current in the primary is steady there is no current in the secondary circuit.

Let us now suppose that the primary circuit is broken, and that the circumstances of the "break" are such that all these M I lines of induction are removed at a uniform rate in a small time $\delta t$ from the secondary circuit.

During this time $\delta t$ an electromotive force will operate upon the secondary circuit equal in magnitude to $-\frac{\mathrm{M\,I}}{\delta t}$, or to the rate of decrease of the included lines of force. We have seen in

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* *See* Boole's "Differential Equations," p. 192, 2nd Edition.

Chap. III. that when an electromotive force E acts on a circuit of inductance L and resistance R that the current $i$ at any time after the commencement of the application of the electromotive force is given by the equation

$$i=\frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right).$$

In the case considered the inductance and resistance of the secondary circuit are L and R, and the impressed electromotive force applied during a time $\delta t$ is $\frac{MI}{\delta t}$. Hence, at the end of the interval of time $\delta t$, the value of the secondary current is given by the equation

$$i=\frac{MI}{\delta t R}\left(1-e^{-\frac{R}{L}\delta t}\right).$$

This gives us the value of the inverse induced current at the instant of breaking the primary. Expand the above expression by the exponential theorem, and it becomes

$$\frac{i}{I}=\frac{M}{L}-\frac{MR\delta t}{L^2 1\cdot 2}+\frac{MR^2\delta t^2}{L^3 1\cdot 2\cdot 3}.$$

At the instant when the removal of lines of force or the cessation of the induction through the secondary takes place the impressed electromotive force ceases and the secondary current begins to die away. If we suppose the "break" of the primary to be very sudden, $\delta t$ becomes practically zero, and we have

$$i=\frac{M}{L}I;$$

that is to say, the secondary current starts with a value equal to $\frac{M}{L}$ of that of the steady primary.

The state of things in the secondary circuit immediately *after* the break of the primary is, then, this: The electromotive impulse due to stoppage of the primary has generated a current of initial value $\frac{M}{L}I$ in the secondary, but there is no impressed electromotive force in the secondary circuit. If at any instant after the break the current in the secondary circuit is $i$, the law of decay of this current is expressed by the equation

$$L\frac{di}{dt}+Ri=0.$$

The solution of this is

$$i = C\,e^{-\frac{R}{L}t}$$

and the constant C is found from the condition that when $t=0$ $i=\frac{M}{L}\,I$. Hence we have

$$i=\frac{M}{L}\,I\,e^{-\frac{R}{L}t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

This gives us the value of the direct or "break" induced current in the secondary at any instant after the break of the primary. Graphically, this may be represented by a curve, such as that in Fig. 85. During the time O T in which the primary is being broken the induced electromotive force is creating an induced current, the rising strength of which is represented by the rise O P. The time occupied by the break $\delta t$ is O T. As O T is diminished in value, the magnitude of the maximum ordinate P T approximates to $\frac{M}{L}I$, and this is the initial value of the inverse secondary current when the break is very sudden. After the break the current decays away along a path represented by P Q, and becomes zero only after an infinite time.

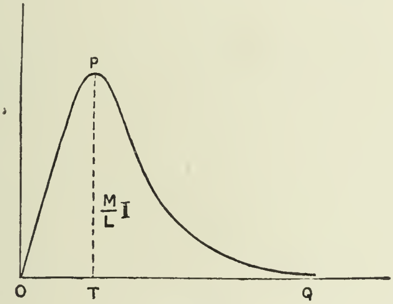

FIG. 85.

The whole quantity of the induced current is obtained by integrating equation (91) with respect to the time from zero to infinity, thus:

$$\int_0^\infty i\,dt = \int_0^\infty \frac{M}{L}\,I\,e^{-\frac{R}{L}t}\,dt = \frac{M\,I}{R}.$$

We see, then, that both the maximum value and whole quantity of the direct secondary current are proportional to the coefficient of mutual induction and to the strength of the primary current, and, moreover, that the whole quantity of electricity set in motion in a secondary circuit of total resistance R by suddenly removing from it M I lines of force is equal to the quotient of number of lines removed by the total resistance of the secondary circuit.

If the induced current is sent through a galvanometer the indications are proportional to the magnitude of $\frac{\mathrm{M\,I}}{\mathrm{R}}$. If, however, the induced current is employed to magnetise steel needles, the magnetisation acquired is dependent upon the magnitude of $\frac{\mathrm{M\,I}}{\mathrm{L}}$, and is therefore greater in proportion as the coefficient of self-induction of the secondary circuit is less. Lord Rayleigh has pointed this out,[*] and shown by experiment that, within certain limits, the magnetising effect of the break-induced current on steel needles is *greater* the smaller the number of turns of which the secondary consists, the opposite being, of course, true of the galvanometer. The galvanometer takes account of the total quantity of the induced current; whilst the magnetising power depends mainly on the magnitude of the current at the first moment of its formation, without regard to the time which it takes to subside.

Returning to the equations (89) and (90), we can now find the constants of integration, counting the time from the instant of "make" of the primary. It is obvious that when $t=0$, $y=0$ and $x=0$, and that the whole quantity of the make-induced current, or $\int_0^\infty y\,d\,t$, must be equal to the whole quantity of the break-induced current, which we have seen is equal to $\frac{\mathrm{M\,I}}{\mathrm{R}}$.

In (90) put $t=0$, $y=0$; we get

$$\mathrm{A}'+\mathrm{B}'=0, \quad \text{or} \quad \mathrm{B}'=-\mathrm{A}'.$$

Hence,
$$y=\mathrm{A}'\left(e^{-\frac{\mathrm{R}t}{\mathrm{L}+\mathrm{M}}}-e^{-\frac{\mathrm{R}t}{\mathrm{L}-\mathrm{M}}}\right)$$

and
$$\int_0^\infty y\,d\,t=-\frac{2\,\mathrm{A}'\,\mathrm{M}}{\mathrm{R}}.$$

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[*] *Phil. Mag.*, Ser. 4, Vol. XXXIX., 1870, p. 429.

Hence the whole quantity of the "make"-induced current is $-\frac{2\,A'\,M}{R}$, and this must be equal to $\frac{M\,I}{R}$, which is the whole quantity of the "break" current. Hence $A' = -\frac{I}{2}$.

Therefore we get for the instantaneous strength of the "make" secondary current

$$y = -\frac{I}{2}\left(e^{-\frac{Rt}{L+M}} - e^{-\frac{Rt}{L-M}}\right) \quad . \quad . \quad . \quad (92)$$

Again, in (89) put $t=0$, $x=0$,

and we get $A+B+I=0$,

or $B = -(I+A)$;

and by substitution in (89)

$$x = A\,e^{-\frac{R}{L+M}} - (A+I)\,e^{-\frac{R}{L-M}} + I.$$

From this equation we can find the value of A by substituting the value of $\frac{dy}{dt}$ derived from equation (92), and $\frac{dx}{dt}$ derived from the above in the original differential equation (83), and we find $A = -\frac{I}{2}$. Hence we arrive at the equation for the value of the primary current at any instant, and it is

$$x = I - \frac{I}{2}\left(e^{-\frac{Rt}{L+M}} + e^{-\frac{Rt}{L-M}}\right) \quad . \quad . \quad . \quad (93)$$

This gives the law according to which the primary current grows up in its circuit. If $M=0$, that is, if there is no secondary circuit; then

$$x = I\left(1 - e^{-\frac{Rt}{L}}\right),$$

which is the ordinary law of current growth. If $M=L$, which is the greatest possible value of M, then

$$x = I\left(1 - \frac{1}{2}\,e^{-\frac{Rt}{2L}}\right).$$

Hence it is obvious that the presence of the secondary circuit hastens the rise of the primary current and operates on it to reduce its inductance.

On making the primary we get a "make" or inverse secondary current according to the law of growth expressed by the equation

$$y = -\frac{I}{2}\left(e^{-\frac{Rt}{L+M}} - e^{-\frac{Rt}{L-M}}\right),$$

and we see that under the circumstances assumed the "make" secondary starts from an initial value zero, rises up to a maximum, and then decays away again. To find the time of reaching maximum, equate $\frac{dy}{dt}$ to zero, and we find

$$t' = \frac{L^2 - M^2}{2RM} \log \left(\frac{L+M}{L-M}\right),$$

and this function *increases* as M *decreases*. So that the more nearly M is equal to L the *sooner* does the secondary reach its maximum. It is not difficult to show that when $M = L$ the above value for $t'$ becomes zero, and when $M = 0$ $t' = \frac{L}{R}$.

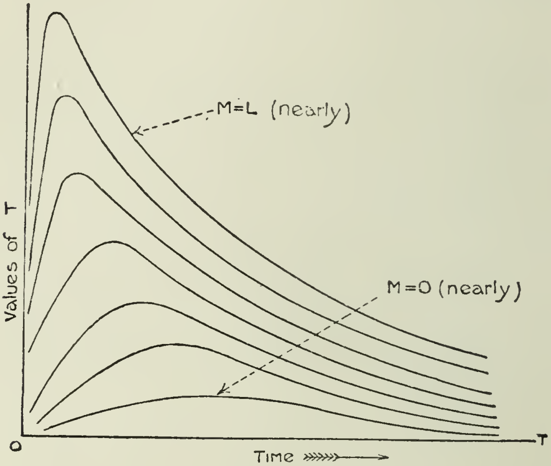

FIG. 86.

Curves representing roughly the current value of the make-induced current for different and increasing values of M.

If, then, we trace a series of curves (Fig. 86) representing the values of $y$, or the make-induced current at each instant for various and increasing values of $\frac{L}{M}$, as the coils are moved further apart, we find a series of curves with decreasing maxima, but the maxima happening later as M decreases.

Lastly, on breaking the primary current we have a break-induced current in the same direction as the primary, which at any instant after the "break" is decaying away according to the law

$$z = \frac{M}{L} I e^{-\frac{Rt}{L}}.$$

If the break was absolutely instantaneous, the induced current would start with a finite value equal to $\frac{M}{L}$ of that of the primary, but as no form of break entirely eliminates sparking, the rise of the direct secondary current is a gradual one. Also we have another element of disturbance which enters into the case. The self-induction of the primary creates direct electromotive force in its own circuit at the instant when the induction through the primary due to its own current vanishes. When the primary is broken either at a mercury cup or at a platinum point the fusion and volatilisation of metal which takes place keeps open for a little time a conductive path through which flows the *extra current* due to the self-induction of the primary. As will be explained later, the decay of the current on breaking a circuit may often be by a series of oscillations or diminishing periodic currents.

This direct extra current in the primary will have its effect in introducing a very short inverse-induced current, which will precede the main direct-induced current due to the decay of the primary current. In any event it will introduce an electrical oscillation tending to render the growth of the direct secondary current a gradual matter. It is an interesting case to examine the relative maximum values and duration of the two induced currents under an assumption very nearly realised when the primary and secondary are wound together on the same bobbin, viz., when M = L. In this case the values of $y$ and $z$ become

$$y = \frac{I}{2} e^{-\frac{Rt}{2L}}$$

$$z = I e^{-\frac{Rt}{L}}.$$

The maximum of the direct currents ("break") is I, and that of the inverse (or "make") is $\frac{I}{2}$. If we wish to know at the end of what times $t$ and $t'$ the strengths of the two induced currents $y$ and $z$ are reduced to $\frac{1}{m}$ of that of the primary we obtain by substitution of $\frac{1}{m}$ for $y$ and $z$ in the two above equations the following:—

$$\frac{1}{m} = e^{-\frac{\mathrm{R}t}{\mathrm{L}}} \quad \text{for the direct-induced current,}$$

and
$$\frac{1}{m} = \frac{1}{2}e^{-\frac{\mathrm{R}t'}{2\mathrm{L}}} \quad \text{for the inverse-induced current,}$$

and therefore
$$\frac{t'}{t} = 2\left(1 - \frac{\log 2}{\log m}\right).$$

We see that $t'$ is always greater than $t$, and that, in proportion as $m$ increases, $t'$ tends towards a limit $2\ t$, or the inverse current has a duration about double that of the direct secondary. We shall now see how this theory is confirmed by experiment.

§ **7. Comparison of Theory and Experiment.**—Masson and Breguet carried out a series of experimental researches on induced currents which illustrate and confirm the foregoing theory. The principal part of their apparatus was a commutator keyed on a revolving shaft, which enabled them to separate the direct and inverse-induced currents. Two brass wheels were keyed on one shaft, but insulated from it, and the wheels had depressions cut in their periphery which were filled up with ivory. These wheels could be shifted relatively to each other, and were insulated from each other and from the shaft (*see* Fig. 87). Two springs pressed against the edge of the wheels, and two against the hub of the wheel. The whole arrangement served as a means to break and make one circuit, and at the same time to control a second circuit so that it was broken at the time when the first was made, and made at the time when the first was broken, or *vice versâ*. One of these wheels was inserted in the circuit of a primary coil and battery, and the other in the circuit of a secondary coil and galvanometer. On rotating the wheel at a certain fixed speed the series of "break" and "make"-induced currents are separated out; all one set are stopped out and all the other are sent through the galvanometer. In this way it was shown that the quantities of the induced currents were equal, but very different in maximum magnitude, and hence in duration, the break-induced currents being greatly superior in making sparks.

Lenz* wound a spiral of wire on the soft iron armature of a magnet and connected the ends of the wire to a ballistic galvanometer. He detached the armature suddenly, and observed the throw of the galvanometer. If $\theta$ denotes the angle of deflection and $x$ the number of windings, he found that the product $\frac{1}{x} \sin \frac{\theta}{2}$ was a constant quantity, which shows that, *cœteris paribus*, the quantity of electricity set in motion was in proportion to the number of lines of induction withdrawn from the circuit. He also established experimentally, in confirmation of Faraday, that the electromotive force of induction was independent of the width, thickness or material of the wire windings,† and by other experimentalists also the fact has been established that the electromotive force is independent of everything except the form of the conductor and the nature of the change it experiences in relation to the magnetic induction through it. Felici‡ carried out an extensive series of experiments on induction, using a form of induction balance.

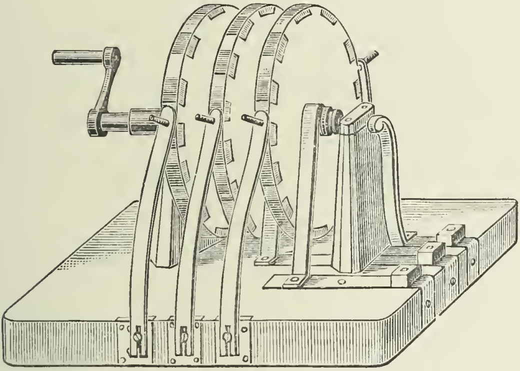

Fig. 87.

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* Lenz, *Poggendorff's Annalen*, Bd. XXXI., 1835, p. 385.

† *See* Faraday, "Exp. Researches," Ser. II., § 193, *et seq.;* also for Electrolytic Circuits, *see* L. Hermann, *Pogg. Ann.*, 1871, p. 586.

‡ Felici, *Nuovo Cimento*, Vol. IX., 1859, p. 345, also *Ann. de Chimie* [3], Vol. XXXIV., 1852, p. 64.

In this apparatus a secondary circuit, consisting of two coils, is arranged in series with a galvanometer. These coils are so far apart as not to influence one another. In contiguity to each secondary coil is a primary coil, and the primaries are wound in opposite directions. The primaries are in circuit with a battery and a key. The circuits can be so arranged, by adjusting the distances of the coils, that the induction of the primaries on their respective secondaries balance each other, and the galvanometer indicates no current, however strong may be the primary current. If three pairs of coils (*see* Fig. 88) are thus taken and balanced, two and two, so that the induction of A on *a* is equal to that of B on *b* and C on *c*, then, if we connect the primary A in series with B and C in parallel, so that the current divides between them in the ratio of their resistances, and connect the secondaries with a galvanometer, all in series, so that the current in *a* is opposed to that in *b* and in *c*, then no induced current is detected when the battery circuit is made and broken. This proves that the quantity of the induction current is proportional to the strength of the primary current.

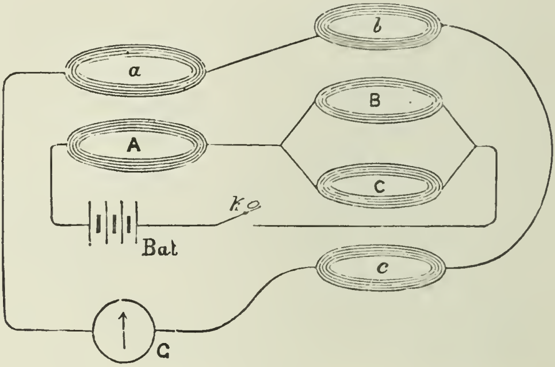

Fig. 88.

If a primary and secondary coil are taken in fixed positions and the "throw" of a galvanometer observed when a definite steady electromotive force E is applied to the primary, then, if the position of battery and galvanometer are reversed, the application of the same electromotive force E to the secondary will give the same "throw" on the galvanometer now attached to the primary circuit, provided that the galvanometer and battery either have equal internal resistance or that their resistance is negligible in comparison with that of the coils. Hence we may assert that the induction of a circuit A upon B is the same as that of B upon A. For, if the resistances are R and S, then we have seen that the total quantity Q of the secondary current is $\frac{M I}{S}$, where I is the steady value of the primary and M is the mutual inductance; but $I = \frac{E}{R}$, hence $Q = \frac{M E}{S R}$. If, then, the positions of battery and galvanometer are reversed, we get a quantity of induced current equal to $\frac{M E}{R S}$, which is the same as before. For any two coils it is possible to find a number of relative positions in which the interruption of a current in one produces no induced current in the other. In such cases the coils are said to be *conjugate* to each other. It is manifest that when in these positions the lines of induction produced by one coil do not pass through the other. It is possible to use one coil in this way to explore the field of another.

Let P be a primary coil and S be a small flat secondary coil, both being shown in section in Fig. 89. Then, if S is placed in a position conjugate to P, it will be found possible to move the coil S along a certain line A B C, maintaining the flat face of the coil always tangent to that line and so that in all these positions P and S are conjugate. It is evident that such a line is a line of induction of the coil P.

When one coil is in a conjugate position to the another, as far as regards inductive action they may be considered to be at an infinite distance apart. It follows, therefore, that if a coil is moved suddenly from a conjugate position to one not conjugate in the field of a primary traversed by a steady current, and then the primary current is stopped at the instant of arriving at the second position, a galvanometer in the second circuit will have its needle jerked from one position of rest to another of rest, because the interruption of the current takes out of the circuit of the second coil just as many lines of induction due to the first coil as the motion from one position to the other put in. A series of well-devised experiments on the conjugate positions of two coils has been carried out by Mr. W. Grant.*

An elaborate investigation into the duration of induced currents was made by Blaserna.†

A commutator was constructed which consisted of two insulating cylinders keyed on one shaft and having on part of their surface brass coverings cut into steps (*see* Fig. 90). These cylinders were capable of being set in any relative position to each other on the shaft. The shaft could be revolved at a high rate of speed, and its velocity ascertained by a siren plate attached to the axis. This siren plate consisted of a disc pierced with holes against which was directed a jet of air. From the pitch of the musical note given out, when ascertained by comparison with standard tuning forks, the speed could be determined. Two springs pressed against the hubs of these cylinders and two against the surfaces of these cylinders, and a current entering by the hub was conducted to the brass coating and escaped by the other spring, if the cylinder was in such a position that this last spring was pressing on the metal part. The apparatus, therefore, formed a device by which each

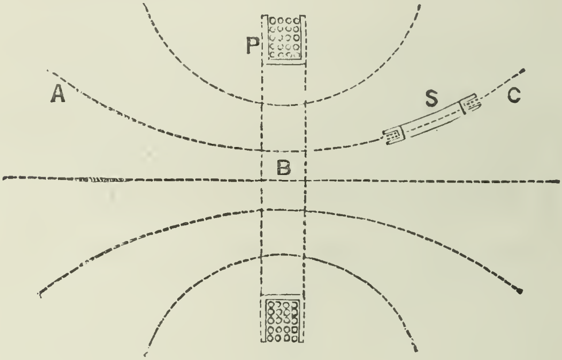

Fig. 89.

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* See *Proc.* Physical Soc., London, Vol. III., p. 121 ; also *Proc.* Physical Soc. London, Vol. IV., p. 361.

† Blaserna, "Sul sviluppo e la durata delle Correnti d'induzione," *Giornale di Scienze Naturali*, Vol. VI. (Palermo, 1870).

pair of springs might be brought into electrical contact for a definite portion of the time of a revolution of the cylinders and be insulated also for a given time, each pair of springs being in connection relatively to the other in a determined manner for a determined time. In the circuit of the one cylinder and pair of springs $m$ M was placed a battery primary coil and tangent galvanometer, and in the circuit of the other pair a secondary coil and sensitive galvanometer. This being prepared, the primary coil P and the secondary S were placed a given distance apart. On revolving the commutator it periodically interrupts the primary current, the time during which the primary current is kept on depending upon the position of the spring M on its cylinder. The other cylinder can be so set as to collect either the direct or

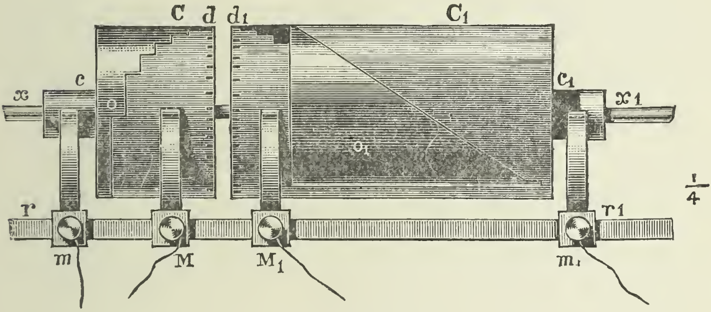

Fig. 90.

inverse secondary currents, and send them in series through the sensitive galvanometer, the time during which this secondary circuit is closed being capable of regulation by the adjustment of the spring $M_1$. In his experiments Blaserna first investigated the duration of each of the induced currents. The interrupters were so arranged relatively to one another that, whilst the primary circuit was made and broken, the secondary circuit was not closed until a small time after "making" the primary, and then broken again before the primary was broken. By adjusting the secondary interrupter a position could be found in which the galvanometer just showed no current. The interval between the closing of the primary and the opening of the secondary was then the interval occupied by the secondary current, and this was the duration of the "make"-induced current. Blaserna found that the "make" secondary (inverse) lasts a longer time than the "break" current (direct). For the coils used the times were—

Inverse secondary lasts ·000485 second.
Direct secondary lasts ·000275 second.

He next proceeded to obtain the curve of each current, and to determine the time of arrival at a maximum.

The secondary interrupter was so set that the secondary circuit was closed just before the primary, and opened after at a certain definite interval of time. The galvanometer thus

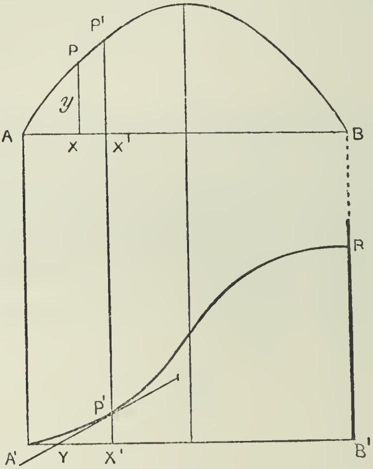

Fig. 91.

received a current which was made up of repeated doses of the whole quantity of the induced current up to a certain fraction. Knowing the speed of the commutator and the coefficient of the galvanometer, the value of the whole quantity of the induced current, extending over a certain fraction of its whole duration, was known; and from those observations, repeated at regular progressive intervals during the whole period of the current, the value of the ordinates of the current curve can be obtained. For, if the curve (Fig. 91) A P P′ B (upper figure) represents the variation of current during a time A B, so that P X = $y$ represents the current strength at a time X, and P′ X′ represents the current strength after a very small interval of time, $X X' = d t$; then the area $P P' X' X = y\, d t$ represents the quantity of electricity which has passed in the time X X′. Call this $d Q$. Hence $d Q = y\, d t$, or $y = \frac{d Q}{d t}$.

Suppose another curve A′P′R (lower curve) is drawn on an equal abscissa A′B′; such that its ordinate at every point represents the whole area of the upper curve up to the corresponding point—that is to say, the lower curve is a curve such that its ordinate P′ X′ is proportional to the area A P X of the upper curve, A X (upper curve) being equal to A′X′ (lower curve), when the time interval $d t$ becomes very small. It is easily seen that if the area A P X (upper curve) is called Q, and the ordinate P X is called $y$, that the tangent of the angle P′YX′ (lower curve) which the geometrical tangent drawn at P′ makes with the axis A′B′, and which is represented by $\frac{d Q}{d t}$, is proportional to the ordinate PX. Hence the upper curve is a derived curve of the lower, and, if we are given a curve like the lower curve, the ordinates of which represent the whole quantity of electricity which has from a given epoch flowed past a point, we can, by drawing a curve whose ordinates represent the *slope* of the first curve, obtain a second curve, which is a curve of current. In this way it is possible to describe the current curve, and to determine its form and position of maximum.

Blaserna found that the greater the distance apart of the primary and secondary—in other words, the less the mutual inductance—the less was the maximum value of the secondary current, and the greater the delay in the appearance of that maximum. This is in accordance with the above elementary theory. In the case of the "break," or direct secondary current, he found the delay in establishing the maximum not so great, and the maximum ordinate was greater though the total duration of the current was less. He established by direct experiment the equality of the quantity of the two induced currents. When the coils were very near together the induced current at starting established itself by a series of *electrical oscillations*.

By the help of the same apparatus Blaserna investigated the rise of a current in a coil when the same is placed suddenly in connection with a constant source of electromotive force. For the "make" extra current only one of the revolving interrupters was used, and the circuit was completed by the means of a battery, galvanometer, and coil. When the commutator was revolved it first started the current and then after an interval cut it off again, and the effect on the galvanometer is due to the sum of all these small quantities of electricity so cut off and integrated whilst the current is in process of increasing. As the duration of the time of contact was increased the galvanometer deflection increased (speed of revolution remaining constant), but when the time of contact was long enough to fully establish the current, then increase of speed of rotation did not increase the galvanometer deflection. By this apparatus the fact was established that the primary current established itself in its coil by a series of oscillations, or short alternating currents.

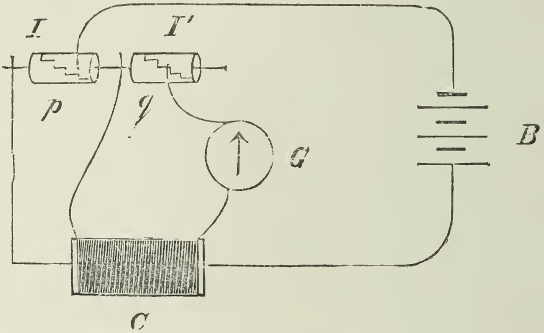

FIG. 92.

Similarly, on breaking the circuit the course of the current was investigated. For this purpose one revolving interrupter, I, was inserted in the circuit of a battery, B, and coil, C, and from the ends of the coil (*see* Fig. 92) other wires were brought and led through the galvanometer G, and other interrupter I′, arranged as a shunt on the coil. The break in the battery circuit at *p* was so arranged that each time the current was fully established before being broken again. The break in the galvanometer or shunt circuit was so arranged relatively to the other that the shunt circuit was closed a little before the battery circuit was broken, and then opened at a definite interval afterwards. In this way there was a little flow of current through the galvanometer due to the steady current, but this could be estimated and allowed for. On plotting out a current curve from the quantity curve it was found that the current decayed away on interrupting the circuit by a series of oscillations which followed each other much quicker than those on the establishment of it, and the whole duration of the extra current at "break," or the time of falling from steady current to practical zero, was less than the time required to fully establish the current. It was found that the first oscillation, on beginning to interrupt the steady current, had a much greater amplitude than any of those on starting the current.

The duration of an oscillation was perhaps three or four ten-thousandths of a second, and about 50 to 100 oscillations probably happened before the current became steady; hence the whole duration of the variable period, or of the extra current, was about two to three-hundredths of a second. Very roughly, the nature of the oscillatory character of the current at the make and break may graphically be represented by the curve in diagram Fig. 93.*

Blaserna drew from his observations the deduction that there is an interval of delay in the starting of the secondary currents, and that a small but measurable time elapses between the instant of making or breaking the primary circuit and the beginning of the secondary current. From this he made a calculation as to the velocity of electromagnetic induction, and he also stated that the interposition of dielectric substances such as glass or shellac between the coils reduced the so-calculated velocity.

Bernstein (Pogg. *Ann.*, Bd. CXLII., 1871, p. 72) repeated these observations of Blaserna, but did not confirm these last results. He found that the first oscillation always began at the instant of breaking or making the primary circuit, and he found no effect produced by the interposition of dielectric media.

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* In *The Electrician* for June 1, 1888, a curve is given by Mr. F. Higgins, showing the rise of current in the magnets of type-printing telegraphs, and the oscillatory character of the current at starting is well marked. Mr. Higgins's curve gives the results of actual observations.

Helmholtz has carefully examined these results of Blaserna and criticised them.* He remarks that Blaserna used for his coils flat spirals of wire with many turns, and also he used the current from several Bunsen cells to create the primary current. Not only do the spirals act like a condenser, giving the whole apparatus a sensible electrostatic capacity, but the use of a battery of high electromotive force causes a considerable spark at the break, which spark has a very sensible and rather irregular duration. Also in Blaserna's experiments, the two circuits were placed at various distances apart. If a current is started in a primary coil the effect of the induced current created in the secondary by its re-action on the primary is to hasten the rise of the primary current, and at the break to accelerate its decay. As the secondary circuit is moved further off this effect is less marked. Hence, the rise and fall of the primary is more gradual and the arrival of the secondary current at its maximum value is more delayed. From this results, then, an apparent retardation of the time of the arrival of the maximum of the induced current.

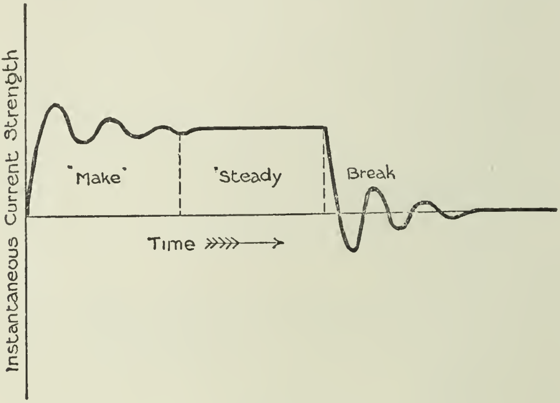

FIG. 93.

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* Helmholtz, "On the Velocity of the Propagation of Electrodynamic Effects," *Phil. Mag.*, Ser. 4, Vol. XLII., 1871, p. 232.

Helmholtz conducted a series of experiments by means of his pendulum chronoscope. A heavy iron pendulum P (*see* Fig. 94), the lower end of which carried two plates of agate, could be made to execute one swing and then be caught by a detent. These plates of agate in the course of the swing were caused to strike against and tip over two little levers $l, l'$. One of these levers was fixed, and the other could be moved forward so as to separate the blows. One was made to break the circuit of a primary coil *Pr*, when tipped over, and the other by its movement separated a connection between a condenser and the ends of a secondary coil, *Sec*, attached to it.

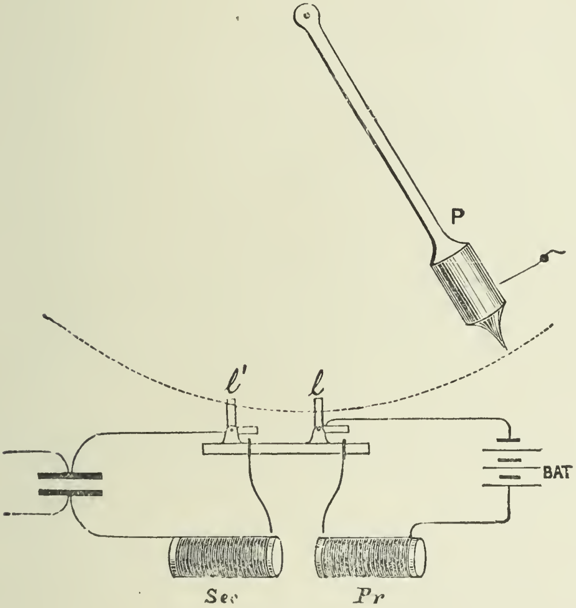

FIG. 94.

These being arranged, the fall of the pendulum executed these two "breaks" successively, separated by an interval of time capable of being calculated from the known motion of the pendulum. The two circuits were placed 170 centimetres apart. The primary consisted of 12 turns of thick wire, and the secondary of 560 turns of fine wire. The current was sent from one Daniell cell. The two ends of the secondary were connected to the two plates of the condenser, and when the pendulum fell it broke the primary current and started in the secondary circuit an oscillatory current reverberating to and fro in the secondary wire, the condenser acting as a resonator. At a definite interval after rupture of the primary, the condenser was separated and examined by a quadrant electrometer. The charge in the condenser showed the *phase* of the electrical oscillation existing at the instant of such separation. In one case Helmholtz observed 35 oscillations in $\frac{1}{80}$th of a second. In order to discover if any retardation took place with increased distance of the coil, it was necessary to fix attention upon some phase in the oscillations. The successive zero points of the current were very sharply defined, and suitable for this purpose. Helmholtz found that alteration of the distance between the primary and secondary coils made no perceptible difference in the position of the zero points, and that, as far as the apparatus he was using could detect, the velocity of the electro-magnetic impulse must be greater than 195 miles per second. He pointed out in this Paper that the commencement of the secondary current is not a sharply marked thing. The spark which takes place at break of the primary lasts an appreciable time, and all this time the primary is dying gradually, and the induced current therefore is increasing. The period of duration of the break spark may be something like $\frac{1}{15000}$th to $\frac{1}{46000}$th of a second, and is, therefore, a large fraction of the duration of a single electrical oscillation, which amounted to about $\frac{1}{2811}$th of a second. The duration of the break spark can be found by observation of the time which elapses from beginning of break up to the first zero point of the secondary current oscillations, as compared with the mean value of the duration of an oscillation. The interval up to the first zero point is the duration of the break spark *plus* the time of half a complete oscillation. The duration of the spark is never constant, and depends a good deal on the amount of platinum thrown off from the contacts each time. The average duration of the spark in Helmholtz's experiments was found to be about one-tenth of the whole period of an oscillation. Helmholtz also noticed in some earlier observations evidence of electrical oscillations set up in a flat spiral, one end of which was insulated. In this case some 45 oscillations were detected in the space of $\frac{1}{80}$th of a second. Henry also noticed that the time of subsidence of the current, when the circuit is broken by means of a surface of mercury, is very small, and probably does not much exceed the ten-thousandth part of a second. It has, however, a quite appreciable duration, for Henry found that the spark at ending presents the appearance of a band of light of considerable length when viewed in a mirror revolving at the rate of six hundred revolutions per second.

Bernstein, with the aid of a contact break somewhat different from that used by Blaserna, also examined the duration of the oscillations set up in a secondary coil. He found that the duration of the first oscillation at breaking primary was longer than that of the subsequent ones. The mean duration when using a single Grove cell in the primary circuit was ·0005 second, and when using a Daniell cell only ·0001 second. We shall return later to consider more recent researches on these electrical oscillations in inductive circuits and point out that they can only occur when some part of the circuit possesses sensible electrical *capacity*. In the case of a coil or bobbin of wire we have not only resistance and inductance, but measurable capacity present in the conductor.

**§ 8. Magnetic Screening and the Action of Metallic Masses in Induction Coils.**—At one stage* of his investigations Henry made the important discovery that, if a primary and secondary coil are separated by a metallic sheet, a notable decrease takes place in the intensity of the *shock* taken from the secondary circuit when a sudden discharge is passed through the primary, or continuous current started or stopped in the primary circuits. A thick copper plate was found more effective than a thin one in thus preventing the inductive effect of the primary upon the secondary coil. If a radial slit was cut in a circular metallic plate the annulling effect was altogether stopped. If the two edges of the gap (*see* Fig. 95) were furnished with wires leading to a magnetising spiral, Henry found he could in this way make evident the existence in the plate of a current induced by the action of the primary.

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* *Phil. Mag.*, Vol. XVI., 1840, p. 257.

A flat coil of insulated wire was substituted for the metal plate, and it was found that the screening action of this coil was only sensible when the two ends were joined so as to complete the circuit. This action, by which the induction of a primary coil on a secondary is prevented by the interposition of a metallic plate, cylinder, or closed circuit of insulated wire, is called *magnetic screening.* The elementary explanation of this effect is not difficult to arrive at. Suppose a small conducting circuit of resistance R to be placed in a magnetic field so that it is traversed normally by N lines of magnetic induction. Let the constant coefficient of self-induction of this circuit be L. If, then, in any small time $dt$ a variation of the lines of induction traversing this circuit takes place, the impressed

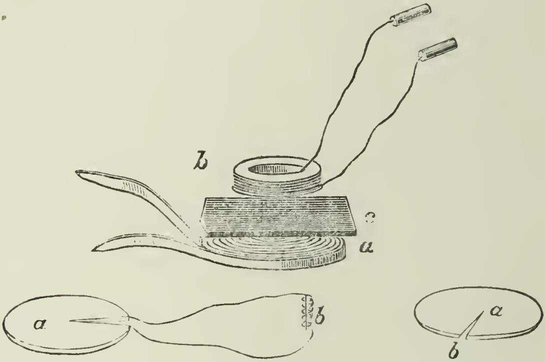

FIG. 95.

electromotive force on that circuit will be represented by $-\dfrac{dN}{dt}$, and if at that instant the current in the circuit is $i$, by the principles laid down in the last chapter the current equation will be

$$L\frac{di}{dt} + Ri = -\frac{dN}{dt},$$

or

$$\frac{d}{dt}\left(Li + N\right) + Ri = 0.$$

Suppose the conductivity of this circuit to be perfect, and R therefore zero, we have, by integration of the above equation, the result

$$Li + N = \text{const.};$$

in other words, the lines of induction L $i$, linked to the circuit at any instant due to the induced current generated in it, are *opposite* in direction to those whose variation is producing the current, and together with them make up a constant number. Hence, if the variation of N is such as to take lines of induction out of the circuit, the action of the current thereby induced is to add or increase them in the circuit at an equal rate. If we suppose our circuit to be a perfectly conducting metal plate, and just behind this metal plate there is another small closed circuit, then any variation of lines of induction passing through this plate will not take effect in producing any induced current in the small circuit, because the inductive action of the current induced in the plate nullifies, as far as the small circuit is concerned, any variation of the external field. It is clear that these conclusions would apply to any surface of finite extent which possessed perfect conductivity; the induced currents which any variation of the external field would produce in this surface would always be such that the induction through each portion would be kept constant—in other words, that the perpendicular component of the magnetic induction at each point on the surface would retain a fixed value. It follows that a closed surface of zero resistance is a complete screen for all points in the interior against the effects of variation of the field on conductors on the outside of the surface; these effects reduce to the production of surface currents in the shielding conductor, which keep the resultant field in the interior constant or at zero.

Faraday describes ("Exp. Researches," Vol. I., §1720 *et seq.*) an experiment which at first sight seems to disprove the fact of magnetic screening. He placed a flat copper wire spiral, which was in connection with a battery and key, between two other flat spirals which were respectively connected with the two coils of a differential galvanometer. The coils were so joined up that the inductive effect of a break and make of the battery circuit produced no movement of the galvanometer needle because it was subjected to two equal and opposite impulses from the two coils. When an exact balance was obtained a flat plate of copper, nearly three-quarters of an inch thick, was interposed between the primary spiral and one of the secondaries. The galvanometer needle was not, however, any more affected than if the copper was absent. To understand this we must bear in mind that the break or make of the primary current produces in the copper a secondary current, but as the effect of the primary coil on the secondary coil on that side is balanced by the other one we may regard the secondary coil next the copper plate as free to receive any inductive effect it can from the eddy current induced in the copper block. This secondary current induced in the copper generates a *tertiary* current in the secondary spiral, and this tertiary current consists, as we have seen, of a double short flux of electricity equal in quantity and opposite in sign. The galvanometer is then traversed by two small equal quantities of electricity in opposite directions, and as this does not sensibly affect a not very sensitive galvanometer no movement of the needle is seen. If, however, instead of the differential galvanometer, Faraday had used a differential telephone, he would have found distinct evidence of a screening action. Again, suppose that, instead of a simple make or break, Faraday had employed a steadily periodic or alternate current in the primary, this would have set up a steady periodic secondary current of equal frequency in the copper plate, and this again would have set up in the secondary coil on that side a steadily periodic tertiary current of equal period, and this might have been detected by the use of a sensitive differential electro-dynamometer or a soft iron needle galvanometer.

Henry found that a sheet of tinfoil afforded a very small amount of screening for shock, but a thick sheet of copper a very considerable one in the case of induction by battery currents, and in the case of induction by Leyden jar discharges the same phenomenon was apparent. In the case of an iron screen there is an additional effect, due to the fact that the iron, by its small magnetic resistance, conducts away the lines of induction somewhat through its mass, and prevents them from extending to the space on the other side. In this case also a considerable thickness of metal is necessary to bring about the effect of annulment. When we are limited to the use, as we are in practice, of materials whose conductivity is far from being perfect, it is found that a thin screen of metal hardly affords any sensible protection from inductive effect.

In other words, the field on the other side of the screen is very far from constant. This has been well demonstrated in certain investigations by Prof. D. E. Hughes in carrying on some highly valuable experimental researches into the means of preventing induction upon lateral telegraph wires.* It has many times been proposed to annul mutual induction between telegraph and telephone wires by covering them over with thin metal covering, which covering is kept "to earth." It is now known, and well exemplified in Prof. Hughes's experiments, that this shielding affords no protection when the covering is not very thick and when the rate of change of the currents is not very rapid. A gutta-percha wire was enclosed in ten coverings of tinfoil, and such arrangement was not found to afford protection to induction, as detected by a telephonic wire stretched alongside. Even when twenty coatings of thin charcoal iron were put round the wire, not only was there found to be a very sensible permanent field outside the iron, but changes of field were made manifest also. It is not to be taken that these experiments disprove the fact of magnetic screening, but only that the low conductivity of the envelopes used is ineffective at the speed of current change employed to render visible the effect of magnetic screening. It is different, however, if the inductive effects are being produced by a *very rapid rate* of change of field. For suppose that a small circuit, as before, is placed in a uniform field, and is traversed by $q$ lines of induction due to this external field. Suppose $q$ varies according to a simple periodic law, so that $q = Q \cos p t$, where $p = 2\pi n$, $n$ being the frequency of the alternations. Then we have

$$-\frac{dq}{dt} = Q p \sin p t;$$

but $-\frac{dq}{dt}$ is the value of the impressed electromotive force in the circuit, and if we call the current at any instant $i$, then, by the principles in Chap. III., we have

$$i = \frac{Q p}{\sqrt{R^2 + p^2 L^2}} \sin (p t - \theta),$$

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* *See* a Paper by Prof. Hughes "On Lateral Induction in Telegraph Wires," read before the Society of Telegraph Engineers, March 12, 1879. Published in *The Electrician*, March 22, 1879.

in which R is the resistance and L the inductance of the circuit, and

$$\theta = \tan^{-1}\frac{\mathrm{L}\,p}{\mathrm{R}}.$$

Suppose that R is very small compared with L $p$, which is the case when $n$ or the frequency of alternation is made very great, then R vanishes compared with L $p$, and if we call $i'$ the value towards which $i$ approximates in this case, we have

$$i' = -\frac{\mathrm{Q}}{\mathrm{L}}\cos p\,t,$$

and
$$\frac{d\,i'}{d\,t} = \frac{\mathrm{Q}}{\mathrm{L}}p\,\sin p\,t,$$

or
$$\mathrm{L}\frac{d\,i'}{d\,t} = \mathrm{Q}\,p\,\sin p\,t = -\frac{d\,q}{d\,t}.$$

Hence
$$\mathrm{L}\frac{d\,i'}{d\,t} = -\frac{d\,q}{d\,t},$$

or
$$\mathrm{L}\,i' + q = \text{constant}.$$

Hence the field due to the current in the circuit, together with the external field, is a constant quantity, and we get the condition of perfect shielding. We may sum up the foregoing by saying that, if a screen of absolutely no electrical resistance is interposed between a primary and secondary coil, it effects a perfect magnetic screening, whatever may be its thickness. If, on the other hand, the screen has a finite conductivity, then the screening will be very imperfect, unless a very great thickness of material is used, and the above will be true when the change of field or the change of primary current is a simple "make" and "break" or a slowly periodic change. When, however, the change of current in the primary is *very* rapidly periodic, then the screening effects of even imperfect conductors will make themselves felt, and a comparatively thin screen of metal will effect a nearly perfect shielding for induction. This theory is strikingly confirmed by some very beautiful experiments of Mr. Willoughby Smith, which are described in the *Journal of the Society of Telegraph Engineers* (November 8, 1883, Vol. XII., p. 458),* and entitled "Experiments on Volta-

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* See also *The Electrician*, November 17, 1883, p. 18.

Electric Induction." Mr. Willoughby Smith's apparatus consisted of two flat coils A and B (*see* Fig. 96), placed a certain distance apart. One of these was a primary coil connected with a battery, and the other was connected with a sensitive galvanometer. In the circuit of both were current reversers, which reversed the galvanometer and battery alternately, and hence made the opposite induced currents both affect the galvanometer in the same direction. This being arranged, the commutator was started so as to reverse the currents very slowly, and a sheet of copper interposed between the spirals. Under these circumstances the interposition of the copper produced but little effect. If, however, the commutator was driven at a very rapid rate the copper plate caused a marked diminution in the galvanometric deflection, and this diminution was greater in proportion as the speed was greater. In the original Paper a curve is given (Fig. 97) which shows the decrease in the galvanometer deflection, expressed as a percentage of the original undiminished deflection, corresponding to various speeds of reversal. It will be seen that the *less* the conductivity of the metal the *greater* must be the speed in order that the magnetic screening may approach perfection. Iron, of course, occupies an exceptional position. It cuts off, even at very low speed reversals, a large portion of the field, not by a true screening action, but by conducting away the lines of magnetic force and preventing their access to the secondary coil. It will be seen that at any given speed the order in which the metals reduce the deflection is the order of their electric conductivity, and that as far as the diagram goes the lines all (except iron) slope upward, indicating that at very

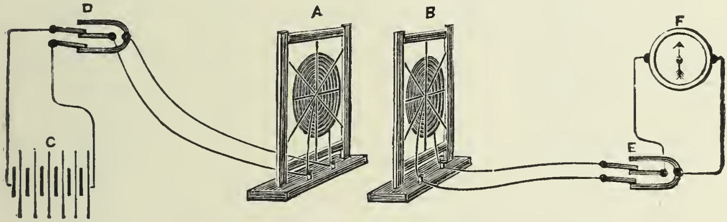

FIG. 96.

high speeds the screening of even the worst conductors will approach perfection. It would no doubt be found that, if the telephone were used as a detector, the magnetic screening of a copper plate or thin tinfoil sheet would become very manifest for high notes when not in any way marked or distinguishable for notes or sounds of low frequency of vibration.*

As far back as 1840 Dove had made experiments† on the effect of the introduction of cores of various materials into the primary circuit of an induction coil. His apparatus consisted of two similar primary bobbins wound on tubes of non-metallic substance and connected in series (Fig. 98). Over each primary bobbin was wound a secondary circuit, and these secondary circuits were connected in series, but so that the induction of the two primary bobbins operated in opposite directions and nullified on the whole secondary circuit each other's effect. Exact neutralisation was obtained by adjusting one of the secondaries. When this was the case, various cores of iron rods of different kinds were inserted in one primary bobbin, and it was found that the induction balance was destroyed.

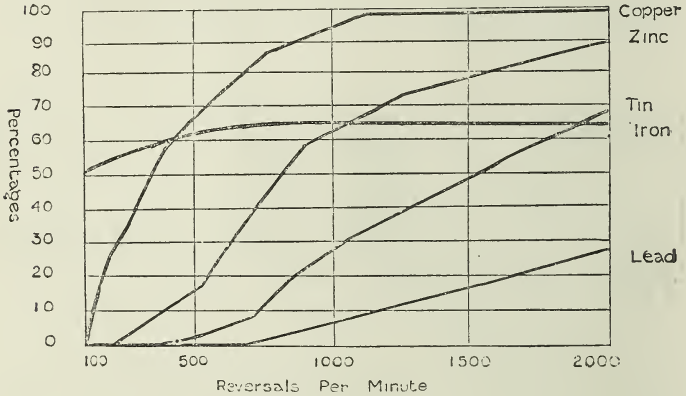

Fig. 97.

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* The above explanation of the cause of the difference between the screening of the different metals is not that given by the distinguished investigator, but it is the explanation which to the author seems most in accordance with known principles.

† Dove, Poggendorff's *Annalen*, Vol. XLIX., 1840.

By inserting iron wires of a certain size in the other core, balance could be again obtained, but not simultaneously, as estimated by the galvanometer and by the shock. Thus, with a bar of forged iron, 110 wires had to be inserted in the other coil to obtain an equilibrium, as estimated by the galvanometer; but, as far as could be judged by the shock, 15 wires were sufficient. With regard to different kinds of iron, experiment shows that if we class them according to galvanometric effect we obtain a different series to that at which we arrive when classifying them in the order in which they create sensation by shock. Thus grey rough cast-iron is the kind

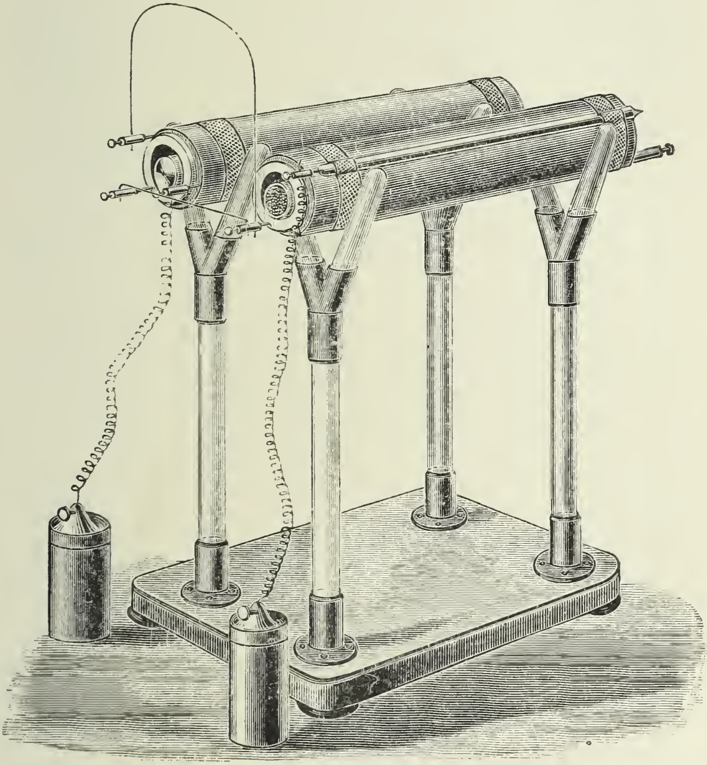

FIG. 93

which approached nearest to bundles of soft iron wire in respect of increasing the shock. Enclosing iron wires in a brass tube reduced the action of the wires in disturbing the inductive balance and rendered them very little better than a bar of solid iron. When the primary current was a discharge from a Leyden jar, Dove found that the physiological effect (shock) of the secondary current, as estimated, was reduced by the introduction into the primary bobbin of non-magnetic conducting cores; in other words, the introduction of a core of non-magnetic but highly conducting material into the primary bobbin reduced the power of a primary discharge to create a secondary discharge. These last results may be obtained in a more modern form by the substitution of a Bell telephone to detect the tertiary currents generated by the metal core.

Let a Bell telephone be connected in series with the secondary coil of a small induction coil, of which the primary is wound on a hollow bobbin and the frames are wholly of wood or non-metallic substance. A convenient form is that known as Du Bois-Reymond's sliding coils. Let an interrupter in the primary circuit make and break the circuit rapidly. This being so, the telephone emits a steady rattle or hum. If a massive copper rod is introduced into the primary bobbin as a core, the telephonic rattle is more or less suppressed; if a core of soft iron wire is introduced the noise is increased; if a core of solid iron or steel is used the noise may be increased, but not so much as when the divided iron is used. The explanation of the exalting effect of the soft iron wire is simple. The presence of the iron reduces the reluctance of the magnetic circuit. More lines of induction therefore flow through the secondary circuit, and hence the strength of the secondary current is increased, and the mean rate of change of induction through it is also increased. The diminishing effect of the copper core is explicable in the light of the knowledge that in such a conducting core the primary current generates induced currents, and these in their turn re-act upon the secondary circuit, inducing in it a tertiary current. The directions of the currents induced by the primary in the solid core and in the secondary circuit are the same. The direction, however, of the first half of the tertiary current developed in the secondary by the current in the copper core is opposite to the direction of the current developed in the secondary by the action of the primary. Hence it results that the current in the secondary circuit is more or less wiped out by the opposing inductions due to the primary circuit and the currents induced in the copper core. Otherwise the operation might be regarded thus:—Suppose the primary circuit to be traversed by a periodic current creating a simple periodic flux of induction through the copper core. As we have seen, under the head of magnetic screening, this variation of induction would induce currents in the copper core, which would themselves generate a flux of induction which would, if the conductivity of the core were perfect, or the rapidity of change of induction infinite, be exactly equal and opposite at each instant to the flux of induction producing those currents.

If the conductivity is not quite perfect, or the rate of variation not very great, yet nevertheless the direction of the field of magnetic force inside the copper, due to the currents induced in its mass, will more or less oppose the field of force at every instant which is by its fluctuations generating those currents. If the thick line 1 1 1 in Fig. 99 represents the sinusoidal or simple periodic change of induction or magnetic field in the interior of the copper, due to the primary helix, and if the dotted line 2 2 represents roughly the changing field due to the eddy currents generated in the core, which are *nearly* 180° behind the primary in phase, the integral or sum of both superimposed fields represented by 3 3 at any instant is less than the original one due to the primary alone at the corresponding instant. Also the *mean rate of change* of the resultant field is less, and the secondary circuit experiences at every instant a less inductive electromotive force. The same reasoning which we have employed in the case of magnetic

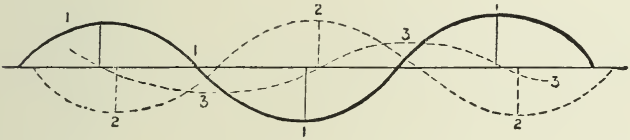

Fig. 99.

shielding applies here, and the differences in the reducing effect of cores of various metals would be found to be less at high speeds of alternation than at low. In some small induction coils used for medical purposes the strength of the secondary current is graduated by drawing in or out of the primary coil a copper tube which slips over the bundle of fine iron wires used as a core. The *rationale* of the action of this copper tube in so operating is in a general way to be found in the principles laid down above.

When Henry obtained possession of the "Experimental Researches" of Faraday, as detailed in the fourteenth series of his "Experimental Researches," he was exercised in his mind to reconcile the results obtained by Faraday on the interposition of metallic screens between inducing and induced circuits with his own. Faraday had found that when the galvanometer was used as a current finder "it makes not the least difference" whether the space between the primary and secondary coils was air, sulphur, shellac, or such conducting bodies as copper and other non-magnetic metals. On the other hand, Henry found that a shock from a secondary coil which would paralyse the arms was so much reduced by the interposition of a metallic plate as hardly to be sensible on the tongue. Here was evidently something to be explained, and in a long memoir (*Phil. Mag.*, Series 3, Vol. XVIII., 1841, p. 492; also *Transactions of the American Philosophical Society*, Vol. VIII., 1840) Henry examined this and other matters. He first verified Faraday's experience by attaching the ends of a secondary coil to a galvanometer and bringing up suddenly towards it a permanent magnet, or a coil traversed by a steady current. The swing of the galvanometer was found to be quite unaffected in extent by the interposition of a plate of copper. Again, in place of the copper plate, a closed metallic conductor (an endless coil) was employed, but whether the circuit of this coil was open or closed it made not the slightest difference on the galvanometer deflection.

Forty feet of copper wire, covered with silk, were wound on a short cylinder of stiff paper, and into this was inserted a hollow cylinder of sheet copper, and into this again a rod of soft iron. When the latter was rendered magnetic, by suddenly bringing in contact with its two ends the different poles of two magnets, a current was generated in the wire, but the strength of this current, as measured in the galvanometer, was the same whether the copper cylinder was present or was removed. Henry then noticed that there was one element of difference between the indications of a galvanometer and that of the magnetising spiral. If the two secondary currents at "break" and "make" of a primary were sent through a magnetising spiral and through a galvanometer, the arrangement might be such that the induced current at "make" of the primary was unable to give any sensible magnetisation to the steel needle enclosed in the spiral, but at "break" was able to magnetise it to saturation. Nevertheless, in both cases the "throw" of the galvanometer was the same. Similarly with the degree of shock felt, the galvanometer indications being alike for the inverse and direct induced current; yet that induced current gave the greatest shock which was able to produce the greatest magnetisation. The explanation of these facts became clear as soon as it was seen that the deflections of the galvanometer depended upon the whole quantity of the discharge, and must necessarily be alike for the inverse and for the direct current, but that the magnetising effect and the physiological shock depended upon the maximum value of the instantaneous discharge current, and might therefore be very different for the two induced currents. It was then evident that any actions by which this maximum value of an induced current was decreased, whilst its duration was increased and total quantity left unaltered, would result in rendering this current less easily detectable by shock or magnetisation, but make no difference in its effect on a galvanometer. Aided by this thought, he repeated Faraday's experiment with the balanced coils referred to in § 8 ("Experimental Researches," Vol. I., § 1,790 *et seq.*). A galvanometer was provided having two equal wires of the same length and thickness wound on the same frame, and also a double magnetising spiral was prepared by winding two equal wires round the same piece of hollow straw. Coil No. 1, connected with a battery, was supported perpendicularly on the table, and coils Nos. 3 and 4 were placed parallel, one on each side, and each coil connected in series with one coil of the differential galvanometer and with one spiral of the magnetising helix. The two outside coils were then adjusted so that when the battery circuit was made and broken, and the current started and stopped in the middle coil, no indication was given by the galvanometer, and no magnetisation produced in a steel needle placed in the double helix. A thick zinc plate was then introduced between the primary coil and one of the secondaries, and it was found that the needle of the galvanometer still remained stationary on making and breaking the primary current, but that the steel needle in the spiral became powerfully magnetic. This indicated that the two secondary currents, whilst still equal in total quantity, had been so affected that one had a less maximum value than the other, and hence a differential magnetising action was produced. A similar effect was observed when a galvanometer and magnetising spiral were together introduced into the secondary circuit of a single primary and secondary circuit. The interposition of a metal sheet considerably reduced the magnetising power or the shock, but left the galvanometer deflection unaltered. In order to increase the number of facts, this last experiment was varied by the exchange of a soft iron needle for the hard steel needle in the magnetising coil, the metal screen being interposed in each case, and it was found that whereas the metal screen cut off almost entirely the power of the secondary current to magnetise hard steel, it could yet slightly magnetise the soft iron. A screen of cast iron half an inch thick, however, not only neutralised the power to magnetise hard steel, but reduced the deflection of the galvanometer as well. The general explanation of the foregoing facts, as due to Henry, is as follows :—The secondary current, as we have seen, is a brief discharge, which rises very suddenly to its maximum value and then fades gradually away. The current curve of the secondary current, due to the rupture of a primary circuit, may be represented by the thick firm line in Fig. 100. If a metal screen is interposed between the primary and the secondary circuit the screen gets a similar secondary current generated in it, and this last again acts by induction to generate a tertiary current in the secondary circuit. This tertiary current consists of two portions—first, an inverse part opposite in direction to the secondary current in the screen, and, secondly, a succeeding direct current. Let the current curve of this tertiary current in the secondary circuit be represented by the fine firm line in Fig. 100. The total quantities of electricity flowing in each part of the two portions of the tertiary current are equal. The resultant effect, then, of the action of the primary current when interrupted is to cause in the secondary circuit the true secondary current, which is an unidirectional flux (thick curve), and a superimposed tertiary current, which is a bi-directional flux, its algebraic total of quantity being zero.

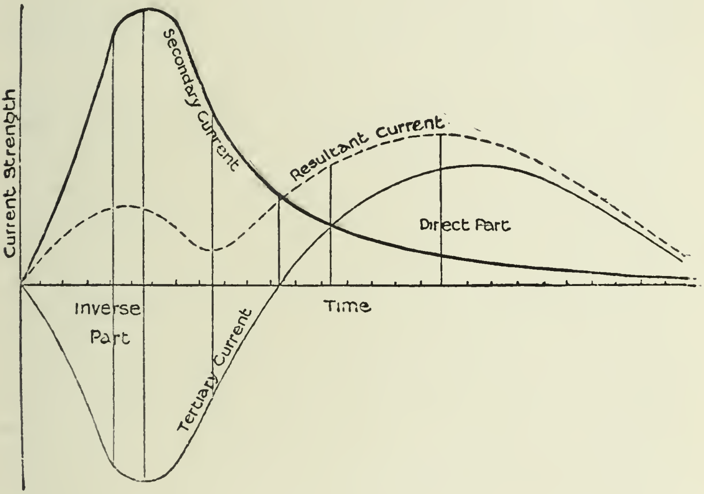

Fig. 100.

If we add together at each instant the ordinates of the two current curves we get a resultant curve (dotted line) which represents the actual current curve in the secondary circuit. The total area (electric quantity) enclosed between the horizontal line and the dotted curve must be equal to the total area enclosed between the thick firm line and the horizontal, because we have added and substracted equal areas; but the *maximum* ordinate of the dotted curve will be less than that of the thick firm line curve, and the *form* of the curve will be very different also. It is, then, clear that the superposition of a complete tertiary current, which is of itself but very little able to affect a galvanometer on a secondary current which gives a definite galvanometer indication, is not able to alter that galvanometer deflection, depending as it does on the total quantity of the discharge. The magnetising power and shock, however, depend upon the maximum value or suddenness with which the induced current rises to its maximum value, and this factor is very much affected by the overlaying of a secondary current by a tertiary. We see, then, that the experiences of Faraday and Henry may be completely reconciled, and that the detection of magnetic screening depends upon the nature of the detecting instrument in the secondary circuit.

The practical outcome of much of the foregoing discussion of magnetic screening is that the use of lead-covered cable for the conveyance of periodic currents of the usual frequency (60 or 100 alternations per second) is of *no advantage* in respect of prevention of inductive disturbance in neighbouring telephone wires. Not only is the lead too poor a conductor, but the frequency of alternation is too small to render the magnetic screening effective. The only effective method of annulling the inductive disturbance is to carry the periodic current along a conductor which lies in the axis of, and is insulated from, a concentric enclosing tube or sheath, which acts as a return. This return must be itself insulated from the earth, and the condition to be fulfilled is that at any instant, and at any section the algebraic sum of the currents in the core and sheath must be zero; reckoning current in one direction positive, and in the other negative.

The whole question of magnetic screening has been worked out mathematically by several mathematicians, and besides the section in Clerk-Maxwell's Treatise (Vol. II. § 654, 2nd Ed.), the advanced student may be referred to memoirs by Prof. Charles Niven "On the Induction of Electric Currents in Infinite Plates and Spherical Shells" (*Phil. Trans.* Roy. Soc., 1881, p. 307), and also to Prof. H. Lamb "On Electrical Motions in a Spherical Conductor" (*Phil. Trans.* Roy. Soc., 1883, p. 519).

§ 9. **Reaction of a Closed Secondary Circuit on the Primary.**—If a Bell telephone is placed in series with a coil of many turns of fine wire wound on a hollow bobbin, and if both are placed in series with the secondary circuit of a small induction coil, the strength of the secondary current can be so adjusted that the telephone emits a low murmur or rattle. This being the case, let a solid bar of copper be introduced into the bobbin of fine wire, and it will be found that the noise of the telephone is *increased.* If a bundle of fine iron wires is substituted for the copper rod it will, on the other hand, *reduce* the noise or stop it altogether. The explanation of this effect is to be found in the reaction which a closed secondary circuit has upon its primary in changing the resultant impedance of the primary. We have shown in Chapter III. (p. 180), that the re-active effect of the secondary is to increase the resistance and reduce the inductance of the primary circuit, and we have deduced two formulæ given by Maxwell for the value of the equivalent resistance R′ and the equivalent inductance L′ of a primary coil of resistance R and inductance L in the presence of a secondary coil of resistance S and inductance N, the magnetic circuit having a constant resistance, and the mutual inductance being M. Hence, the equivalent impedance of the primary coil in presence of the secondary is $\sqrt{R'^2+p^2L'^2}$, and that which we may call its isolated or intrinsic impedance is equal to $\sqrt{R^2+p^2L^2}$. For brevity we may write the symbol Im for $\sqrt{R^2+p^2L^2}$ and Im′ for $\sqrt{R'^2+p^2L'^2}$, also $\text{Im}_2$ for $\sqrt{S^2+p^2N^2}$. The question then arises, which is the greater—Im′ or Im? To discover this, take for R′ and L′ the values given on page 180, and we have

$$R' = R + \frac{p^2M^2S}{S^2+p^2N^2},$$

and

$$L' = L - \frac{p^2M^2N}{S^2+p^2N^2}.$$

Forming from these the function $R'^2+p^2L'^2$, we have

$$R'^2+p^2L'^2 = R^2+p^2L^2+\left[\frac{2p^2M^2RS-p^4M^2(2LN-M^2)}{S^2+p^2N^2}\right]$$

or $$(\mathrm{Im}')^2 - (\mathrm{Im})^2 = \frac{2\,p^2\,M^2\,R\,S}{S^2 + p^2\,N^2} - \frac{p^4\,M^2\,(2\,L\,N - M^2)}{S^2 + p^2\,N^2};$$

or $$(\mathrm{Im}')^2 = (\mathrm{Im})^2 - \frac{p^2\,M^2}{(\mathrm{Im}_2)^2}\{p^2\,(2\,L\,N - M^2) - 2\,R\,S\}.$$

If $S = \infty$, or the secondary circuit is open, the right-hand side of the above equation is zero, and we find that the impedance of the primary circuit is not altered by the presence of the open secondary, as of course it should not be.

If S is not infinite, that is if the secondary circuit is closed, then the above equation shows us that, if the quantity $2\,R\,S$ is greater than the quantity $p^2\,(2\,L\,N - M^2)$, then Im′ is greater than Im, or the impedance of the primary circuit is *increased* by closing the secondary. But if $2\,R\,S$ is less than $p^2\,(2\,L\,N - M^2)$, then Im′ is less than Im, or the impedance of the primary is *decreased* by closing the secondary circuit.

If $\alpha_1$ stands for $\frac{p\,L}{R}$, and $\alpha_2$ for $\frac{p\,S}{N}$, and also if $\beta$ stands for $\frac{M}{\sqrt{L\,N}}$, it is not difficult to show* that to make Im′ greater than Im we must have

$$\alpha_1\,\alpha_2 \text{ less than } \frac{2}{2 - \beta^2}.$$

When the secondary circuit has a certain critical value it is possible to show experimentally that above this value closing the secondary circuit increases the primary impedance, but below this value closing the secondary circuit decreases the primary impedance.

The following experiment was made in the laboratory of Prof. Elihu Thomson †:—A small induction coil had its primary circuit arranged in series with nine incandescent lamps joined in parallel, thus exciting it with an alternating current of about ten amperes. When the secondary circuit was closed by means of a vacuum tube of high resistance, a marked *fall* occurred in the candle-power of the lamps used as a resistance in the primary circuit. The impedance of the primary was thus increased. When the secondary circuit was closed through a water resistance, the lamps brightened up, thus showing that the primary impedance was decreased.

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* *See* Mr. E. C. Rimington, "On the Behaviour of an Air Core Transformer when the Frequency is Below a Certain Critical Value," *Proc.* Physical Soc., London, October 27, 1893.

† See *The Electrician*, Vol. XXXII., p. 225.

Hence the closing of the secondary circuit does not always decrease the primary impedance. Mr. Rimington (*loc. cit.*) quotes an experiment with an air core transformer or induction coil consisting of two circuits without iron core, in which closing the secondary circuit had the effect of decreasing the primary current by about 3 per cent., thus showing an increased primary impedance. Generally speaking, however, the closing of the secondary circuit so that the total secondary circuit resistance is small has the effect of decreasing the primary circuit impedance.

Hence also holding a conductor or conducting circuit of low resistance near a primary coil has the effect of decreasing the impedance of that coil and therefore increasing the flow of primary current through it under the influence of a constant impressed primary electromotive force.

The explanation of our experiment with the induction coil and the copper rod is now simple. The introduction of the copper rod into the fine wire helix is equivalent to approximating to a primary coil a closed secondary circuit. The impedance of the fine wire circuit to the alternating current from the secondary circuit of the induction coil is hence reduced; it gets more current, and the telephone is made to emit a louder sound. If, however, a core of divided fine iron wire is introduced into the fine wire helix, the result is simply to increase the impedance of that circuit, and therefore to reduce the current actuating the telephone. When considering in particular the theory of the induction transformer as applied to electric distribution we shall see the above principles have important practical bearings.

In a Paper recording some experimental results on the self-induction and resistance of compound conductors* Lord Rayleigh has given some comparisons of the results of theory and experiment on Maxwell's formulæ above alluded to. By the use of a resistance and inductance bridge, very similar to one designed by Prof. Hughes, the measurements of the inductance and resistance of a circuit can be made separately with ease. A pair of wires was wound on one bobbin; each wire had a resistance of nearly ·1 ohm, and a diameter of ·037in. Each coil consisted of nine double convolutions. In certain arbitrary units the resistance of one of these copper wires to steady currents was 1·75, and its inductance 11·2°. These values were obtained when the other coil was on open circuit. On closing the unused coil, the resistance of the first rose to 2·67 and its inductance fell to 4·7°.

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* See *Phil. Mag.*, December, 1886, p. 469.

To compare this with the theory.

The formulæ are

$$R' = R + \frac{p^2 M^2 S}{S^2 + p^2 N^2}$$

$$L' = L - \frac{p^2 M^2 N}{S^2 + p^2 N^2}.$$

Now $R = S = 1{\cdot}75 \times {\cdot}0492 \times 10^9$ absolute C.-G.-S. units of resistance,

and $L = N = 11^\circ{\cdot}2 \times 1553$ centimetres,
$M = 11^\circ \times 1553$ centimetres,
and $p = 2\pi n = 2 \times 3{\cdot}1415 \times 1050$.

The periodic current used had a frequency of 1050 per second;

hence

$$\frac{p^2 M^2}{R^2 + p^2 L^2} = {\cdot}6.$$

Therefore $R' = R\,(1 + {\cdot}6) = 1{\cdot}6\,R$,
and $L' = L\,(1 - {\cdot}6) = {\cdot}4\,L$;
but $1{\cdot}6 \times 1{\cdot}75 = 2{\cdot}8 = R'$,
and $\cdot 4 \times 11^\circ{\cdot}2 = 4^\circ{\cdot}5 = L'$.

These calculated values compare very favourably with the observed values, viz.:

$$R'' = 2{\cdot}67,\ L' = 4{\cdot}7^\circ,$$

and experimentally confirm the truth of Maxwell's formulæ for the increased resistance and diminished inductance of a circuit when placed near a closed secondary circuit.

§ 10. **Hughes's Induction Balance and Sonometer.**—In 1879 Prof. Hughes constructed and described a very perfect induction balance, with which he was able to conduct researches of an exceedingly interesting character. In order to have a perfect induction balance he found it necessary to make all the four coils exactly similar.* Four boxwood bobbins (*see* Fig. 101) are each wound over with 100 metres of No. 32 copper wire. These coils are arranged in pairs at a considerable distance apart, so that the coefficient of mutual induction between the separated pairs is negligible. Two of the coils, A and B, are joined in series with each other and with a battery and interrupter I, and the other two coils, C and D, are employed respectively as secondary coils to these two. These secondary coils are in series with each other and with a telephone receiver T, and are so joined up that the direction of the induction of A on C is opposite to that of B on D. One pair of coils is placed in a fixed position, and the other pair can be slightly moved to or from each other by means of a micrometer screw. The coils are first adjusted so that the inductions are equal and opposite, and on listening at the telephone the opposing secondary currents produce at best but a very slight sound, which can be perfectly abolished by adjusting the distance of one pair of coils. When this is the case, if we insert in the opening of the bobbin of one of the primary coils a disc or piece of metal *d*, the balance is destroyed, and we hear sounds more or less intense. In order to get some comparative measurements, Prof. Hughes designed

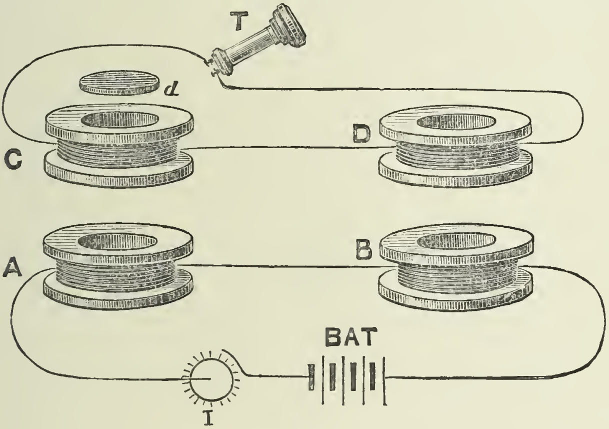

Fig. 101.

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* "On an Induction Current Balance." By Prof. D. E. Hughes. *Proc. Roy. Soc.*, No. 196, May 5, 1879.

a companion instrument, called a *sonometer*. In this instrument a pair of primary coils are, as before (*see* Fig. 102), joined in series with each other and with a battery. The coils are fixed at the extremities of a bar. Between these primary coils slides a single secondary coil, and the primary coils are so wound that their inductions on this secondary coil are equal and opposite. When this secondary coil is exactly between the two primary coils, a telephone placed in series with the secondary coil gives out no sound when the primary current is rapidly interrupted. If, however, the secondary coil is slid from one primary and towards the other, the differential action creates an induced current detected by the telephone. By reading off on the bar the extent of displacement necessary to create in the telephone a sound of a certain magnitude an arbitrary reading can be obtained corresponding to every different value of the secondary current. A switch is provided, by means of which the same telephone can be shifted rapidly from the induction balance secondary circuit to the sonometer secondary circuit. The experiments first performed consisted in placing within one primary coil of the induction balance certain equal-sized discs of different metals, and then so arranging the sonometer secondary coil that the noise in the telephone produced by the current in the secondary of the sonometer was judged by the ear to be equal to the sound produced in the telephone when it was shifted to the secondary circuit of the induction balance, and in which the inductive balance had been broken

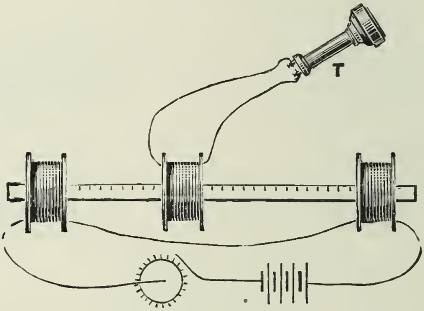

Fig. 102.

down by the insertion of the disc of metal. Discs of various metals the size and shape of an English shilling were made, and, when inserted in the induction coil, the sonometer bar readings, reckoned from the centre or absolute zero of sound given in certain arbitrary degrees, were as follows:—

| | | | |
|---|---|---|---|
| Silver (chemically pure) | 125 | German Silver | 50 |
| Gold | 117 | Iron (pure) | 40 |
| Silver coin | 115 | Copper (alloy) | 40 |
| Aluminium | 112 | Lead | 38 |
| Copper | 100 | Antimony | 35 |
| Zinc | 80 | Mercury | 30 |
| Bronze | 76 | Bismuth | 10 |
| Tin | 74 | Zinc (alloy) | 6 |
| Iron (ordinary) | 52 | Carbon | 2 |

This list does not agree in order entirely with that of any of the lists of electrical conductivity. In some degree it evidently has reference to conductivity, because, roughly speaking, the best conductors come at the top and the worst at the bottom; but whilst it is headed by silver, which has the highest conductivity per unit of volume, we find aluminium, which has the highest conductivity per unit of mass, occupying a position above that of copper. The disturbing effect of the metal on the inductive balance is not, however, simply proportional either to the conductivity per unit of mass or per unit of volume. In more recent experiments a graduated zinc wedge pushed in more or less between one pair of coils of the induction balance was employed to obtain comparative numbers representing the disturbance produced when discs of various metals are inserted in the other coil. The elementary theory of the induction balance is of course contained in all that has gone before in this and the last chapter. It is, generally speaking, dependent for its action on effects similar to those producing magnetic shielding. If the discs are slit so as to prevent circumferential electric currents in their mass, their action in disturbing the inductive balance is mitigated or annulled. If the metal disc is replaced by a copper coil with open extremities no effect is observed on the inductive balance. If the ends of the coil are joined, the coil behaves as if it were a metallic disc and causes loud sounds in the telephone. The effect due to the iron disc is a mixed one. It in part acts like any other metal disc, but it differs from them in one respect. If any non-magnetic disc is placed edgeways in the centre of the primary bobbin it has a diminished effect in disturbing the balance; in the case of iron the disturbance is increased by turning the disc edgeways. In order to have before us a typically simple case, imagine an induction balance made of two very long primary helices and each embraced near the centre by a small secondary coil. Let the primary coils be traversed by a simple periodic current. We have then in the interior of the primary coil a uniform magnetic field varying synchronously with the primary current in a simple periodic manner, and the rate of change of the magnetic field at any instant will be a measure of the electromotive force acting in the secondary circuit. Suppose into one primary helix is inserted a thin copper tube; this will form a closed secondary circuit, and secondary periodic currents will be induced in it, flowing round the cylinder in directions parallel to the turns of the primary helix. As this copper cylinder possesses a very sensible *time constant*, the phase of these secondary currents in the copper cylinder will be nearly opposite to that of the primary current. The resultant magnetic field in the interior of the cylinder is therefore that due to the resultant of these two simple periodic currents which are nearly opposed in phase. Hence the absolute magnitude of the interior field and its rate of variation will be less than if the copper cylinder was removed. It results, therefore, that the induction through the secondary helix and the electromotive force impressed on it will be diminished by the presence in the primary coil of this copper cylinder. The diagram given on page 177, showing a geometrical construction for the magnitudes of the primary and secondary currents in an induction coil without iron, shows us why the primary and secondary currents are thus more or less opposite in phase. Since, in a general way, the higher the conductivity of the tube or disc introduced into the primary the greater the time constant, and the greater the lag in phase of the currents induced in this metallic circuit behind the phase of the inducing primary, it follows that the resultant interior field acting to produce inductive electromotive force in the secondary helix will be diminished by the introduction of discs of very high conductivity more than by discs of very low conductivity.

From the principles discussed under the head of magnetic shielding it would appear that the differences between various metals inserted as discs in the induction balance would be *less* marked at very high speeds of interruption than at very low ones. With respect to the action of iron, two effects have to be considered which are the results of very different actions. The introduction of the iron into the primary coil reduces the magnetic resistance of the circuit of induction of that coil, and this cause, if it operated alone, would destroy the inductive balance by *raising* the inductive electromotive force in that secondary circuit corresponding to the primary into which the iron is introduced; but the iron disc, like every other disc, gets circumferential induced currents created in it, and these, if they acted alone, would destroy the inductive balance by *lowering* the inductive electromotive force in that secondary coil.

These two effects conflict, and it is an interesting confirmation of theory to find that Prof. Hughes says it is possible to introduce into one primary coil of the induction balance a disc of iron and some soft iron wires in such positions that these opposite actions nullify each other, and, though each mass of iron separately would destroy the induction balance, yet the two together being introduced complete silence in the telephone is the result. The sensibility of the induction balance to minute differences of electric conductivity and magnetic permeability is very remarkable. If into one coil of a carefully-adjusted balance we place a good sovereign, or shilling, and into the other a bad one, the telephone detects the base coin with unerring certainty by the loud noise given out. In the same way, if two pieces of soft iron are introduced into the two primary coils, and a balance is obtained, the mere magnetisation of one of them will be at once detected, because that magnetised piece becomes thereby less permeable, and destroys the balance. We may present the rough general theory of the induction balance in another way. Let the " coin " be simply regarded as a closed circuit, between which and the primary circuit surrounding it there is a certain coefficient of mutual induction. The two primary coils forming one primary circuit have, on the whole, no action on the two secondary coils forming one secondary circuit, and we may therefore consider the primary circuit as if it were in a position *conjugate* to the secondary. The coin, however, is acted upon inductively by the primary circuit, and the eddy currents or secondary currents generated in it react on the secondary circuit, causing in it tertiary currents, which affect the telephone. Looking at it from this point of view, we might construct an induction balance thus. Let A (*see* Fig. 103) be a single primary coil, and B a secondary coil, having a telephone in series with it. Place the coil B in a position conjugate to A—that is, with its axis at right angles to that of A. Then let variation of current in A produce no current in B. Now hold a sheet of copper anywhere, say at C, and the telephone will be caused to sound. For A, though it cannot affect B inductively directly, yet it can produce a secondary current in C held at a non-conjugate position, and these secondary currents in C will create other tertiary currents in B. The experiment thus appears to indicate a sort of *reflection* of inductive power.*

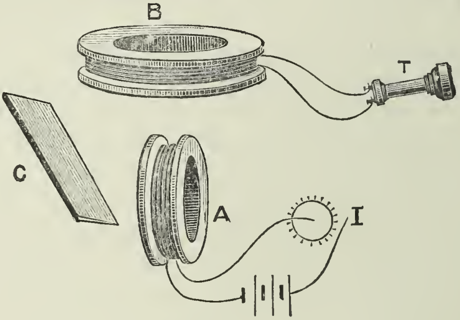

Fig. 103.

This was experimentally shown by Mr. Willoughby Smith in his Paper on "Volta-Electric Induction" (see *Journal of Society of Telegraph-Engineers*, Vol. XII., page 465).

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* The full theory of the induction balance has been given by Prof. Oliver Lodge. See *Proc. Phys. Soc. London*, Vol. III., p. 187. Also in the same volume is a Note by Prof. J. H. Poynting "On the Graduation of the Sonometer."

An interesting experiment due to Mr. Willoughby Smith is to employ a simple Bell telephone receiver, unconnected with any circuit, as an induction finder. If a coil of wire is traversed by an electric current, either rapidly intermittent or alternating, then a Bell telephone held anywhere in the magnetic field emits a sound. The pulsating field disturbs the magnetism of the telephone magnet, and enables us, therefore, to detect rapid electromagnetic disturbances at the place where it is held. It is obvious, then, that the induction balance, combined with a telephone, is an apparatus of extreme sensitiveness. It can render evident the smallest differences of weight, nature, degree of purity or temperature of two conductors of identical dimensions, such as two coins placed in identical conditions in respect of the two systems of coils.

It enables us to detect very small masses of metal in a badly conducting body, and may be employed with much advantage in verifying the insulation of the different windings of a coil, the ends of which are open. At the same time, however, it lends itself better to qualitative than to quantitative work, as it is difficult to interpret rigorously the results obtained.*

§ **11. The Transmission of Rapidly Intermittent or Alternating Currents through Conductors.**—Some experiments by Prof. Hughes in 1886 on the self-induction of metallic wires were the means of directing the general attention more closely than before to the nature of the propagation of electric currents of high frequency through metallic conductors, and although mathematical writers, particularly Maxwell and Oliver Heaviside, had previously considered the problem theoretically, the experimental results drew the attention of many to this question to whom the more recondite mathematical investigations were unknown. Prof. Hughes's experiments* on the self-induction of metallic wires were made with a combined resistance and induction bridge of somewhat novel form. Suppose that a quadrilateral arrangement be formed of four conductors P, Q, R, S, only one of which, P, has any sensible self-induction, and let the diagonals be completed by a telephone T, and battery B, with interrupter I. In the first place, let the resistance-balance be obtained for steady currents. This can be achieved by placing the telephone with the interrupter as a conjugate circuit to the battery (*see* Fig. 104), and altering one resistance, say, R, until a balance is obtained. By a suitable

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* For further information on the use and theory of the induction balance the student may consult, with advantage, Mascart and Joubert's "Electricity," Vol. II., § 986 ; also Hughes, *Phil. Mag.* [5], Vol. II., p. 50, 1879. On the differential telephone, *see* Chrystal, *Phil. Trans. Roy. Soc. Edin.*, Vol. XXIX., p. 609, 1880. O. Lodge, *Proc. Physical Soc. London*, Vol. III., p. 187, on intermittent currents and the theory of the induction balance.

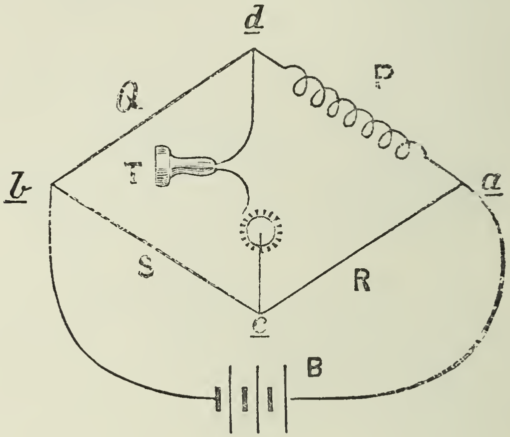

Fig. 104.

adjustment of the four resistances complete silence can be obtained in the telephone.

Next, let the interrupter be removed to the battery circuit, all the other arrangements remaining the same (*see* Fig. 105). It will be found that the balance is destroyed, and that no mere change in the value of the resistance R will enable a perfect balance to be obtained. The reason for this is that, on closing the battery circuit, the inductance of P introduces a counter electromotive force into P and the potential rises at *c* faster than at *d*, and on breaking the circuit the potential at *c* dies down faster than at *d*; and hence at each make and break the telephone is subjected to an alternate flux of current which causes it to emit a sound. Supposing that an attempt is made to get rid of this sound by shifting the point *c* so as to alter R, the steady balance will be destroyed, and the telephone will be traversed by a current during the time when all the currents have become steady; but no such change in the value of R will prevent a *variation* of current taking place through the telephone during the complete period from the first instant when the battery circuit is closed to the instant when it is opened again.

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* These experiments formed the subject of Prof. Hughes's Inaugural Discourse to the Society of Telegraph-Engineers on the occasion of his election to the office of President. See *Journal of the Society of Telegraph Engineers*, January 28, 1886, "The Self-Induction of an Electric Current in Relation to the Nature and Form of its Conductor."

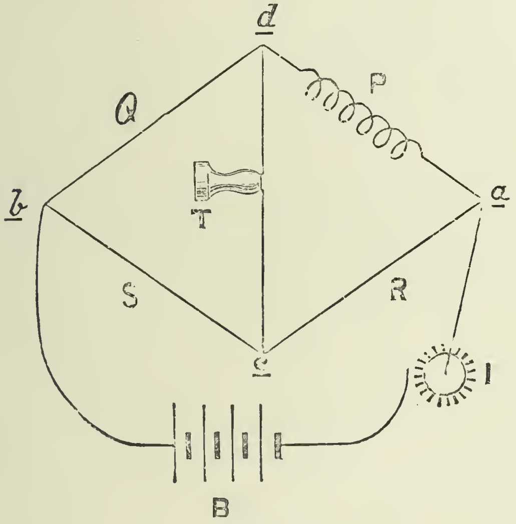

Fig. 105.

The only way in which a balance can be obtained in this last arrangement is by introducing into the telephone circuit an electromotive force which shall be capable of being made to balance at every instant the inductive electromotive force due to the inductance of P. Prof. Hughes does this very ingeniously by introducing a pair of mutually inductive coils into the battery and telephone circuits, and the final arrangement is as shown in Fig. 106. $M_1$ and $M_2$ are a pair of coils, one of which, $M_2$, is in the battery circuit and is fixed, and the other, $M_1$, is in the telephone circuit, and can be placed so that, whilst its centre coincides with that of $M_2$, its axis makes any required angle with that of $M_1$. In this way the mutual inductance between $M_1$ and $M_2$ can be varied from zero when the coil axes are at right angles to a definite maximum value when they are co-linear.

It is found that, when the coils $M_1$ $M_2$ are in certain positions, the inductive electromotive force set up in the telephone circuit by the induction of $M_1$ on $M_2$ can be made to neutralise the electromotive force of self-induction due to the inductance of P, when, in addition, a certain value is given to the resistance R. Under these circumstances the bridge can be balanced and the telephone completely silenced, both when the interrupter is in the battery circuit and also in the telephone circuit; in other words, the bridge can be balanced both for steady and for variable currents.

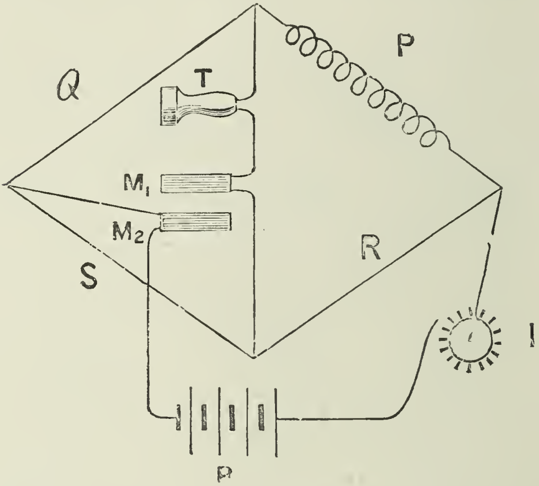

Fig. 106.

In the arrangement adopted by Prof. Hughes the resistances Q, R, and S, were sections of one and the same fine

German silver, 1 metre long, and having a total resistance of 4 ohms (*see* Fig. 107). The ends of this wire were joined to the conductor P under investigation, and the rest of the apparatus was arranged as described.

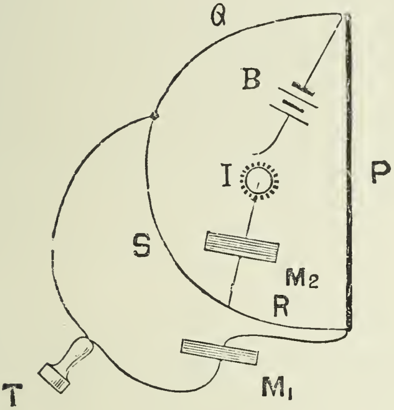

FIG. 107.

In order to investigate the relation between the resistances and inductances which holds good when the bridge is balanced for steady and also for variable currents, a diagram must be drawn (Fig. 108) representing the network of conductors. Then call

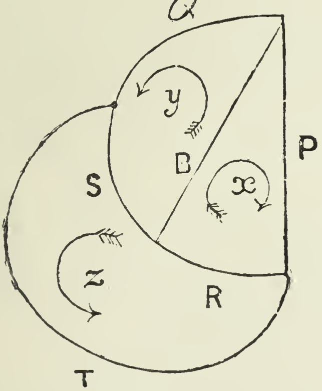

FIG. 108.

the current at any instant in the inductive branch P, $x$, that in the branch Q, $y$, and that in the telephone circuit $z$. The current in the branch battery is then $x+y$. Let L be the inductance of P, and M the mutual inductance of the coils placed in the circuits B and T, and let all the other circuits, Q, R, and S, have no sensible inductance. Let $e$ be the electromotive force of the battery at any instant $t$. Then the currents in the various branches at that instant are as follows :—

In the branch P the current is $x$
,, ,, R ,, ,, $x+z$
,, ,, S ,, ,, $y-z$
,, ,, Q ,, ,, $y$
,, ,, B ,, ,, $x+y$
,, ,, T ,, ,, $z$

Applying Kirchhoff's corollaries to each of the three meshes of the network, we have three equations, viz.,

$$\mathrm{P}\,x+\mathrm{B}\,\overline{x+y}+\mathrm{R}\,\overline{x+z}=e-\mathrm{L}\,\frac{dx}{dt}-\mathrm{M}\,\frac{dz}{dt} \qquad (94)$$

$$\mathrm{B}\,\overline{x+y}+\mathrm{S}\,\overline{y-z}+\mathrm{Q}\,y=e-\mathrm{M}\,\frac{dz}{dt} \quad . \quad . \quad . \qquad (95)$$

$$\mathrm{R}\,\overline{x+z}+\mathrm{T}\,z-\mathrm{S}\,\overline{y-z}=-\mathrm{M}\,\frac{d\,\overline{x+y}}{dt}, \quad . \quad . \qquad (96)$$

and these three equations enable us to find at any time $t$ the current in any branch.* If we suppose the bridge to be balanced for variable currents, then $z$ is zero, and on making this limitation we find the above equations reduce to the two,

$$\mathrm{Q}\,y-\mathrm{P}\,x-\mathrm{L}\,\frac{dx}{dt}=\mathrm{R}\,x-\mathrm{S}\,y, \quad . \quad . \quad . \qquad (97)$$

and

$$-\mathrm{M}\,\frac{dx}{dt}-\mathrm{M}\frac{dy}{dt}=\mathrm{R}\,x-\mathrm{S}\,y. \quad . \quad . \quad . \qquad (98)$$

Furthermore, let us assume that the currents vary according to a simple periodic law. In this case, if X is the maximum value of $x$, then we can write

$$x=\mathrm{X}\sin p\,t,$$

---

* The general method of finding the current equations for any network is given in Maxwell's "Treatise on Electricity," 2nd Edition, Vol. II., § 755. *Also see* "Problems on Networks of Conductors," by J. A. Fleming, *Phil. Mag.*, September, 1885, Vol. XX., p. 221 ; or *Proceedings* Phys. Soc., Lond., 1885.

where $p$ as usual $=2\pi n$, $n$ being the frequency of the alternation. Hence

$$\frac{dx}{dt}=p\,\mathrm{X}\cos pt,$$

and

$$\frac{d^2x}{dt^2}=-p^2\,\mathrm{X}\sin pt$$
$$=-p^2x.$$

Adopting the fluxional notation, it is convenient to write $\dot{x}$ for $\frac{dx}{dt}$ and $\ddot{x}$ for $\frac{d^2x}{dt^2}$. Hence, for simple periodic variation of a current $x$, we always have the condition

$$-\ddot{x}=p^2x.$$

If we differentiate with respect to $t$ the two equations (97) and (98), and eliminate $\ddot{x}$ by the help of the equation $\ddot{x}=-p^2x$, we obtain two other equations, which, together with the original two (97) and (98), give us the necessary four equations for eliminating the four variables $x$, $y$, $\dot{x}$, $\dot{y}$. We have thus,

$$\mathrm{Q}y-\mathrm{P}x-\mathrm{L}x=\mathrm{R}x-\mathrm{S}y \quad . \quad . \quad . \quad . \quad (99)$$
$$-\mathrm{M}\dot{x}-\mathrm{M}\dot{y}=\mathrm{R}x-\mathrm{S}y \quad . \quad . \quad . \quad . \quad (100)$$
$$\mathrm{Q}\dot{y}-\mathrm{P}\dot{x}+\mathrm{L}p^2x=\mathrm{R}\dot{x}-\mathrm{S}\dot{y} \quad . \quad . \quad . \quad . \quad (101)$$
$$\mathrm{M}p^2x+\mathrm{M}p^2y=\mathrm{R}\dot{x}-\mathrm{S}\dot{y} \quad . \quad . \quad . \quad . \quad (102)$$

The student who has mastered the elements of determinant analysis will recognise that the variables $x$, $y$, $\dot{x}$, $\dot{y}$ can be eliminated from these equations, and the relation which must always hold good between the constants can be found by equating to zero the determinant of these four equations. We have then

$$\begin{vmatrix} -\mathrm{L}, & \mathrm{O}, & -(\mathrm{P}+\mathrm{R}), & (\mathrm{Q}+\mathrm{S}) \\ -\mathrm{M}, & -\mathrm{M}, & \mathrm{R}, & \mathrm{S} \\ -(\mathrm{P}+\mathrm{R}), & (\mathrm{Q}+\mathrm{S}), & \mathrm{L}p^2, & 0 \\ -\mathrm{R}, & \mathrm{S}, & \mathrm{M}p^2, & \mathrm{M}p^2 \end{vmatrix}=0.$$

This determinant writes out into the sum of three terms, viz. :—

$$\begin{aligned} & p^2\mathrm{L}\,[\mathrm{L}\,(\mathrm{M}^2p^2+\mathrm{S}^2)-\mathrm{M}\,(\mathrm{S}^2+\mathrm{S}\mathrm{Q}+\mathrm{R}\mathrm{Q}+\mathrm{S}\mathrm{R})\,]+ \\ & -(\mathrm{P}+\mathrm{R})\,[\mathrm{M}^2p^2\,(\mathrm{P}+\mathrm{Q}+\mathrm{R}+\mathrm{S})-\mathrm{S}\,(\mathrm{Q}\mathrm{R}-\mathrm{S}\mathrm{P})\,]+ \\ & +(\mathrm{Q}+\mathrm{S})\,[\mathrm{M}\mathrm{L}p^2\,(\mathrm{R}+\mathrm{S})+\mathrm{R}\,(\mathrm{S}\mathrm{P}-\mathrm{R}\mathrm{Q})-\mathrm{M}^2p^2\,(\mathrm{P}+\mathrm{Q} \\ & \qquad +\mathrm{R}+\mathrm{S})]=0. \end{aligned}$$

This long equation reduces to the simpler form

$$[(p\,S\,L)^2 - (p\,M\,(P+Q+R+S)\,)^2] + [(M\,L\,p^2)^2 - (Q\,R - S\,P)^2] = 0.$$

In order that the sum of the two left hand terms in the above equation may always be zero, each factor in the square brackets must be separately zero, and it will be seen that each of these factors equated to zero are equivalent to the two equations:—

$$Q\,R - S\,P = M\,L\,p^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (103)$$

and

$$M\,(P+Q+R+S) = S\,L. \quad . \quad . \quad . \quad . \quad . \quad (104)^*$$

These equations express the relation which holds good between the resistances of the branches and the self and mutual induction coefficients of a Hughes bridge when the bridge is balanced for variable currents.

It will be seen that the ordinary relation of the resistances for steady balance, viz., $P : Q = R : S$ is departed from, and that we have for the resistance of branch P, when traversed by variable currents, the value

$$P = \frac{Q\,R - M\,L\,p^2}{S} = \frac{Q\,R}{S} - \frac{M\,L\,p^2}{S}, \quad . \quad . \quad (105)$$

and for the inductance of branch P under these circumstances, the value

$$L = \frac{M\,(P+Q+R+S)}{S}. \quad . \quad . \quad . \quad (106)$$

In some of his experiments Prof. Hughes interpreted his results on the assumption that P was always equal to $\frac{Q\,R}{S}$, and L was equal to M; but the complete investigation shows that this is not the case. A very full theoretical and practical examination of the induction bridge has been given by Prof. H. F. Weber, for which the student is referred to the pages of the *Electrical Review*, Vol. XVIII., p. 321, 1886, and Vol. XIX., p. 30, 1886.†

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* These equations were given by Lord Rayleigh in the discussion on Prof. Hughes's Paper. See also Lord Rayleigh "On the Self-Induction and Resistance of Compound Conductors," *Phil. Mag.*, Dec. 1886, p. 471. Equivalent equations have been also arrived at by Prof. H. F. Weber and Mr. Oliver Heaviside.

† *See also* Mr. Oliver Heaviside in the *Phil. Mag.*, August, 1886.

The whole method of the construction and use of the induction bridge has been the subject of elaborate examination by Lord Rayleigh in a Paper on the self-induction and resistance of compound conductors (*Phil. Mag.*, December, 1886), from which we shall quote freely in what follows. Lord Rayleigh discarded the tooth-wheel interrupter, as it does not give a regular variation of current corresponding in period to the passage of a tooth; and he substituted a harmonium reed, the vibrating tongue of which made contact once during each period with the slightly-rounded end of a brass or iron wire advanced exactly to the required position by means of a screw cut upon it. Blown with a regulated wind, such reeds are capable of giving interruptions of current up to about 2,000 per second. The one usually employed had a frequency of 1,050 vibrations per second. The induction compensator consisted of two circular coils, one of which was fixed and the other movable round an axis, so placed that the flat circular coils could be placed either with their planes coincident or at right angles. If the inner coil is very small compared with the other, and the coils are placed with centres coincident and axes inclined at any angle, $\theta$, and if $M_0$ be the maximum mutual inductance and M the inductance in any position, $\theta$, then

$$M = M_0 \cos \theta.$$

This law is, however, not followed when the coils are sensibly of the same size. In this case Lord Rayleigh has shown that the mutual induction is very approximately proportional to the angle between the axes of the coils for a range between 40° and 140°. In the actual experiments the mutual inductance of the coils was determined for each degree of angular displacement of the axes by comparing it with the calculated coefficient between two wires, wound in measured grooves, cut in a cylinder, and it was found that every degree of movement of the movable coil, when the axes were not far removed from perpendicularity, was equal to 776·3 centimetres of mutual induction, the maximum when $\theta = 0$ being 56,100 centimetres. The first experiment described in the Paper referred to is one on the self-induction and resistance of a coil of copper wire. In the bridge used the resistances $Q + R + S$ were together equal to 4·00 ohms. Resistances were, however, measured in scale divisions of the bridge wire, each one equal to $2{\cdot}04 \times 10^6$ centimetres per second. The copper coil being balanced on the bridge, it was found that the readings of the three resistances and of M were as follows:—

$$Q = 610,\ R = 190,\ S = 1{,}160,$$
$$M = 36^\circ = 36 \times 776 \text{ centimetres},$$

and the frequency $n$ of the vibrations $= 1{,}050$. Hence $p = 2\pi \times 1{,}050$. Taking the equations (103) and (104) on page 288, and eliminating L, we have for the value of P, the equation

$$P = \frac{Q \,.\, R}{S}\left[\frac{1 - \dfrac{p^2 M^2 (Q + R + S)}{S \,.\, Q \,.\, R}}{1 + \dfrac{p^2 M^2}{S^2}}\right].$$

Substituting the values above, we find

$$P = {\cdot}876 \frac{Q\,R}{S} = 87{\cdot}5 \text{ scale divisions.}$$

This gives the value of the real resistance of P for the periodic currents used; and we see that if we neglected the peculiarity of the bridge, and simply assumed the ordinary law, that the resistance of P was equal to $Q\,R \div S$, we should make an error of some 12 per cent. On actually balancing the bridge for steady currents the resistance of P was found to be 87·3 scale divisions, thus indicating that for this copper coil at the frequency employed the resistance to variable currents was the same as to steady ones.

On inserting a solid copper rod into the aperture of the coil and measuring again the resistance and self-induction, it was found that the values of the reading were $Q = 660$, $R = 190$, $M = 295^\circ$, instead of as before, $Q = 610$, $M = 36^\circ$. Hence the introduction of another closed secondary circuit (viz., the copper rod) increased the real resistance and diminished the real self-induction in accordance with the principles explained on page 180, at which place we demonstrated Maxwell's equations for the increased resistance and diminished self-induction of a primary circuit when in contiguity to a closed secondary circuit.

The next example selected was that of a soft iron wire, 160 centimetres long and 3·3mm. dia. Here, with the variable currents from the reed interrupter of the same period as before, a balance was obtained for

$Q=178,\quad R=190,\quad S=1{,}592,\quad M=8\times 776$ centimetres,

from which we find

$$P=\cdot 985\,\frac{Q.R}{S}=20\cdot 93 \text{ scale divisions.}$$

The resistance of the same wire to steady currents was

$$P_0=\frac{100\times 190}{1{,}670}=11\cdot 38 \text{ scale divisions.}$$

Hence the effective resistance to variable currents having a frequency of 1,050 was 1·84 times the resistance to steady currents. We have presented to us here the phenomena characteristic of the behaviour of conductors to electric currents rapidly intermittent or reversed. The real resistance of the conductor is increased. This is not to be confused with the fact that for intermittent currents the impedance $(\sqrt{R^2+p^2L^2})$ measured in ohms is greater than the ohmic resistance (R); but it is to be understood as a real increase in the rate at which energy is dissipated per unit of current. It is now well understood that such increase of resistance is due to the fact that the current density for rapidly periodic currents is not uniform over the cross-section of the wire, but is greatest along the outer layers of the wire. Hence, under rapidly periodic currents the inner portions of a conducting wire are never reached by the current, and, as far as current carrying duty is concerned, might as well be away. This difference may be graphically represented thus: Let relative density of current or quantity passing per second through unit of cross-section of a conductor per unit of time be represented, like relative density of population, by degree of density of shading. Then the flow of a steady current through the section of a wire might be represented as in Fig. 109; and the flow of current over the cross-section when the current is rapidly periodic might be represented as in Fig. 110.

We must consider that the current in beginning in a conductor starts its flow first on the outside, and soaks or penetrates inwards into the deeper layers by degrees. We see that, in consequence of this, if the current is reversed in sign, or rapidly intermitted, it will not have time to soak or diffuse very far into the mass of the conductor before it is, so to speak, re-called, and its operations will be confined to the outer layers. This is a rather broad way of stating modern views on the *modus operandi* of current flow. According to these views the current in a wire is not established by a process analogous to starting a flow of water in a pipe by a push applied one end, but it is put into the wire at all points of its surface by energy absorbed from the surrounding dielectric. Other things being equal, the rate at which this equalisation of current across the cross-section of the conductor goes on will be a function of the magnetic permeability of the material. The current in flowing along a magnetisable circuit magnetises it circularly. This magnetisation involves work, and the impressed electromotive force which is increasing the current has to do work, not only against that which may be called the formal inductance of the circuit, or against that part of the counter electromotive force of induction which depends on the form of the circuit, but has to create this circular magnetisation.

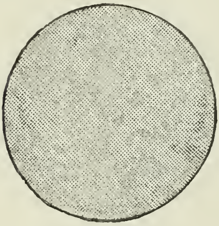

Fig. 109.

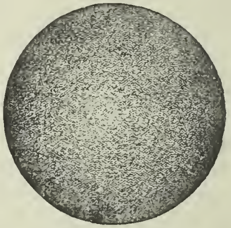

Fig. 110.

By keeping to the outer layers of the conductor the periodic current avoids magnetising the deeper layers of the material. Proof will be given later in describing the remarkable investigations of Hertz that this description of the mode of establishment of a current is one supported by experimental facts. We are thus able to offer a consistent theory of the real increase of resistance which we find for rapidly periodic currents. The inner core or central portion of the conductor is not used by the current, and, so far as conducting it goes, might as well be absent; hence the solid conductor does no more, or not much more, in the way of carrying the current than a hollow or tubular conductor would do: and, accordingly, the real or ohmic resistance of the conductor for such variable currents is greater than it is for steady currents.

Another way of regarding this inequality of current distribution over the cross-section of a wire is as follows:—The counter electromotive force arising from self-induction is greater at the axis or central portion of the wire than it is near the surface. If we consider the whole current flowing across any section of the conductor as made up of little streamlets of currents flowing parallel to each other, the central streamlets or filaments of current experience more opposition in reaching full magnitude than do the outer ones, because of the mutual induction with those surrounding them. The current therefore arrives at its maximum value at the surface of the conductor before it does at the deeper or central portions. If the current is periodic or transitory the central streamlets or current filaments are always greatly inferior in strength to those at the surface. There is reason, then, to believe that a sudden rush of current, very brief in duration, such as the discharge from a Leyden jar or condenser, moves chiefly along the surface of a discharging wire, and the same statement holds good for very rapid pulsatory or alternate currents. Although it may be said that the general principles governing the behaviour of alternating current flow as conductors were virtually given by Maxwell,* they have been subsequently chiefly developed mathematically by Mr. Oliver Heaviside and Lord Rayleigh, and were brought to the notice of practical electricians principally by the experiments of Prof. Hughes previously mentioned.

This increase of the resistance proper of a wire for rapidly periodic currents is one of the most striking of the results of Prof. Hughes's researches. The full mathematical development of the problem, even for comparatively simple cases, leads to some very complex mathematical expressions. Lord

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* Maxwell's "Electricity," Vol. II., § 689-690. In this paragraph it is shown that the counter electromotive force of self-induction at any point in a conductor is a function not only of the time but of the position of the point considered, and varies over the cross-section of the conductor.

Rayleigh has, however, treated with great fulness* one or two cases of practical importance. If R and L are the true ohmic resistance and inductance of a cylindrical straight wire of length $l$ and magnetic permeability $\mu$ to steady currents or currents of very slow alternations, and if an alternating current of simple periodic form and frequency $n$ is sent through it, then the resistance is increased to $R^1$ and the inductance diminished to $L^1$ in such wise that if $p=2\pi n$, as usual, we have

$$R^1 = R\left[1 + \frac{1}{12}\frac{p^2 l^2 \mu^2}{R^2} - \frac{1}{180}\frac{p^4 l^4 \mu^4}{R^4} + . \;\; . \;\; . \;\; \&c.,\right] \quad (107)$$

and $$L^1 = l\left[A + \mu\left(\frac{1}{2} - \frac{1}{48}\frac{p^2 l^2 \mu^2}{R^2} + \frac{13}{8640}\frac{p^4 l^4 \mu^4}{R^4}, \;\; . \right)\right] \quad (108)$$

A being some constant depending on the position of the return wire.

These formulæ express the fact that the resistance is increased and the inductance diminished in proportion as the frequency of alternation gradually increases from zero to infinity.

At slow rates of alternation the chief opponent with which the impressed electromotive force has, so to speak, to contend is the ohmic resistance; and the distribution of current across the cross-section of the conductor under these conditions is such as to make that resistance a minimum, and this is known to be so when the distribution is a uniform distribution. The current is then taking the greatest advantage of the conductor, and the heat generated and dissipated per unit of time is less under these conditions than if the same total current were distributed in any other way over the cross-section of the conductor. This last statement can be easily proved. Let the cross-section of the conductor, supposed to be a cylindrical wire, be divided into two equal zones by a circular line. Let the resistance per unit of length of the conductor be $r$ for each portion corresponding to the outer and inner zone. Call the outer portion the sheath and the inner the core of the conductor for brevity. If a total quantity of current, $x$, flows through the conductor, then the rate of dissipation of energy as heat is $\frac{r x^2}{4}$ for each portion per unit of length, or $\frac{r x^2}{2}$ for the whole conductor, on the assumption that the current is equally divided between the sheath and the core.

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* "On the Self-Induction and Resistance of Straight Conductors," *Phil. Mag.*, May, 1886, p. 382.

If, however, we suppose the total current, $x$, to be distributed so that a portion, $y$, travels by the sheath, and the remainder, $z$, travels by the core, then the heat generated per unit of length per unit of time is $r y^2$ for the sheath and $r z^2$ for the core. Hence, for the equi-distribution of current, the energy dissipation is $\frac{r x^2}{2} = \frac{r (y+z)^2}{2}$, and for the unequi-distribution it is $r(y^2+z^2)$. Which, then, is greater, $\frac{r (y+z)^2}{2}$ or $r(y^2+z^2)$?

Consider the following inequalities :—

$$(y-z)^2 \text{ is greater than } \frac{1}{2}(y-z)^2,$$

or $$y^2+z^2-2yz \text{ is greater than } \frac{1}{2}(y-z)^2;$$

but $$\frac{1}{2}(y-z)^2 = \frac{1}{2}(y+z)^2 - 2yz.$$

Hence, $$y^2+z^2-2yz \text{ is greater than } \frac{1}{2}(y+z)^2 - 2yz.$$

Adding $2yz$ to both sides, we have

$$y^2+z^2 \text{ is greater than } \frac{1}{2}(y+z)^2.$$

Accordingly it follows that

$$r y^2 + r z^2 \text{ is greater than } \frac{r}{2}(y+z)^2,$$

or $$r y^2 + r z^2 \text{ is greater than } \frac{r}{2}x^2;$$

that is to say, the rate of energy dissipation is greater for the assumed unequal distribution than for the distribution in which the current is equal in density over the cross-section of the conductor. The same kind of proof may be extended to any other arbitrary distribution of current over the cross-section, and the reasoning will lead to the conclusion that the equi-dense distribution is that which causes the *least* rate of dissipation of energy per unit of current.

For slow alternations, therefore, the current adopts that mode of distributing itself over the cross-section of the conductor which makes the rate of energy dissipation a minimum. On the other hand, for rapid alternations the current meets with its greatest obstacle from the counter electromotive force of self-induction, and it accordingly distributes itself over the cross-section of the conductor, so as to get as much to the outside as possible, and thus avoids, in the case of magnetic conductors, magnetising the inner layers or portions of the conductor. The endeavour is to make the self-induction a minimum irrespective of resistance. This is only an instance of the broad, general principle that behaviour of current for very rapid pulsations, or alternations, is determined by the inductances rather than the resistances, whereas for steady or slowly periodic currents the behaviour is governed by resistance rather than by self-induction.

In order to see under what conditions the alteration of resistance and self-induction becomes sensible, we have to examine the value of the term $\frac{1}{12}\frac{p^2 l^2 \mu^2}{R^2}$ in the above-given series for $R^1$. We will first take the case of an iron wire 0·4 centimetre, say, 0·16 inch diameter (No. 8 B.W.G.). The specific resistance of iron in C.G.S. measure is about $10^4$; so that

$$\frac{R}{l}=\frac{10^4}{\pi\times 0{\cdot}04}=\frac{10^6}{4\pi},$$

$p^2=4\pi^2 n^2$, $n$ being the frequency.

Let us take $n=100$, so that there are supposed to be 100 complete alternations per second. The value of $\mu$ is more difficult to assign. For small degrees of magnetisation and solid iron, we may, perhaps, take $\mu=300$;

then $$\frac{1}{12}\frac{p^2 l^2 \mu^2}{R^2}=\frac{1}{12}\frac{4\pi^2 n^2 \mu^2 l^2}{R^2}=\frac{5{\cdot}2\,\mu^2 n^2}{10^{10}}.$$

If $\mu=300$, $n=100$, $\mu^2 n^2=9\times 10^8$, and $\frac{1}{12}\frac{p^2 l^2 \mu^2}{R^2}=0{\cdot}47$ $=0{\cdot}5$ nearly.

Accordingly, for this case $R^1=R\ (1+0{\cdot}47)$ nearly, or the resistance is increased to about half as much again.

If $n=1{,}000$ we should find $R^1=48\ R$, or the resistance would be increased nearly fifty times.

Consider next the case of copper. The specific resistance is 1,640 C.G.S. units. If $a$ be the radius of the wire in centimetres, then we have

$$\frac{1}{12}\frac{p^2 l^2 \mu^2}{R^2} = \frac{\pi^4}{3}\frac{a^4 n^2}{(1{,}640)^2} = \frac{1{\cdot}2\, a^4 n^2}{10^5}.$$

If, as before, $n=100$, this fraction becomes equal to $0{\cdot}12a^4$. This shows that for a diameter of one centimetre we should have

$$R^1 = R\,(1+0{\cdot}12);$$

and hence for diameters of one centimetre and upwards the resistance of round copper rods becomes very sensibly increased for alternating currents of a frequency about 100 and upwards. The practical conclusions of importance in electrical engineering from the above investigation are these:—*First*, copper rods or conductors should be used, and not iron, for transmitting alternate or intermittent electric currents having a moderate frequency, say of 100 to 1,000 per second; *secondly*, to avoid, as far as possible, the increase of resistance due to the current keeping to the outer portions of the conductor, the conductor should be in the form of a thin strip, or better, a tube having walls thin in proportion to the radius. It is to be noted that mere stranding of the conductor, or building it up of separate insulated conductors joined in parallel, will not prevent this augmentation of resistance, unless the stranding is of such a kind that portions of the cable which at one point of its length form the inner parts or heart of the cable at another part of its length form the outside.

The object to be achieved is to construct some kind of stranding by which all portions of the conductor are brought as near as possible to the dielectric, so that the energy arriving from the dielectric finds all parts of the mass of the cable, both surface and interior, equally accessible. In order to avoid external inductive disturbance, the proper form to give to a cable intended to convey rapidly intermittent or alternate currents is a couple of rather thin concentric tubes of copper well insulated from each other, and both insulated from the earth, of which one forms the *lead* and the other the *return*. By this device the metal will be most economically employed. An equivalent device used in practice is a concentric cable, which consists of a central core of stranded copper cable covered with insulation, and then plaited over with a sheath of other copper wires which form the return conductor.

In a further experiment, Lord Rayleigh (*loc. cit.*) examined the resistance of an iron wire of hard Swedish iron 10·03 metres long and 1·6 millimetre in diameter. In arbitrary units the resistance of the wire to steady currents was 10·4 units or 0·51 ohm, and to currents of 1,050 complete alternations per second its resistance was 12·1 units, or 0·595 ohm, which is an increase of about 20 per cent. In the case of a stouter wire, 18·34 metres long and 3·3 millimetres in diameter, the resistance to steady currents was 4·7 units, and the resistance to the interrupted currents of the above-mentioned frequency was 8·9 units, or nearly double. This illustrates the fact that, for a given frequency of alternation, the ratio in which the resistance is increased is greater the greater the diameter of the conductor, assuming it to be a round solid rod.

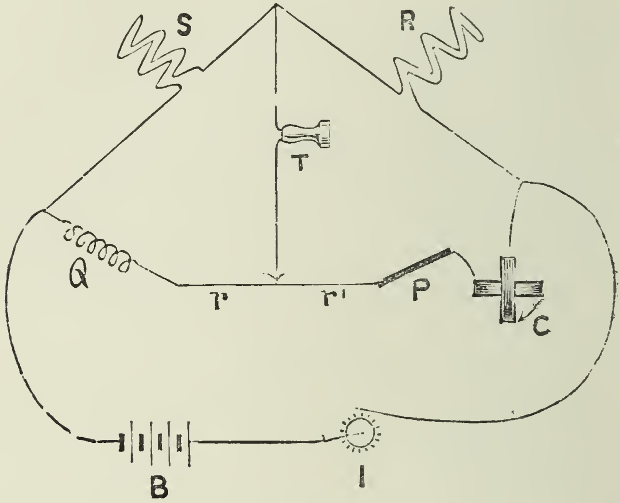

Fig. 111.

Lord Rayleigh found it more convenient in many researches to slightly alter the arrangement of the induction balance as described by Prof. Hughes, and to make it as follows (Fig. 111).

Two arms of a quadrilateral, R and S, consist of equal resistances of German-silver wire, wound double, so as to have negligible inductance. One arm, Q, consists of a coil having inductance and resistance greater than that of any conductor, P, to be placed in the fourth arm. B and I are a battery and an interrupter, T is a telephone in the "bridge," and $r r^1$ is a German-silver wire of appropriate resistance, along which slides the contact of the bridge. The arm P includes a pair of coils joined in series, and which act upon each other by mutual induction, so that the resulting self-induction of the two coils in series can be varied within certain limits by turning one coil round within the other. For the resulting self-induction of such a pair of coils used in this manner may be regarded as made up of the component self-inductions of each coil taken separately and of twice the positive or negative mutual self-induction, depending upon which faces of the coils are presented to each other. It is possible, then, within certain limits to vary the inductance of the branch P C, and to vary also the resistance of the branches Q and P C by shifting the contact of the telephone along $r r^1$.

The condition for obtaining a true balance when the current is periodically interrupted is that the resistances and inductances of the branches Q and P C shall be separately equal. Suppose a balance has been obtained without the use of P, in which the resultant self-induction of C is made to balance the inductance of Q, and the resistance of $C + r^1$ is made to be equal to that of $Q + r$. Let, now, any conductor, P, be inserted as in the figure; the telephone contact will have to be shifted, and *also* the inductance of C will have to be changed to re-obtain a balance. The inductance of P is measured by the amount by which that of C has to be *reduced* on inserting P, and the resistance of P is measured by twice the resistance of that length of the German-silver wire $r r^1$ by which the telephone contact point has to be shifted to regain the balance. This method of employing the induction balance separates out at once the real resistance of P from its effective induction.

With the aid of this balance an interesting experiment was made, showing the effect of a closed secondary circuit on the resistance and inductance of the primary. The frequency was again, as usual, 1,050 per second. A coil was prepared of two copper wires, wound side by side on one bobbin. The diameter of each wire was about 0·08in., and the length of each wire 318in. There were 20 (double) turns, so that the mean diameter of the coil, wound as compactly as possible, was about 5in., and the resistance of each wire was 0·05 ohm.

The coefficient of mutual induction of the two wires was determined by comparison of the self-induction L of one wire with that of the two wires connected oppositely in series, viz., $(2L - 2M)$. In this way it appeared that

$$M = 43{\cdot}1^\circ = 43{\cdot}1 \times 1{,}553 \text{ centimetres.}$$

Observation showed that closing of the circuit of one wire reduced the self-induction of the other from 44·4° to 3·4°. The resistance to steady currents was 0·92 (arbitrary units). The resistance to the periodic currents was 0·97 with the secondary circuit open, and 1·74 with the secondary circuit closed.

Hence, $L = 44{\cdot}4 \times 1{,}553$ centimetres, and

$$R = 0{\cdot}97 \times 0{\cdot}0492 \times 10^9 \text{ centimetres per second.}$$

From Maxwell's formulæ, page 180, we get

$$\frac{p^2 M^2}{R^2 + p^2 L^2} = \frac{10^{17} \times 1{\cdot}951}{10^7 \times 0{\cdot}023 + 10^{17} \times 2{\cdot}071} = 0{\cdot}932.$$

Hence, $L^1 = L\,(1 - 0{\cdot}932)$,

where $L^1$ is the decreased inductance. Hence,

$$L^1 = 0{\cdot}068\, L,$$

or $L^1 = 0{\cdot}068 \times 44{\cdot}4^\circ = 3^\circ$,

and the observed value is 3·4°, which is in very tolerable agreement.

Again, the steady resistance with secondary open is 0·92, and hence the resistance $R^1$ with secondary closed is

$$R^1 = 1{\cdot}932 \times 0{\cdot}92 = 1{\cdot}77;$$

and observation gives the value 1·74. We see, then, that observations with this bridge confirm, with a considerable degree of accuracy, the deductions from the theory of simple periodic currents, that the closing of a secondary circuit increases the resistance and diminishes both the inductance and the impedance of an adjacent primary circuit.

From a practical point of view the most important difference between the conduction of steady electric currents and rapidly periodic currents is that of the *locale* of the currents in the conductor and the consequent rise in the ohmic resistance of the conductor as a whole when employed with such periodic currents. Prof. Hughes called attention in 1883 to this great difference in the resistance of an electrical conductor if measured during the *variable* instead of the *stable* condition of the current.*

In experiments with his induction bridge Prof. Hughes was able to assure himself that the resistance of an iron telegraph wire of the usual size was more than three times greater for rapid periodic currents of about 100 per second than for steady currents. The full elucidation of the propagation of currents in conductors under periodic electromotive force is not to be attempted without following out some very elaborate mathematical analysis. The subject has received its most complete treatment perhaps in the published writings of Mr. Oliver Heaviside† and all that can be attempted here is to give a slight sketch of the views which are now very generally held on this subject.

Consider a long level tank or canal full of liquid. There are, amongst others, two ways in which we might suppose this liquid to be set in motion. A paddle or the hand might be placed in the liquid, and by giving the liquid bodily a push it might be made to move forward; or we might suppose some body floating on the surface, such as a plank of wood to be dragged along the surface. The friction between the plank and the layer of water beneath it would then cause the subjacent layer of liquid to move with the plank, and the motion of this layer would be gradually communicated to the other and deeper-lying layers by reason of the viscosity of the fluid. Or take the case of a basin containing water. The liquid might be set in rotation by stirring it with a paddle or the hand, but it might also be set in rotation by twisting the basin rapidly. In this last case the rotation of the basin would be communicated by friction to the water in contact with its sides, and then handed on from layer to layer of the water by internal fluid friction. Thus the twist or spin of the basin would be gradually propagated inwards from the circumference to the centre. Imagine the whole mass of the liquid divided up into very thin concentric shells, like the coats of an onion. If the liquid were a perfect fluid there would be no friction between these layers, but since every liquid possesses some degree of viscosity or internal fluid friction, the sliding of one layer of fluid over another gradually causes the second layer to partake of the motion of the first. Hence, when the rotation of the basin commences the friction between its sides and the first layer of fluid starts that gradually in motion; this motion is then transmitted to the second layer, and so on, until the whole mass of the liquid possesses an equal angular velocity round the axis of rotation. The greater the fluid friction or viscosity the more rapid will be this equalisation of the angular velocity of all parts of the fluid, and so a rotating vessel full of tar would arrive at a stationary condition as regards angular velocity sooner than one filled with a limpid liquid as alcohol or ether.

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* Discussion on a Paper by Mr. W. H. Preece on "Electrical Conductors," *Proceedings* Inst. Civil Engineers, Vol. LXXV., 1883.

† "Electromagnetic Theory," by Oliver Heaviside in *The Electrician* Series, and "Electrical Papers," published by Messrs. Macmillan.

Just as the angular velocity diffuses inwards from the circumference to the centre in the case of such a revolving basin of liquid, so, according to modern views, does the current diffuse inwards from the circumference to the axis of the electric conductor, The student who has been accustomed to think of a current as produced in a conductor by a sort of push given to it *in* the conductor—such conception being based on a rough working hypothesis of a hydrodynamic nature—will perhaps have some difficulty in discarding this notion and realising that the current in a wire may perhaps be generated in it by an action taking place at all parts of the *surface of the wire* which gradually soaks or diffuses into the conductor out of the surrounding dielectric, but he will find that this new hypothesis serves to establish a mode of viewing the induction phenomena which makes various experimental results much more easily correlated. It was well demonstrated by the experiments of Prof. Hughes and others that a flat sheet or strip of metal has a less self-induction than a round wire of equal cross-sectional area. On the present hypothesis, this is explained by saying that the flat strip offers a greater absorption surface to the dielectric; the current therefore soaks in more quickly to the centre and arrives at a uniform distribution over the cross section very soon—in other words, the variable state is sooner over, and we express this fact by saying that the self-induction is small. Again, if the electromotive force is oscillatory or rapidly periodic, we see at once that the current has not time to penetrate right into the core of the conductor before its sign or direction is reversed. It has hardly started on its journey inwards, soaking from surface to centre, before it is recalled; hence the flow of a current when very rapidly periodic is confined to the surface of the conductor, the real or ohmic resistance is *increased*, and the self-induction is *diminished*.

Lord Kelvin has shown (Bath British Association Meeting, 1888) that for alternating currents of a frequency equal to about 150 complete alternations per second, the depth to which the currents penetrate into the substance of the copper is about three millimetres, so that portions of the conductor beyond this distance from the surface are almost useless for conduction. The practical moral of this is that the proper form for a conductor for alternate currents is either a flat sheet of copper or a copper tube, in which, for the above frequency, the thickness of material is not more than one-quarter of an inch. It is useful in this connection to note a few facts with regard to cables as used for alternating currents. A seven strand cable has an overall diameter of three times one strand. A nineteen strand cable has an over-all diameter of five times one strand. A No. 12 wire (S.W.G.) has a diameter of 0·109 inch. Hence a 19/12 cable has a diameter of 0·5 inch, and a cross-sectional area of 0·1645 square inch. At a current density of 600 amperes per square inch this cable will carry 100 amperes, and it has a resistance of one-sixth of an ohm per 1,000 yards. For alternating currents, therefore, of about 100 frequency, a 19/12 stranded cable is about the largest size that should be employed. For alternating currents of 100 frequency, beyond about 100 amperes, a form of cable must be employed in which the thickness of the conductor is not at any part greater than about one quarter of an inch; and this is only to be achieved by the employment of concentric tubes or concentric stranded cables in which the core or central strand is not of greater thickness than half an inch, and which condition necessitates, therefore, that when above a certain cross-sectional area the central conductors should also be of tubular form. One of the advantages to be gained by the employment of alternating currents of low frequency is that the limiting diameter of the conductors is much larger for low than for high frequency. To return to our illustration of the twisting basin of fluid. Suppose the action on the vessel consists in rapidly twisting it through a small angle, first one way and then the other, the liquid in the interior would be subjected to a strain which would consist in the various concentric layers of the liquid sliding backwards and forwards over each other. The interior of the liquid would be thrown into stationary waves, in which the nature of the wave motion consisted in each particle of water being displaced first one way and then another along an arc of a circle described on a horizontal plane, with its centre in the axis of rotation. The more rapid the motion the greater would be the rate of decrease in the amplitude of each wave in passing from the circumference to the centre of the vessel; in other words, for very rapid oscillations the bulk of the water in the centre of the basin would remain nearly at rest.

Every experiment as yet made on the self-induction or change of self-induction in conductors is consistent with the above hypothesis. It shows, for instance, why a conductor composed of thin insulated wires or thin insulated strips has a less self-induction than a solid conductor of equal cross-section. Prof. Hughes says*:—" We can reduce the self-induction of a current upon itself to a mere fraction of its previous value by simply separating the contiguous portions of a current from each other, the results proving that a comparatively small separation, such as is obtained by employing ribbon conductors in place of a wire of the same weight, reduces the self-induction 80 per cent. in iron and 85 per cent. in copper, and if we still divide the current by cutting the ribbon into several strips (separating the strips at least 1 centimetre from each other), then the combined but separated strips show a still greater reduction, being 94 per cent. in iron and 75 per cent. in copper."

These, and many other experiments of a similar sort, indicate that we may regard the inductance of a conductor as an effect which is due to the fact that the current takes *time* to penetrate into the conductor, and that a reduction of the time required to arrive at an equal current density in all parts of the conductor can be effected by any change of form which brings the inner parts of the conductor nearer the surface, or makes them more get-at-able from the dielectric, The better the conductor the slower is the rate of equalisation of current density over its cross-section—in other words, the less rapid is the rate of diffusion of the current inwards from circumference to centre; and the "time constant" of the circuit, or the time in which, under the operation of a constant electromotive force, the current will rise to a definite fraction of its maximum value, is a quantity proportional to the conductivity of the circuit, and to another factor (the formal inductance), which may be considered as expressing the accessibility of the conductor as regards geometrical form to the entrance of the current into it; and finally, in the case of magnetic conductors, to a quantity (the permeability) determined by the capacity of the conductor to utilise part of this incoming energy in producing magnetisation of its substance.

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* Inaugural Address, *Journal Soc. Tel. Eng.*, 1886.

We are indebted to a Paper read before the Austrian Academy by Prof. Stefan for a simple and intelligible analogy helping to comprehension of the electrical distribution of current in a conductor. Imagine a cylinder or cylindrical wire heated throughout to a uniform temperature; let it be suddenly brought into a chamber where the temperature is higher. The outer layers of the cylinder will rise first in temperature, and gradually convey the heat to the successive interior layers. Precisely the same order of phenomena occurs if an E.M.F. is suddenly set up between the ends of the wire or cylinder. The current during the variable state passes first through the outer layers alone, and gradually penetrates the inner layers. When the external E.M.F. is suddenly removed, the action, of ceasing in the current resembles the *cooling* of the cylinder. The current ceases first, or, rather, most quickly, in the outer layers.

Now, let us imagine the cylinder transferred to and fro from a very hot place to a cool one. It is easy to see that waves of heat will pass in and out radially, and also that the condition at any instant will depend largely upon the rate of transference.

When the rate of motion is sufficiently slow, the waves of heat passing any given point in the radius of the wire follow exactly with the periodic changes of position. The *amplitudes* of these variations have values which decrease from the surface inwards. When the rate of change is increased, the amplitude of the waves gets shorter and shorter, and at an infinite velocity of transference the wire would acquire an equable temperature throughout. In the electrical analogue the rate of transference corresponds to the inverse of the periodic time of an alternating current. The heat *conducting* power of the material corresponds to electrical *resistance.*

Prof. Stefan gives some numerical illustrations which are useful. If an alternating current have a frequency of 250 per second and is passed through an *iron* wire of 4mm. diameter, the amplitude of the waves of current density is about twenty-five times greater upon the surface than at the axis of the wire. For double the number of vibrations per second the external amplitude becomes only six times as great. The difference of phase is one-third the duration of the vibration in the first case and one-half in the second. The latter statement implies that the external current is at a given moment actually in the reverse direction to the internal current.

For non-magnetic wires the difference is not nearly so marked, and it decreases as the specific resistance increases. For a copper wire of 4mm. diameter, with a periodic time of one 500th second, the difference between the current density at the surface and at the centre is only 14 per cent. If, however, the copper wire be increased to 20mm. diameter, then we should get the same difference as in the particular iron wire quoted.

It is obvious that this non-homogeneous distribution of current must increase dissipation of energy, which is, of course, proportionate in each transverse section to the square of the current strength at that spot. In the case of the iron wire quoted, the increase of resistance is 48 per cent. at the 250 per second frequency, and 100 per cent. at the higher speed. As the frequency of alternation is increased, the resultant self-induction of the circuit is lessened, but although the true resistance is increased, the impedance may be diminished on the whole.

§ 12. **Electromagnetic Repulsions.**—The effects of self and mutual induction in conducting circuits are well illustrated in studying the dynamical actions taking place between conductors conveying currents and other circuits. On the 2nd day of October, 1820, Ampère presented to the Royal Academy of Sciences in Paris an important memoir, in which he summed up the results of his own and Arago's investigations in the then new science of electro-magnetism, and crowned that labour by the announcement of his great discovery of forces of attraction or repulsion existing between conductors conveying electric currents.* Respecting that achievement, when developed in its experimental and mathematical completeness, Clerk Maxwell speaks of it as "one of the most brilliant in the history of physical science." Our wonder at what was then accomplished is increased when we remember that hardly more than two months before that date John Christian Oersted had startled the scientific world by the announcement of the discovery of the magnetic qualities of the space near a current-traversed conductor. Oersted named the actions around the conductor, which we now refer to as the magnetic field, the *electric conflict*, and in his first Paper,† in describing the newly-observed facts, he says: "It is sufficiently evident that the *electric conflict* is not confined to the conductor, but is dispersed pretty widely in the circumjacent space." "We may likewise collect," he adds, "that this conflict performs circles round the wire, for without this condition it seems impossible that one part of the wire when placed below the magnetic needle should drive its pole to the east and when placed above it to the west." These words are taken from the original paper, which stimulated the philosophic thought of

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* Mémoire présenté à l'Académie Royale des Sciences le 2 Octobre, 1820, où se trouve compris le résumé de ce qui avait été lu à la même Académie les 18$^{me}$ et 25$^{me}$ Septembre, 1820, sur les effets des courants Electrique, par M. Ampère. *See* Vol. XV. *Annales de Chimie*, 1820.

† In the *Annals of Philosophy* for October, 1820, Vol. XVI., p. 274, is to be found an English translation of Oersted's original Latin essay, dated July 21, 1820, describing his immortal discovery. This Paper is entitled "Experiments on the Effects of a Current of Electricity on the Magnetic Needle," by John Christian Oersted, Knight of the Order of Danneborg, Professor of Natural Philosophy in Copenhagen.

Ampère, and finally led him to the valuable discovery of the electro-dynamic actions between conductors conveying currents.

Referring the student to text-books on Physics for the complete statement of Ampère's work, we may describe briefly some illustrations of the interactions of two circuits traversed by currents in the same or opposite directions. Holding a circular coil traversed by a continuous electric current near to a similar circuit free to move, we find that when the circuits are parallel to each other there is an attractive force between them if the currents in adjacent parts of the circuits flow in the same direction, and a repulsive effect if they flow in the opposite. This is the electro-dynamic action discovered by Ampère and utilised in the construction of instruments for the measurement of electric currents in practical work. If one conducting coil, such as that of an electro-magnet, is traversed by an alternating current, and the other is simply a closed circuit or coil placed a little distance off, but in its field, it has been previously explained that the closed circuit becomes the seat of an alternating induced current, which, if the inducing current is sufficiently powerful, can be made to render itself evident by illuminating a small incandescent lamp placed in the secondary circuit.* We notice, however, that in performing the experiment the secondary circuit must be so placed that the magnetic induction of the primary coil perforates through the secondary circuit. If the secondary circuit is held in such a position that the reversal of direction of the primary current causes no reversal of direction of the magnetic field traversing the secondary circuit, because it is not linked with any of the lines of induction of the primary, the secondary circuit is no longer the seat of any induced current.

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* The experiments described in the following paragraphs can be shown with an alternating current magnet, having a core formed of a bundle of fine iron wire about 3in. in diameter and 12in. long, excited by an alternating-current dynamo, giving a current at an electromotive force of about 100 volts. A small shelf around the core a little above the middle serves as a support for rings, &c., to be projected. The performance of these experiments on a scale suitable for large audiences requires from 10 to 15 horse-power at least, and can hardly be shown well unless the alternator can provide a current of 100 amperes at 100 volts available at the moment of maximum demand.

This electromagnetic induction thus taking place across space is not stopped by the interposition of a non-conducting screen. The magnetic induction passes freely through a deal board or a plate of glass, but if we interpose a thick sheet of copper (Fig. 112) we thereby screen the secondary circuit from the inductive action of the primary. The rapid heating of this copper screen makes us aware that the secondary currents are induced in the copper sheet in the form of eddy currents, and it therefore screens the secondary circuit, as already explained, because the inductive action of these eddy currents on the side remote from the magnet is exactly equal and opposite in inductive effect to that of the primary circuit on the secondary coil.

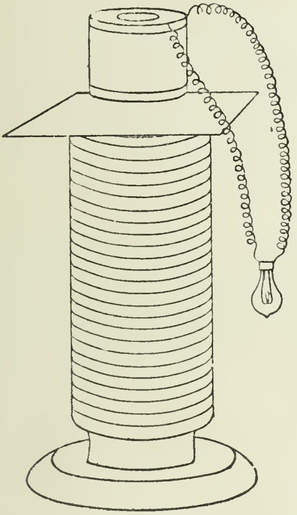

Fig. 112.—Copper Plate interposed between a Primary and a Secondary Coil and shielding the Secondary from Induction.

If a continuous current is sent through the coils of an electro-magnet, and magnetises its iron core very powerfully, it is found to be impossible to strike the pole of the magnet a sharp blow by means of a sheet of copper. Holding a sheet of copper over such a magnetic pole, and exciting the magnet, the hand holding the copper sheet feels a repulsive action at the moment when the current is put on and an attractive action when it is cut off. If we try to slap the magnet pole sharply with the copper sheet, it is found that this repulsive force prevents anything like such a sharp blow being given to the pole when the current is on as can be given when the current is off. Moreover, when a very powerful electro-magnet is employed, it is found that a disc of copper let fall over the pole does not fall down sharply and quickly on to it when the current is flowing through the coils of the magnet, but settles down softly and slowly as if falling through some viscous fluid. The correct explanation of these facts is to be found in the statement that the motion of the conductor towards the magnetic pole causes eddy electric currents to be generated in it by electro-magnetic induction, and that these, being in the opposite direction to the exciting current of the magnet, cause a repulsive force to exist between the inducing and secondary circuits, which creates the apparent resistance we feel.

In order to exhibit the stress brought into existence between an electro-magnet and a metal sheet held near it when induced currents are set up in the disc, we may arrange the following experiment:—Over the pole of a powerful electro-magnet we balance a small disc of copper, the size of a penny, carried on one end of a delicately-balanced bar. A mirror attached to that bar serves to reflect on to a screen a ray of light indicating the smallest motion of the copper disc. On magnetising the magnet the copper is suddenly repelled, but comes to rest again immediately in its initial position. When the magnet is demagnetised the copper experiences a momentary attraction. Or we may illustrate the same action in another way. Consider, for instance, a ring of copper hanging in front of the pole of an electro-magnet (*see* Fig. 113), having the plane of the ring perpendicular to the lines of magnetic force proceeding out from the pole. Let the magnet be an electro-magnet, and let the pole be suddenly made a north or marked pole. Lines of magnetic force are thrust into the aperture of the ring. This magnetic flux, in accordance with a well-known law, generates an inductive electromotive force, which causes a transient current to flow round the ring in a counter-clockwise direction, as looked at *from* the north magnetic pole. The ring becomes virtually a *magnetic shell*, having a north pole facing the north pole of the exciting magnet. By the fundamental laws of action between currents and magnets established by Ampère, the ring experiences a slight repulsive force, due to the electro-dynamic action between the current in the ring and the magnetic pole. The generation of the momentary induced current in the ring is accompanied by an electro-dynamic impulse tending to thrust it away from the pole.

Suppose, next, that the electro-magnet is demagnetised. The ring has generated in it a reverse induced current flowing in the direction the hands of a clock move when looked at from the magnetic pole. This is also accompanied by an electro-dynamic attraction of the ring towards the pole, but which is much more feeble than the previous repulsion. These attractions and repulsions are obviously due to the Ampèrian stress set up between the magnet and the metal by reason of the induced currents set up in the latter. It has been pointed out by Prof. S. P. Thompson that Ampère himself probably observed an effect of this kind (*Proc.* Phys. Soc. of London, Vol. XIII., p. 493, "Note on a Neglected Experiment of Ampère."). Impulsive effects of this nature have been also studied by Prof. Vernon Boys (see *Proc.* Phys. Soc. of London, Vol. VI., p. 218, "A Magneto-electric Phenomenon").

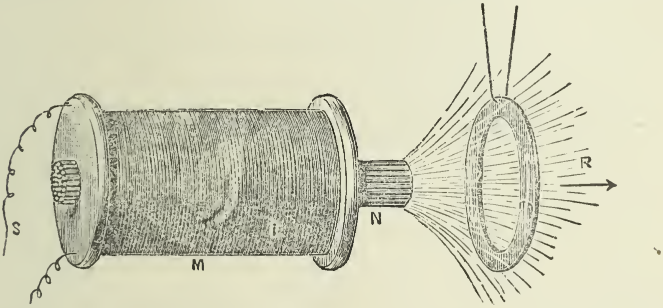

FIG. 113.—Copper Ring hung in the Field of an Electro-magnet, and Repelled or Attracted when the Current is put on or cut off.

Let us in the next place consider a circuit, say a closed conducting ring, suspended in front of the pole of an electro-magnet, and let the coils of the electro-magnet be transversed by an alternating current of electricity (Fig. 113). The magnetic field of the magnet is then an alternating field. We shall suppose it to vary in strength according to a simple periodic law. The closed circuit is therefore subjected to an inductive action, and we know that the induced electromotive force in that circuit is at any instant proportional to the rate of change of the magnetic field in which it is immersed. If, therefore, the variation in strength of that field is represented geometrically by the ordinates of a periodic curve, the varying electromotive force acting in the ring circuit is represented by the ordinates of another such curve of equal wave length, shifted a quarter of a wave length behind the first. In the diagram (*see* Fig. 114) the variation of the induced magnetic field, and the induced electromotive force in the circuit, are represented as usual by two harmonic curves. This induced electromotive force creates an induced current flowing backwards and forwards in the ring, and we shall, in the first instance, suppose that this current flows in exact synchronism with its electromotive force. The induced current and the inducing magnetic field may therefore be represented as to relative phase and strength by the curves in the diagram (Fig. 114). The dynamical action,

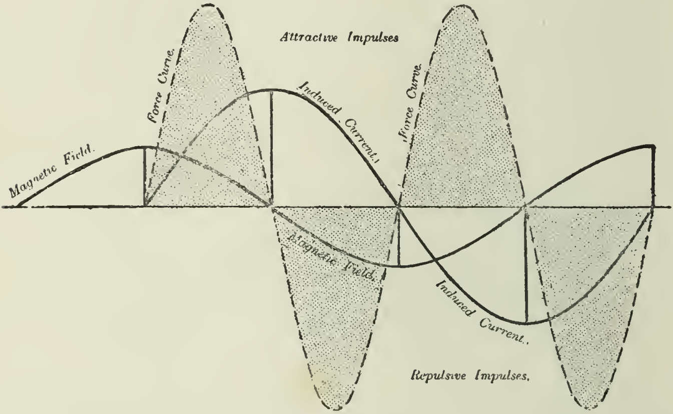

FIG. 114.—Diagram showing the Equality of the Attractive and Repulsive Impulses in a Non-inductive Circuit when held in an Alternating Magnetic Field.

or the force which the ring experiences, is at any instant proportional to the product of the strength of the magnetic field in which the ring is immersed and to the strength of the induced current created in it. If we multiply together the numerical values of the ordinates of these two curves at any and every point on the horizontal line, and set up a new ordinate at that point representing this product, the extremities of these last ordinates define a curve, which is a curve representing the *force* acting on the secondary circuit; and it is seen from the diagram (Fig. 114) to be a wavy curve having a wave length equal to half that of the first two curves. Moreover, the whole area enclosed between the outline of this force curve and the horizontal line represents to a certain scale the time integral of that force, or the *impulse* acting on the secondary circuit, and the theory shows us that, under the assumptions made, the secondary circuit so acted upon experiences in each period of the current four impulses, two positive or repulsive, and two negative or attractive. Hence, it follows that such an ideal conducting circuit held in front of an alternating electro-magnet should experience a rapid alternate series of equal pushes and pulls, or of little impulses to and from the magnet. These equal and opposite impulses in quick succession would neutralise one another, and our supposed circuit would not, on the whole, be subject to any resultant force.

When we present a real conducting circuit to the pole of an electro-magnet traversed by a powerful alternating current, we find that under the actual circumstances there is a powerful repulsive action between the pole and the circuit. With a powerful alternating current electro-magnet striking effects of repulsion may be thus shown.

If we hold a copper ring over the pole of a powerful vertical alternating electro-magnet, we find at once that there is a perceptible and strong repulsion. Letting the ring go, it jumps up into the air, impelled so to do by the electro-magnetic repulsion acting upon it (Fig. 116). All good conducting rings will execute this gymnastic feat, and rings of copper and aluminium are found to be most nimble of all. Rings of zinc and brass are sluggish, and a ring of lead will not jump at all. Prof. Elihu Thomson was the first to call attention to this strong repulsive action between conducting rings and an alternating electro-magnet. He has thus described his first notice of these effects:—"In 1884, while preparing for the International Electrical Exhibition at Philadelphia, we had occasion to construct a large electro-magnet, the cores of which were about 6in. in diameter and about 20in. long. They were made of bundles of iron rod about $\frac{5}{16}$in. in diameter. When complete the magnet was energised by a current from a continuous-current dynamo, and it exhibited the usual powerful magnetic effects. It was found also that a disc of sheet copper of about $\frac{1}{16}$in. in thickness and 10in. in diameter, if dropped flat against a pole of the magnet, would settle down softly upon it, being retarded by the development of currents in the disc, due to its movement in a strong magnetic field, and which currents were of opposite direction to those in the coils of the magnet. In fact, it was impossible to strike the magnet pole a sharp blow with the disc, even when the attempt was made by holding one edge of the disc in the hand and bringing it down forcibly towards the magnet. In attempting to raise the disc quickly off the pole a similar but opposite action of resistance to movement took place, showing the development of currents in the same direction as those in the coils of the magnet, and which current, of course, would cause attraction as a result. The experiment could be tried in another way. Holding the sheet of copper by one

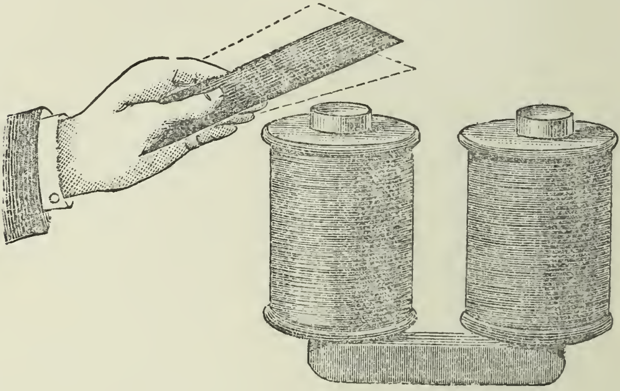

Fig. 115.

edge, just over the magnet pole (*see* Fig. 115), the current in the magnet coils was cut off by shunting them. At that moment was felt an attraction of the disc, or a dip towards the pole. On starting the current the plate experienced a powerful repulsion." The question may then be asked: Why is it the metal rings are always repelled by the alternating magnet? The explanation is not difficult to find. The real ring possesses a quality, called its inductance, of which we took no account in our examination of the case a moment ago. As a consequence of this inductance we have seen that the current induced lags in phase behind the inducing electromotive force.

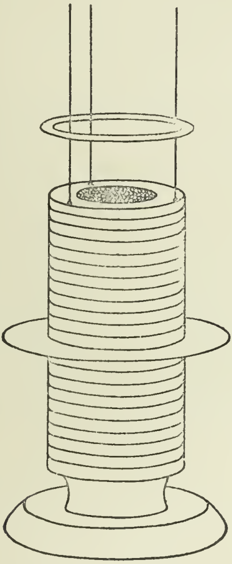

FIG. 116.—Aluminium Ring projected from the pole of an Alternating Electro-magnet, and floating over the pole when restrained by three strings.

We have then to correct the diagram considered just now, to make it fit in with the facts of nature, and we must represent the periodic curve which stands for the fluctuations of the induced current in the ring as shifted backwards or lagging behind the curve which represents the electromotive force in the circuit brought into existence by the fluctuating magnetic field. Making this change (Fig. 117), and forming, as before, a force curve to represent the impulses on the ring, we then find that, owing to the "lag" of the secondary current, one set of the impulses, namely, the positive or repulsive impulses, has been enlarged at the expense of the negative or attractive impulses. Theory, therefore, points out that, as a consequence of the self-induction of the ring, the balance between the attractive impulses and the repulsive impulses is upset, and that the latter predominate. The real ring behaves therefore, very differently to the ideal non-inductive ring. The real ring is strongly repelled, because the resultant action of all the impulses is to produce, on the whole, an electro-magnetic repulsion. This repulsion is evidence of the self-induction or inductance of the circuit exposed to the magnetic field, and it forms a new way of detecting it. But although this is part of the truth, it is not the whole truth. The lag of the induced current in the ring, and hence the predominance of the repulsive impulses, depends on the conductivity of the material of which the ring or circuit is made; and the better

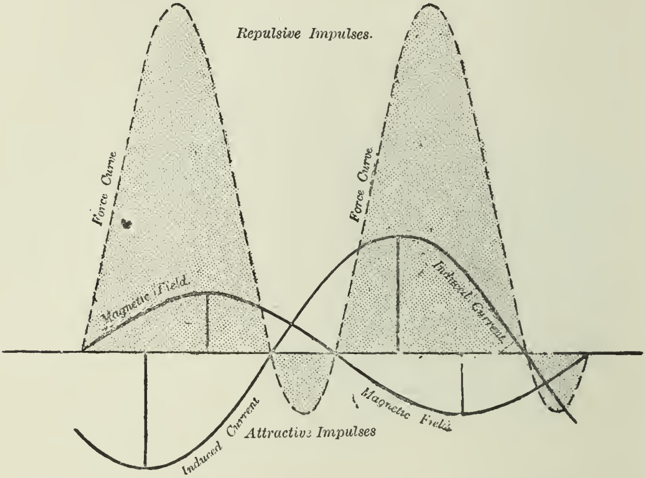

Fig. 117.—Diagram showing the Inequality of the Attractive and Repulsive Impulses in the case of an Inductive Circuit when held in an Alternating Magnetic Field.

this conductivity the greater is that repulsion, because both the induced current and the lag are thereby increased. Hence it comes to pass that there are two factors involved in making this repulsive effect, the conductance of the ring or disc and its inductance. For equal conductivities, the greater the self-induction the greater the repulsion. For equal self-inductions, the greater the conductivity of the circuit so much the more repulsive effect will be produced.

We can show the effect of the relative conductivity of discs of equal size, and therefore of equal self-induction, by *weighing* similar discs of various metals over an alternating pole. If we take discs of copper, zinc, and brass of equal form and size, and weigh these discs on the scale pan of a balance placed over the pole of an alternating current magnet, the scale pan being made of a good non-conductor, we can measure the electro-magnetic repulsion on the disc by the loss in weight it experiences.*

The same result can be illustrated by placing over the pole of our alternating magnet a paper tube. Taking one of the copper rings, and first exciting the magnet, we let the ring drop down the tube. It falls as if on an invisible cushion that buoys it up, and it remains floating in the air. If rings of different metals and equal size are placed on the tube, they float at different levels like various specific-gravity beads in a liquid. The greater the conductivity of the ring the greater is the repulsion on it in any given part of the alternating field, and hence the highly conducting rings will be sustained in a weaker field than the feebler conducting ring, assuming the rings to have about equal weights. Moreover, we are able to show by another experiment the fact that these rings are traversed, when so held, by powerful electric currents. If we press down the copper ring upon the zinc or brass ring floating beneath it, the rings are attracted together and the copper ring holds up the zinc. This is obviously because the rings are all traversed by induced currents circulating in the same direction.

It is, of course, an immediately obvious corollary, from all that has just been said, that any cutting of a ring or disc which hinders the flow of the induced currents causes the whole of the repulsion effects to vanish. We illustrate this by causing a ring of copper wire to jump off the pole, and then cutting it with pliers, find it has ceased to be capable of giving signs of life. When the metallic masses or circuits which are presented to the alternating magnetic pole are of very low resistance the electro-magnetic repulsion may become very powerful, many pounds of thrust or push being produced by apparatus of quite moderate size. It is, in fact, quite startling to hold over the pole of a very powerful alternating magnet a very thick plate of high conductivity copper. It would greatly surprise anyone not acquainted with these principles to be told that a massive copper ring weighing eight or ten pounds could be made to float in the air, but it is possible to show this easily. The ring needs to be tethered by light strings (Fig. 118) to prevent it from being thrown off laterally, although these strings in no way support its weight.

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* Experiments of this kind have been made by M. Borgman. See *Comptes Rendus*, No. 16, April 21, 1890, p. 849, and also February 3, 1890, Vol. CX., p. 233.

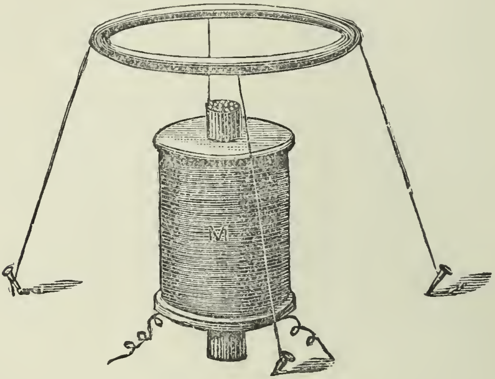

FIG. 118.—Copper Ring "floating" in air over the pole of an Alternating Current Electro-magnet, when restrained by strings.

One of the most beautiful of Prof. Elihu Thomson's experiments exhibits this effect of electro-magnetic repulsion on a closed coil, which is buoyed up in water by a small incandescent lamp in circuit therewith. In a glass vase is floated a little glow-lamp like a balloon (Fig. 119). The car consists of a coil of insulated wire, and the ends of this coil are connected with the lamp. The whole arrangement is accurately adjusted to just, or only just, float in water. Placing the vase over an alternating magnetic pole, the magnetic induction creates a current in the coil which lights the lamp, and, moreover, the electro-magnetic repulsion on the coil causes the lamp and coil to rise upward in the water.

There is also another class of actions—namely, deflections and rotations—produced by electro-magnetic repulsion on highly conducting discs or rings. If the conducting ring or disc which is presented to the alternating magnet is constrained by being fixed to an axis around which it can rotate, the action may reduce to a deflective force. On presenting a flat suspended disc to the pole, the disc is prevented by its constraint from being repelled bodily; so it sets its plane parallel to the lines of magnetic induction, and places itself in a position such that the induced currents in it are reduced to a minimum. On this principle, before becoming acquainted with Prof. Elihu Thomson's original work, the author devised

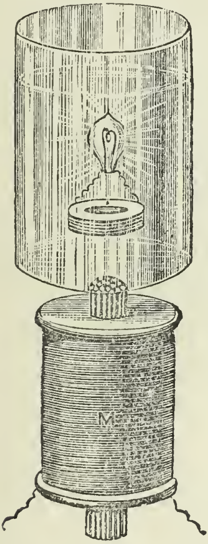

FIG. 119.—Incandescent Lamp and Secondary Coil floating in water and Repelled by an Alternating Current Electro-magnet placed beneath.

a little copper disc galvanometer for detecting small alternating currents.

This deflection by an alternating current of a copper disc suspended within a coil with its plane inclined to the plane of the coils was, in March, 1887, noticed independently by the author, who subsequently described a copper disc galvanoscope for alternating currents based on this fact (see *The Electrician*, May 6, 1887). He did not at the time know how thoroughly Prof. Thomson had explored the phenomena, but the substantial explanation of the facts as above given had already occurred to him.

More interesting than the deflective actions are those which result in the production of continuous rotation in highly conducting bodies placed in an alternating field. We employ for this purpose an electromagnet having a laminated iron core (*see* Fig. 120), the ends of the iron circuit being provided with copper bars, which embrace and cover portions of the polar terminations of the magnet. When the magnet is excited by a periodic current, these secondary circuits become the seat of powerful induced secondary currents. Taking in hand a large copper disc pivoted at the centre and held in a fork, we hold this wheel so that part of the disc is inserted between the jaws of the electro-magnet. Immediately, rapid

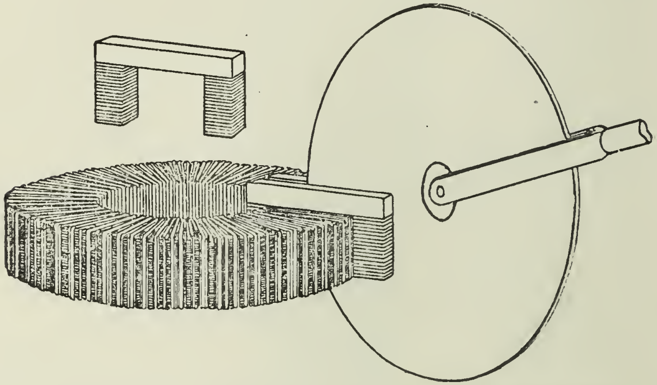

Fig. 120.—Alternating Electro-magnet with Shaded Poles, causing a Copper Disc placed between the Jaws to revolve.

rotation is produced. The reason is not far to seek. The alternating field creates induced currents, both in the closed coils and in the neighbouring portions of the disc; and the conductors in which they flow are therefore drawn together. If the polar coils are so placed as to partly shield the poles these attractive actions act unsymmetrically on the disc and pull it continuously round. The action is, perhaps, better illustrated by a simpler experiment. If we hold a pivoted copper disc (Fig. 121) symmetrically over an alternating pole, the action of the pole is one of pure repulsion on the disc, and it causes no rotation in it. When a copper sheet is so placed as to shield or "shade," as Prof. Thomson calls it, part of the magnetic pole, currents are induced both in the fixed plate and in the movable one. The fixed disc shields part of the other from the induction of the pole, and hence causes the induced currents in that plate and disc to be so located that they are in positions to cause continual attraction between the conductors and to continuously pull round the movable disc into fresh positions, so creating regular rotation.

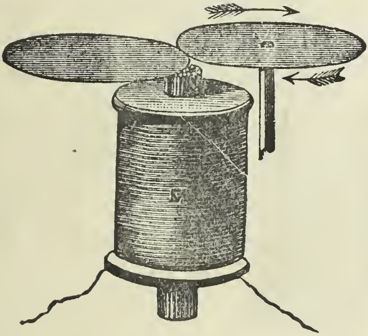

FIG. 121.—Revolution of a "Shaded" Copper Plate held over the Pole of an Alternating Current Electromagnet.

This principle of "shading" a portion of a conductor from the inductive action of the pole, and so causing the eddy currents in it to be located in a portion of its service and to cause attraction between that conductor and the shading conductor, is capable of being exhibited in various ways.

We place on a copper plate a light, hollow, copper ball (*see* Fig. 122), and support it in a little depression in a copper plate. Holding the arrangement over the alternating magnet, the ball begins to spin round rapidly when the magnet is excited. This rotation is caused by the continual attraction of the eddy currents induced in the fixed plate and in that part of the ball which is not shielded from the pole by the plate. We may vary the experiment, and exhibit many more or less curious and amusing illustrations of it. If we float these copper balls in water (Fig. 123), and place the glass bowl containing them over the alternating pole, the interposition of a copper sheet between the pole and the balls causes the latter to begin to spin in a highly energetic manner.*

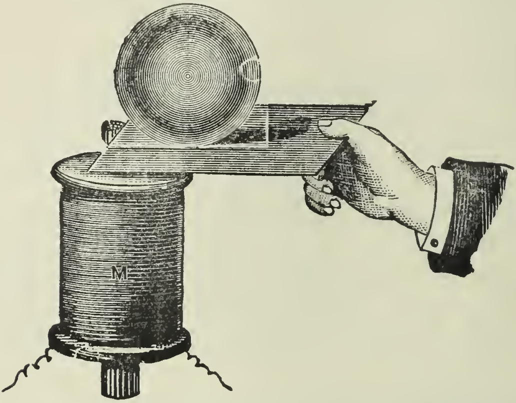

Fig. 122.—Light, Hollow, Copper Ball standing in a depression on the edge of a Copper Plate, and set in rotation when held over the Pole of an Alternating Electro-magnet.

Amongst other illustrations of the principles above described Prof. Elihu Thomson invented a novel form of electromagnetic gyroscope (Fig. 124). Over the alternating magnet a gyroscope of the usual form is suspended. The wheel of the gyroscope is made of iron, and the tyre of the wheel is a thick copper band. Immediately the iron core is magnetised, the gyroscope begins to rotate with great rapidity over the pole. In this case the unsymmetrical disposition of the eddy currents in the copper band around the wheel is sufficient by itself to cause the rotation to occur. The phenomenon, however, which lies at the bottom of all these effects is that the self-induction of the secondary circuit causes the eddy currents to be delayed in phase behind the magnetising field, and hence to persist into the period of reversal of that field, and so produce the repulsion between the primary conducting circuit and that part of the secondary conducting circuit in which the eddy currents are set up.

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* For a mathematical discussion of these electro-magnetic rotations, see *Phil. Trans.* Royal Soc., Vol. CLXXXIIIa., 1892, p. 279, Mr. G. T. Walker on "Repulsion and Rotation produced by Alternating Electric Currents."

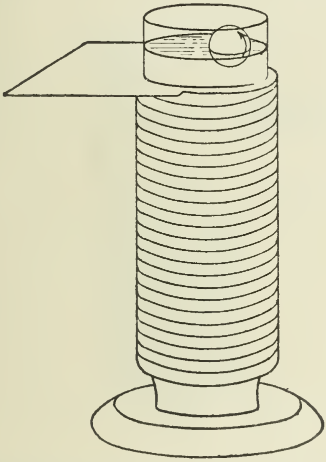

Fig. 123.—Hollow Copper Ball floating in water over an Alternating Current Electro-magnet, and caused to revolve by the interposition of a "shading" copper plate.

One more experiment in this part of the subject may be referred to. Returning to the use of the electro-magnet, in which the iron circuit is all but complete, we find that, when a highly-conducting disc is put between the closely approximated half-shielded jaws of this electro-magnet, and an alternating current employed to excite it, the conducting disc is held up in the air gap by reason of the attraction set up between the currents induced in the disc and the shielding polar plates. If, however, the disc has a relatively poor conductivity the attraction is not nearly so marked. A good or bad silver coin can thus be discriminated, because the good silver coin has sufficient conductivity to be the seat of powerful induced currents, but the bad coin has not.

Closely akin to the foregoing, but more difficult to explain, are the rotations in copper and iron discs which can be caused by the approximation to them of a laminated iron bar alternately magnetised. These actions have been carefully studied by Prof. Elihu Thomson, and applied by him and others in many practical devices. Across the top of an electro-magnet is placed a long bar of laminated iron with the plane of the lamination vertical (Fig. 125). This bar is surrounded at intervals by copper bands, which form small closed secondary circuits upon it. If we excite the magnet and hold near the bar an iron disc capable of free rotation, it begins to rotate rapidly. Not only can this be done with a laminated bar throttled by conducting circuits, but even a solid bar of hard steel will serve the same purpose, and a couple of steel files placed across the poles can cause rapid rotation in pivoted discs of copper or of iron held with their edges close to the bars so alternately magnetised.

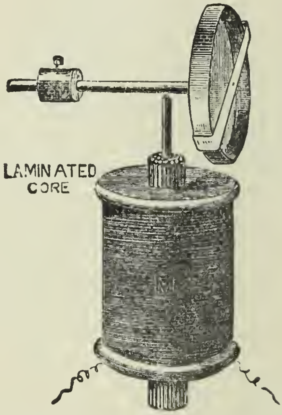

Fig. 124.—Electromagnetic Gyroscope Revolving over the Pole of an Alternating Current Electro-magnet.

To understand the operations which produce this rotation, we have to return to some elementary principles. Consider a conducting ring held in front of the electro-magnet as in Fig. 113. Let a sudden flux of magnetic induction be made through the ring aperture, that is, in common parlance, let "lines of force" be thrust through the opening of the ring. If these lines proceed out from a north pole, they will create an electromotive force in the ring in such a direction as to make a current flow counter-clockwise round the ring as seen from the pole. This current in the ring itself creates a magnetic field round the ring, and a consideration of the direction of the current will show that in the central aperture of the ring the direction of the inducing field and the field due to the induced current are in opposite directions. The effect of this opposition is to retard the formation of the field due to the magnet in the aperture of the ring. In other words, the current induced by the internal field causes the lines of induction due to the external pole to be, as it were, momentarily thrust out laterally, and resisted in their endeavour to pass through the aperture of the ring.

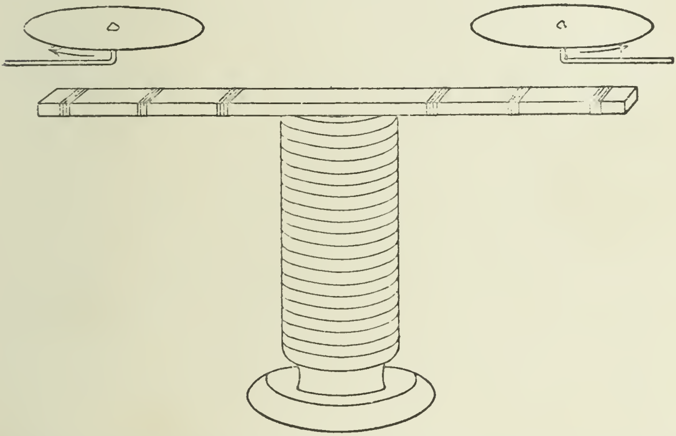

Fig. 125.—Alternating Current Magnet with Laminated Iron Bar across its pole causing Revolution of two Iron Discs held near its extremities.

If we consider a bar of iron surrounded with a copper band, and imagine that this bar is suddenly acted upon by a magnetising force at one end, the result of the current induced in the ring will be that for a brief time the magnetic induction in the bar will be caused to leak out laterally, and go round the copper ring on the outside, its passage through the ring being resisted. If, then, we throttle a magnetic circuit, such as a laminated iron bar, with copper coils closed upon themselves, and place a magnetising coil at one end, the closed conducting circuits hinder the rise of magnetic induction in the bar; or, in other words, give it what may be called *magnetic self-induction*. If the source of magnetism is a rapidly-reversed pole, the consequences of this delay or "lag" in the induction is that a series of alternating magnetic poles are always travelling with retarded speed up the bar, and these may be considered to be represented by tufts of lines of magnetic induction which spring out from and move laterally up the bar. If the bar is not laminated and not throttled, the eddy currents set up in the mass of the bar itself act in the same way, and operate to resist the rise of induction in the bar and to delay the propagation of magnetism along it. Hence we must think of such a throttled bar, when embraced by a magnetising coil at one end, as surrounded by laterally moving bunches of lines of magnetic induction, which move up the bar. Each reversal of current in the magnetising coil calls into existence a fresh magnetic pole at the one end of the bar, which is, as it were, pushed along the bar to make room for the pole of opposite name, which appears the next instant behind it. When an iron disc is held near such a laminated and throttled bar, these laterally moving lines of force induce poles in the disc which travel after the inducing poles, and hence the disc is continually pulled round. If the disc is a copper disc, the laterally moving lines of magnetic force induce eddy currents in the disc, and these, by the principle already explained, create a repulsion between the pole and that part of the disc in which the eddy currents are set up, causing revolution of the disc.

An interesting application of the above principle has been made in the meter of Messrs. Borel, Wright and Ferranti for measuring alternating currents. It consists of a pair of vertical electro-magnets (*see* Fig. 126), with laminated iron cores, and each magnet bears at the top a curved horn of laminated iron which is throttled by copper rings. These curved horns, springing from the magnets, embrace and

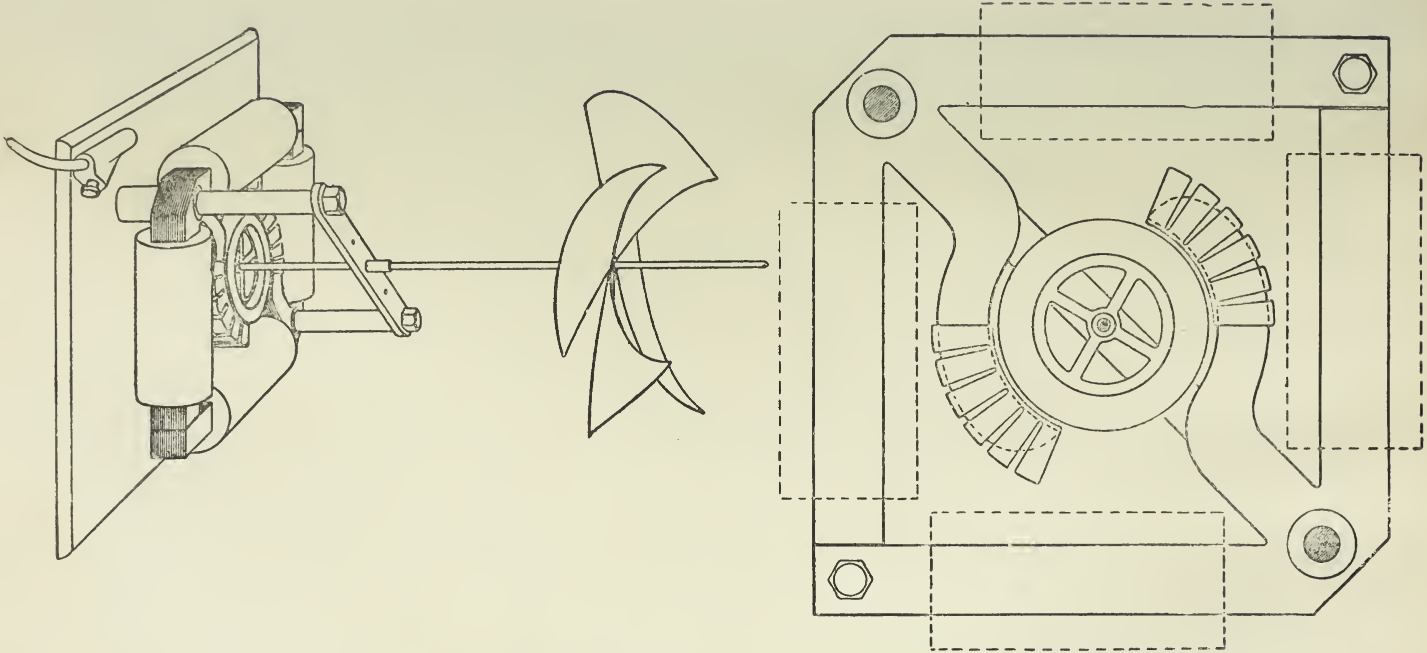

FIG. 126.—Plan and General View of Wright-Ferranti Self-starting Alternating Motor working a Fan.

nearly touch a light iron-rimmed wheel, free to turn in the centre. The actions just explained drive the wheel round, when the magnet coils are traversed by an alternating current. The iron wheel carries on its shaft a set of mica vanes, which retard the wheel by air friction. Under the opposing influences of this retardation and the electro-magnetic rotation forces, the wheel takes a certain speed corresponding to different current strengths in the magnetic coils, and hence the total number of revolutions of the wheel in a given time, as recorded by a counter, serves to determine the total quantity of alternating current which has passed through the meter.

The rotation of iron discs can be shown also by means of a badly-designed transformer. If a closed laminated iron ring (Fig. 127) is wound with a couple of conducting circuits, such an arrangement constitutes a transformer. If these two circuits are wound on opposite sides of the iron ring, the previous explanations show that the arrangement will be productive of great magnetic leakage across the iron circuit. In designing transformers for practical work, one condition amongst others which must be held in view is to so arrange the conductive and magnetic circuits that a great magnetic leakage of lines of force across the air does not take place. If, however, this leakage exists, it indicates that the secondary circuit is not getting the full benefit of the induction created by the primary. To detect it we have merely to hold near the

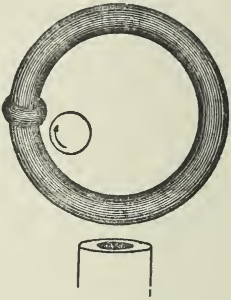

Fig. 127.—Magnetic Leakage across a Throttled Iron Ring, causing rotation of an Iron Disc placed near the Secondary Coil.

iron circuit a little balanced or pivoted iron disc, and if it is set in rapid rotation it indicates that there are laterally-moving lines of magnetic force outside the iron, which have escaped from the iron in consequence of the back-magneto-force of the secondary circuit.

The above described phenomena have been utilized in the construction of measuring instruments of various kinds, and the effects due to the magnetic leakage of magnetic fields will be found to have applications which will be considered more carefully in discussing the action of transformers.

§ **13. Symmetry of Current and Induction.**—A consideration of the effects described in the present chapter will have disclosed to the careful reader that there is a complete symmetry between the two fundamental quantities, electric current and magnetic induction. Let the diagrams in Fig. 128 represent a circuit of iron (magnetic circuit) linked with a circuit of copper (conductive circuit) and let the iron circuit have wound on it a magnetising coil capable of imposing a magneto-motive force (M.M.F.) on it, whilst the conductive circuit has a source of electromotive force (E.M.F.), say a battery, introduced into it. Suppose, then, that this E.M.F. is suddenly introduced in the conductive circuit, the linking of this circuit with the iron circuit bestows considerable self-induction on the conductive circuit, and this operates to delay the rise of the current strength in the conductive circuit when the E.M.F. is suddenly applied. If the two circuits were plunged into a good conducting liquid medium, the action of the iron circuit would be to cause a leakage of current across the conductive circuit. Quite similarly, we find that if a magnetomotive force is suddenly applied to the iron circuit, the induced current set up in the conductive circuit opposes the growth of the induction in the magnetic circuit, and, as air is not an insulator for magnetic induction, it causes a leakage of magnetic induction across the circuit. Hence the growth of electric current in the conductive circuit is hindered by linking it with an iron circuit, and the growth of magnetic induction in an iron circuit is hindered by linking it with a copper circuit. This is only one instance of the fact that the laws of current establishment in conductive circuits are similar to the laws of establishment of magnetic induction in magnetic circuits.

The growth of current from surface to centre of conductors has been described in sections of this chapter. The gradual soaking in, or growth of the magnetic induction, from the surface to the centre of the iron cores of electro-magnets, when a sudden external magnetising force is applied, has been experimentally examined with great skill by Dr. J. Hopkinson and Mr. E. Wilson, and the reader is referred for a full account of their work to *The Electrician*, Vol. XXXIV., 1895, p. 510. Very briefly it may be said that these experiments consisted in showing experimentally that in the case of an electro-magnet with a very large solid iron core the magnetic induction in the iron is not established instantaneously at its full value at all points in the iron when the magnetising force is

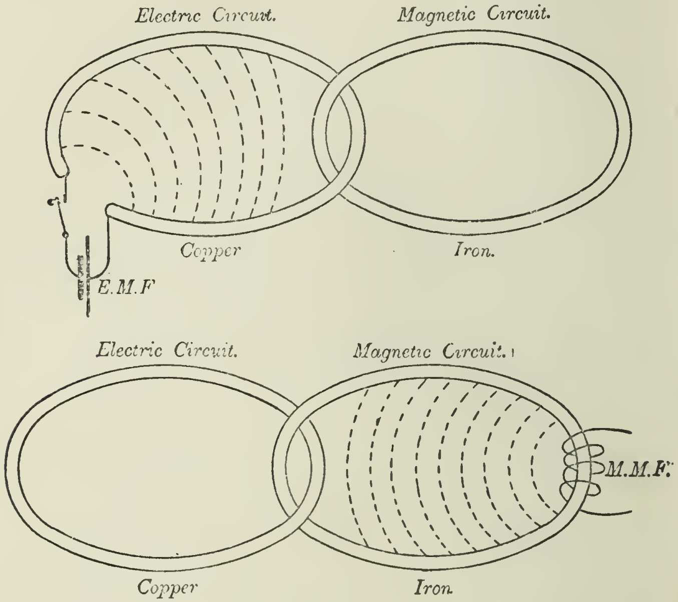

FIG. 128.—Diagrams illustrating the Symmetry in relation between Electromotive Force and Electric Current, and Magnetomotive Force and Magnetic Induction.

applied, but it begins at the surface of the core and slowly works inwards to the centre. In an entirely similar manner we know that in a conductor of large cross-section the actions involved in the production of a current in the conductor establish the current first at the surface of the conductor, and the central portions of the conductor are only reached by it after a certain finite time. We shall return in the next chapter to the discussion of some of these theoretical questions.

# CHAPTER V.

## DYNAMICAL THEORY OF INDUCTION.

§ **1. Electromagnetic Theory.**—In the matter so far before the reader attention has been directed to the chief facts of electromagnetic induction without any inquiry into the possible mechanism by which this may be effected. Attention may at this stage be directed to modern views of the subject, which have been the outcome of the work of Faraday and Maxwell and all their illustrious followers in this field of study. The cardinal principle of these methods of viewing the phenomena is the denial of action at a distance. That is to say, if at any point in a field we find a force due to a current flowing in some conductor, this force cannot be regarded as appearing there without anything happening in the interspace, but must be the consequence of successive changes in closely contiguous places, and not the result of operations at a distance without intermediate machinery. Whenever we find an electromagnetic effect taking place at any locality we are directed therefore by these notions to look for its antecedents or consequences at the adjacent places, and the apparent phenomenon is not to be regarded as the whole of it, but to be taken as a portion of the whole of the effects which are produced in every part of the region or medium. The finite velocity of light, and the impossibility of accounting for its propagation on any other hypothesis than that of actual transmission of something across space, or the propagation of a state of stress and strain or periodic change of some kind through a medium, led to attempts to settle between the rival hypotheses by crucial experiment, with the result that the vast bulk of the accumulated evidence decides in favour of the existence in space of a *medium* which has properties not possessed by the ordinary atomic matter, but which may certainly be called a material substance in the sense that it can be the recipient or vehicle of energy. The study of the phenomena of light indicates that along the path of a ray there are certain *changes* which are periodically repeated, such that at portions of the medium separated by a distance called a wave-length, changes of a similar kind are being coincidently effected. The application of mathematical analysis to optical phenomena has led to the conclusion that we can offer a tolerably satisfactory account of them by making the supposition that there exists such a universally diffused *ether* or medium in which these changes go on. At this point, however, the profound difficulties of the subject begin. To offer a complete account of the phenomena of light, and to deduce all the observed effects from a fundamental principle, we have to construct a hypothesis as to the structure of this ether and the nature of the periodic changes which constitute the wave motion in it. The periodic changes which in the case of sound and fluid waves are known to exist suggested that in the case of the ether the periodic changes are motions of the parts of the ether relatively to one another, and that these motions are the result of displacements taking place under certain stresses. We cannot even attempt here a sketch, however brief, of the various hypotheses which have formed as to the sort of motions which may occur. On one assumption the ether has been regarded as capable of having displacements or deformations made in it against internal forces, resisting these changes similar to the shearing strains and stresses in solids. From this point of view, now sometimes called the elastic solid theory of light, we may picture this ether to ourselves as a distortable but incompressible jelly-like solid, which exists everywhere and penetrates into the interior of all material bodies. As long as the hypothesis of a universal ether was demanded merely to correlate the observed phenomena of light a limited order of facts had alone to be considered; but the conception that electric and magnetic effects also required a similar assumption, increased the difficulties to be dealt with. The mind of Faraday continually turned to the thought that the medium assumed in both these regions of phenomena might be the same; and his great disciple, Maxwell, was led more definitely to formulate a similar conception. If an ether or medium is demanded as a fundamental cause of two or more classes of facts, then it is certainly unscientific to fill space several times over with ethers of different kinds until the attempt has been made to ascertain if one alone cannot be found to fulfil all the required functions. Maxwell was led therefore to the conclusion that both luminous, electric, and electromagnetic phenomena might be explained by the supposition of one single *medium* capable of certain internal changes, and possessing certain mechanical properties, and he thus avoided the unscientific process of thought of postulating two different *ethers* by boldly adopting the hypothesis that the medium on which electric effects and optical phenomena depend for their existence is one and the same. We shall see later how this supposition has been supported.

One important element in Maxwell s electric theory is his conception of *electric displacement.* When an electromotive force acts upon any part of a dielectric which is uniform and non-crystalline it is assumed that at all points along the line of electrostatic induction there is an *electric displacement,* as Maxwell calls it. The theory does not tell us what is the physical nature of this displacement. We may, in the first place, merely for the purpose of illustration, suppose that the unknown something which we call electricity is moved along a line of induction, and that on the removal of the electric force it returns to its original position, and that a dielectric or insulator is a material in which the electricity, when displaced by the application of an electrostatic stress or force, resists this displacement in virtue of an *electric elasticity.* The apparent charge on conductors, according to this view, is the electricity displaced out of, or into, the dielectric, and positive charge or electrification may be regarded as the possession of an excess which is extruded from the dielectric on to the conductor, and negative as a deficit when the conductor gives up some to the dielectric.

Maxwell's next principle is that change in electric displacement *is* an electric current whilst the change lasts. He calls this a *displacement current,* to distinguish it from a current in conductors called the *conduction current.* The displacement current is supposed to have, however, all the properties of an electric current. Conducting bodies must be regarded as those in which there is no elastic force resisting displacement, or, in other words, have no electric elasticity, and in which, therefore, electric displacement can go on continuously. The existence of a current of conduction is recognised by two co-existing effects—first, the dissipation of energy into heat; and second, the existence of magnetic force the direction of which is along closed lines described around the line of the current. The displacement current in dielectrics, which takes place at the instant of applying or changing the electric force, is also considered to be accompanied by magnetic force. In other words, we must consider the displacement current which takes place in a dielectric when electrostatic force acts on it as a very brief conduction current, and as originating a system of lines of magnetic induction—surrounding it, just as a conducting wire is so surrounded, by its loops or closed lines of magnetic induction. Conversely, when lines of magnetic force penetrate through an insulator or dielectric, any change in the density of these lines creates eddy displacement currents in the mass. If the lines penetrate through a conductor they produce, under similar circumstances, eddy currents of conduction, whose energy is ultimately frittered down into heat. Also, if a conductor is moved across a magnetic field so that it "cuts" lines of induction we have seen that if the conductor is a portion of a closed circuit it has a current of conduction produced in it. Similarly, if a dielectric body is moved in a magnetic field in a like manner it has during the continuance of the motion a displacement current produced in it. Since a dielectric circuit is always a closed circuit, a displacement current, or the production of electric displacement in it is always the result of any change in the magnetic field in its interior. For the purpose of obtaining a rough illustrative working model of the actions going on in a dielectric submitted to the action of electric force, it is necessary to fall back on some *material* hypothesis of electricity—that is, we must conceive of electricity as a *something* which can be displaced relatively to the molecules of the dielectric, and that it resists this displacement, and that when this displacement is made under the action of electric stress the removal of this stress causes a disappearance of the displacement. Dr. Lodge has suggested a form of apparatus serving as a rough working model of this dielectric action, in which buttons sliding along a rod, and held in certain positions by elastic strings, represent the electric particles capable of elastic displacement.*

We may quantitatively define *electric displacement* by saying that in a homogeneous non-crystalline dielectric, if a plane be drawn perpendicular to the line of action of the resultant electric force, then under the operation of this electric force the quantity of electricity displaced normally across a unit of area of this plane is called the *electric displacement*. This displacement is of the nature of an elastic strain, and is removed when the electric force is removed. Let us fix our ideas by imagining a sphere immersed in a dielectric medium to receive an electric charge of quantity Q. Suppose this sphere to be surrounded by a concentric spherical shell (Fig. 129) also immersed in the dielectric. On giving the central sphere a charge + Q we know that on the inside surface of the insulated concentric shell will appear an inductive charge − Q of equal quantity and opposite sign, and a charge + Q on the outside surface.

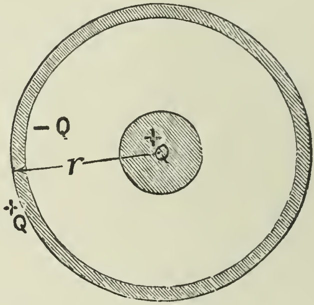

Fig. 129.

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* *See* Dr. O. J. Lodge "On a Model Illustrating Mechanically the Passage of Electricity through Metals and Dielectrics," *Phil. Mag.*, November, 1876.

Let this spherical shell be very thin and be placed at a distance $r$ from the central sphere, supposed to be very small. The electric force due to the central charge Q at the surface of the concentric shell is represented by $\frac{Q}{K r^2}$, and this force exerts a displacing action on the electricity of the shell, causing positive electricity to be displaced outwards or in the direction of the force and negative electricity to be displaced inwards or against the force.

The quantity K which appears in the above expression for the magnitude of the electric force is called the *dielectric constant*, or the *specific inductive capacity*, according to Faraday, of the medium. If the dielectric is air or other gas, K is very nearly unity, and the law of the force becomes the ordinary Newtonian law, viz., *force* varies as *quantity* divided by *square of distance*—that is, the electric force at any point due to a small quantity, Q, collected on a sphere is numerically equal to $\frac{Q}{r^2}$, where $r$ is the distance of the point from the centre of the sphere. The quantity K is assumed to have a value of unity in the case of a vacuum, and varies for known dielectrics from a little above unity for dry air up to a value of 6 to 10 for glass. In the case of metals and conducting bodies we may consider K to be infinitely great, and generally K is a number which expresses the ratio of the displacement in the given dielectric to the displacement which would take place under the same electric force if the dielectric was removed and a vacuum left in its place. The whole quantity of electricity displaced outwards through the conducting shell is $+Q$, and since the radius of this spherical shell is $r$, its surface is $4\pi r^2$ and the quantity displaced through unit of area of this shell in the direction of the force is $\frac{Q}{4\pi r^2}$. This quantity, then, Maxwell calls *the electric displacement*, and denotes by the symbol **D.**

The electric force or resultant electric intensity at all points over the spherical shell is $\frac{Q}{K r^2}$, and this quantity Maxwell calls the *electromotive intensity* at that point, and denotes it by **E** We may also speak of **D** as the electric strain and **E** as the electric stress by an extension of usual mechanical language.

The quotient of a *stress* by its corresponding *strain* is, in mechanics, called the *coefficient of elasticity* corresponding to that stress. For instance, the quotient of stretching force by longitudinal extension in the case of solid rods subjected to extending forces is called Young's Modulus of Elasticity, or the longitudinal elasticity. By a similar use of language the quotient electric stress by electric strain may be called the *electric elasticity*, and we have

$$\frac{\dfrac{Q}{K r^2}}{\dfrac{Q}{4\pi r^2}} = \frac{4\pi}{K} = \textit{the electric elasticity.}$$

Hence the series of numbers obtained by dividing the number 4 × 3·1416 by the specific inductive capacities give a series of numbers which are the *electric elasticities* of these substances.*

§ **2. Displacement Currents and Displacement Waves.**— Maxwell's second fundamental conception, as we have mentioned, is that a displacement of electricity *whilst it is* taking place is an electric current. That is to say, the variation of displacement, whether of increase or decrease, is a movement of electricity which is in effect an electric current. A dielectric must, however, be considered as a body which does not permit any but a very transient electric current passing through it. If continuous electric force is applied to it the dielectric is soon strained to its utmost extent, and no more current or flow or displacement takes place through it until the sign or direction of the electric force is reversed. A dielectric may be considered to be pervious to very rapidly reversed periodic currents, but very opaque or impervious to continuous currents. This is familiarly illustrated by the fact that a condenser inserted in a telephonic circuit does not stop telephonic communication, but does stop continuous currents. If **D** be at any instant the displacement at any point in a dielectric, and if **D** varies with the time so that $\frac{d\mathbf{D}}{dt}$ is its time rate of variation, then $\frac{d\mathbf{D}}{dt}$, or as it may be best written in Newtonian fashion $\dot{\mathbf{D}}$, is the *displacement current*, or rate of change of displacement. If at any point in a dielectric rapid changes of displacement are taking place, these variations of electric displacement are in effect electric currents producing magnetic induction in the surrounding portions of the dielectric. When we come to discuss the investigations of Hertz we shall see that this view receives support from experimental research. An electric displacement taking place all along a certain plane is equivalent to a *current sheet*, and an electric displacement taking place along a certain line is a linear current. Electrostatically speaking, lines of electric displacement are lines of electrostatic induction, and these lines, when the displacement is changing, become lines along which electric current flow is taking place. The denial of action at a distance involves the assumption that the only portions of a dielectric which can act directly upon each other are those which are in immediate contact or are contiguous.

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* The occurence of this $4\pi$ in electric and magnetic equations is an objection from some points of view. Mr. Oliver Heaviside has discussed the subject fully in his writings in *The Electrician*, and proposed a system of rational units in which it is suppressed.

§ 3. **Maxwell's Theory of Molecular Vortices.**—Given a medium possessing certain mechanical qualities, such as elasticity, a definite density, a capability of relative displacement of its parts, we may ask, is it possible to imagine a structure which will mechanically account for the effects we have to consider in electrical phenomena? A full discussion of ether theories is not possible here, but it may be of assistance to the student to place before him a general account of one such attempt to construct a mental imagery of its mechanism. We should always remember, however, that even if we are able to imagine a mechanism capable of producing even all the effects we find in Nature in any region of fact, it does not in the least follow that the real state of affairs agrees with our conception of it. Maxwell put forward his theory of *Molecular Vortices* in the *Philosophical Magazine* for 1861 and 1862. A general account of this theory has been given in the "Life of James Clerk Maxwell," and as the limits of such an elementary treatise as the present one preclude any detailed account of the mathematical portion of this theory, we shall borrow the language of the authors of the above-mentioned work* in describing it. Maxwell supposes that any medium which can serve as the vehicle of electro-magnetic energy consists of a vast number of very small bodies called *cells*, capable of rotation, which we may consider to be spherical, or nearly so, when in their normal position. When magnetic force is transmitted by the medium or acts through it, these *cells* are supposed to be set in rotation with a velocity proportional to the intensity of the magnetic force, and the direction of rotation is related to the direction of the force in the same manner as the *twist* and *thrust* of a right-handed screw. We have thus all the magnetic field filled with *molecular vortices*, as Maxwell calls them, all rotating round the lines of forces as axes. These cells as they revolve tend to flatten out like revolving spheres of fluid, and to become oblate spheroids; they thus contract along the lines of force and expand at right angles, creating a *tension* along the lines of force, and a *pressure* at right angles to them. These cells are supposed to be elastic spheres closely packed together and incapable of separating from each other. If any line of cells is set rotating the contraction of each cell along its axis of revolution must set up a tension or pull along that line, it behaves like a filament of muscular tissue, and contracts in length and swells out or increases in thickness. If several adjacent lines of cells or vortices are all set revolving in the same direction, the swelling out of each line causes them to press on each other; hence there is a lateral pressure and a longitudinal tension. In any space filled with these cells so revolving the lines of tension or axes of revolution of the cells will take up certain positions, depending on the necessity existing for the stresses to adjust themselves to equilibrium, and Maxwell has shown mathematically that such a system of cells in tension and pressure is a system which will behave in a manner similar to that in which we find actual lines of magnetic force do, and that the behaviour of magnetic poles to each other can be explained fully by the assumption of a tendency on the part of the lines of force between them to contract like elastic threads along their length, and to push one another apart when laid parallel and proceeding in the same direction. To account for the transmission of rotation from one cell to another in *the same direction*, and from one line of cells or vortex to the next, Maxwell supposed that there exists between the cells a number of extremely minute spherical bodies which can roll without sliding in contact with the vortex cells. These bodies serve the same purpose as "idle wheels" in machinery, which coming between a driving wheel and a following wheel serve to cause both to turn in the same direction. These minute spheres Maxwell supposed to constitute *electricity*. We shall speak of them collectively as the electric matter. These electric particles are furthermore supposed to be free to move in conductors; but in dielectrics they are tethered to one spot, or rather into one molecule of the substance, and can only be displaced a little way against an elastic resilience, which brings them back to their original position when the displacing force is withdrawn. Furthermore, we must assume that both cells and particles are very small, compared with the molecules of matter. The passage of electric particles from molecule to molecule in conductors, however, sets up molecular vibration, or generates heat. Something of the nature of friction must, therefore, be also postulated to account for the fact that the electric particles, when set moving in a conductor, give up energy to the molecules, and the energy is in them dissipated in the form of heat. That there is some kind of rotation going on along the lines of magnetic force has been held by Maxwell to be indicated by the behaviour of a ray of polarised light when passing through a dielectric along a line of magnetic force, and he states* that Faraday's discovery of the magnetic rotation of the plane of polarised light furnishes complete dynamical evidence that wherever magnetic force exists there is matter small portions of which are rotating about axes parallel to the direction of that force. The further assumption is made that the *cells* are composed of an elastic material, and that they can be distorted or squeezed slightly, returning again in virtue of their resistance to their original form. In order to obtain a clear conception of the interrelation of the idle wheels or the electric particles and the revolving cells or lines of induction, we may construct a mechanical illustration of one element of the mechanism as it is supposed to exist in the dielectric.

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* "Life of James Clerk Maxwell." By Lewis Campbell and William Garnett. 1st Edition. 1882. The general description of Maxwell's views in the above-mentioned work is due to Prof. Garnett, and in the annexed paragraphs the expository account of this theory is taken in part almost verbatim from the pages of this book.

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* Article "Faraday," *Encyclopædia Britannica*, 9th Edition.

Consider A and B (*see* Fig. 130) to be two wheels of india-rubber, and that C is another small wheel lying between A and B and transmitting motion from one to the other. Let C be tethered to a fixed point, D, by an elastic spring, and let C be at the same time capable of rotation round its centre. Suppose A is set in rotation, clock-hand wise, whilst B is held fast, and that the wheel C cannot slip on A, the result will be to drag down C to the position of $C_1$, stretching the spring and displacing C. Let B be then set free; the wheel C continues to roll on A, and transmits its rotation to B. Owing to the assumed elasticity of the discs A and B, the wheel C can be drawn down between them, and yet within the limits of its displacement equally transmit the rotation of A to B without slip. The same action of a preliminary displacement of C and subsequent rotation of B will take place if the wheel B possesses *inertia*—that is, if we assume it to be a heavy wheel which cannot in virtue of its mass be set rolling with finite speed in an infinitely short time.

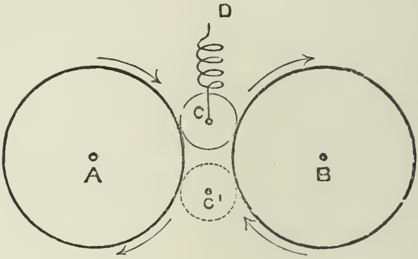

Fig. 130.

If, then, we suppose a long row of such wheels with intermediate displaceable idle wheels, the main wheels being heavy bodies, the result of causing the first wheel to rotate would be to propagate along the line a successive displacement of the idle wheels, and to set the main wheels successively in rotation. Translating these mechanical concepts into their electrical equivalents, Maxwell considers that the heavy wheels are the analogues of the molecular vortices or lines of force, and that their density is determined by what we call the magnetic permeability of the medium; the elastically displaceable idle wheels are the electricity in the dielectric; and that when a line of force is brought into existence in a dielectric, or, in other words, when a line of cells is set rotating, this action propagates itself outwards, producing successive displacements of the electric particles, or generates a displacement wave, and is accompanied by the successive appearance of rotation in the cells, or by the propagation of a wave of electromagnetic force.

The velocity of propagation of this wave will depend on the elastic forces restraining displacement, and on the inertia of the revolving vortices. We have seen that the elasticity of the dielectric is expressed by the quantity $\frac{4\pi}{K}$, where K is the specific inductive capacity. We shall see later on that the electromagnetic density of the medium is expressed by $4\pi\mu$, where $\mu$ is the magnetic permeability.

The velocity of propagation of a disturbance through an elastic medium is numerically equal to the quotient of the square root of its effective elasticity $e$, by the square root of its density $d$, or by $v=\sqrt{\frac{e}{d}}$.

If, then, for the electromagnetic medium $e=\frac{4\pi}{K}$ and $d=4\pi\mu$, we have $v=\frac{1}{\sqrt{K\mu}}$, or the velocity of lateral propagation of a wave of electric displacement or of magnetic force in a medium is numerically equal to the square root of the reciprocal of the product of its specific inductive capacity and its magnetic permeability. Such a mechanical hypothesis shows us how the spin of one line of vortices results in producing displacement of the idle wheels or electricity along lines which are circles described round the initial vortex as axis, and in propagating outwards the vortex spin or magnetic force with a finite velocity from one line of molecular vortices to another.

By the aid of the ideas which were discussed in the last section we are enabled to arrive at a mechanical conception which helps us to connect together observed facts, and which, even if not a real representation of what is taking place, is at least a working model, which may assist us to correlate the actions taking place when an electric current is started in a wire.

An electric current on this hypothesis is a flow or progression of the electric particles which are free to move forward in a conductor, and which only can move steadily forward, owing to their incompressibility, when the circuit in which they flow is a complete circuit. Suppose a thin conductor bent into the form of a very large circle, and that an electromotive force urges a procession of electric particles round it. As these particles go forward they cause the electric *cells* next them to rotate, and the motion of this line of cells embracing the line of current will be just like that which would take place if a bracelet of spherical beads strung on an elastic thread were rolled along a round rod which it closely embraces. Each bead would turn over and over, rolling on the rod, and the motion of the whole bracelet would be like that of a tightly-fitting india-rubber umbrella ring pushed along a round ruler. The progression of the electric particles would start circular vortex rings revolving round the line of motion. This corresponds to the fact that a linear current creates a magnetic field composed of circular embracing lines of forces. The first or adjacent line of vortices would, by the intervention of the idle wheels, set in rotation another set of cells lying on a concentric line, and cause them to rotate in the same manner as the first ones. Also, it would cause a backward displacement of the intermediate idle wheels, if we consider that only the central line of electric particles are conducting matter, and that the next and all succeeding rows are in a dielectric. The starting of the progressive movement of the line of electric particles in the conductor will result in an elastic displacement in the *opposite* direction of all surrounding electric particles in the dielectric along lines parallel to the line of current; and also in setting up a system of molecular vortices composed of revolving cells, the axes of these vortices being co-axial circles described round the line of flow, the rotations and displacements being propagated out laterally from the line of current. In consequence of the fact that the revolving cells are supposed to possess inertia or mass, and that all the mechanism is supposed to be rigidly connected together, a steady force applied to set the central line of electric particles in motion will not be able to produce in them the full velocity until time has elapsed sufficient to allow the inertia of the connected mechanism to be overcome. We are thus able mechanically to imitate the phenomena of *self-induction* of the circuit and the gradual rise of current strength in an inductive curcuit under the operation of a steady impressed electromotive force, and to deduce it as a consequence of the fundamental hypothesis.

Our theory, then, points out that a current should rise gradually in strength, and also that the embracing lines of magnetic force must be considered to come into existence successively as the rotation is taken up in ever-widening circles by the molecular vortices successively receiving motion of rotation. Also, on withdrawing the impressed electromotive force the inertia of the mechanism tends to make it run on for a little and the electric particles, which by their motion started the vortices, are now themselves urged forward for a little in the same direction, and this constitues the *extra current* at "break."

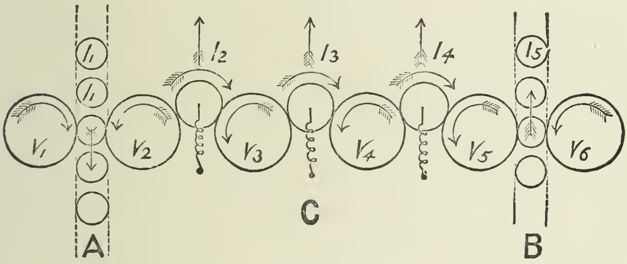

Fig. 131.

Let us next endeavour to see what ought to happen on the supposition that there are two conducting circuits in the field, both forming closed circuits, and to one of which an impressed electromotive force can be applied. Let $V_1$, $V_2$, $V_3$, &c. (Fig. 131), represent the sectional view of a series of vortex lines of electric cells, and let $I_1$, $I_2$, $I_3$, &c., be the idle wheels or electric particles. Let the row of electric particles $I_1$ be supposed to be lying inside a conducting circuit, A, represented by the dotted lines, and by our fundamental supposition, the particles $I_1$ are quite free to move along the conductor, and to rotate on their axes. Let there be another conductor, B, placed parallel to A, and let $I_5$ be the electric particles in it. The space C between is supposed to be occupied by a dielectric, and in it the electric particles can only be displaced elastically from a fixed position. We may regard these idle wheels $I_2$ $I_3$ $I_4$ as tethered by springs to one spot. Such being the mechanism, imagine that the row of particles $I_1$ is urged forward in a downward direction. As the row of particles pass between the cells $V_1$ $V_2$ they will set them in rotation in opposite directions. Owing to the inertia of the vortices the first effect of the rotation of $V_2$ will be to cause $I_2$ to roll over $V_3$ and be displaced in an upward direction; its displacement is resisted by the elastic force of the spring. The rotation of $I_2$, however, sets $V_3$ in rotation, and after a short interval $V_3$ is rotating at the same speed and in the same direction as $V_2$. $I_2$ then ceases to be displaced, because the action of $V_2$ on $I_2$, and the reaction of $V_3$ on $I_2$, simply amount to a *couple* or *twist* on $I_2$. The same sort of action results in a gradual handing on of the rotation from vortex to vortex, and a propagation of displacement from one idle wheel to the other. When the motion reaches the conductor B, the first result is to cause a displacement of the electric particles upwards, the rotation of $V_5$ not being instantly acquired by $V_6$. This amounts to a current in the upward or opposite direction. As soon, however, as the vortex $V_6$ has accepted the full speed of rotation, then the forces on the electric particles $I_5$ amount only to *twists*, and not to forces of displacement; hence the particles $I_5$ cease to experience any force impelling them forward, and come to rest in virtue of the fact that the conductor offers a resistance to their motion. They fritter down their energy of motion into heat, and come to rest. Hence the induction current in the conductor after a short flow ceases, and the vortex spin becomes equal in the vortices on either side of it. Suppose now that the impressed force in the circuit A is withdrawn, the electric particles in the A circuit are driven forward for a short time by the energy stored up in the adjacent vortices; these last, however, give up one by one their energy to the circuit A, where it is dissipated as heat. This surrender of velocity is propagated outwards until at the surface of the circuit B the state of things finally is, that when the vortex $V_5$ has come nearly to rest, the motion of $V_6$ still continues. The energy of $V_6$ and of vortices beyond expends itself in moving forward the electric particles in circuit B in the same direction as that in which the current in A was travelling originally—in other words, part of the energy of the field is spent in making a transitory current in B as well as in A in the same direction. It follows, therefore, that there is a less induction current in A at breaking circuit when a closed circuit B is present than if B were not there—that is to say the presence of a closed secondary circuit B diminishes the self-induction of the primary circuit, as is known to be the case. We see, therefore, that the theory is so far in accordance with observed facts.

The theory must, however, be taken for no more than it is worth, viz.: an attempt to construct a mechanical system which shall act in the manner in which we find electro-magnetic fields and circuits do act. The true mechanism *may* be very different; the one described has at least the utility that it shows a way in which the observed effects might be produced. The various dynamical elements in the supposed mechanism have their equivalents in the recognised electrical and electro-magnetic qualities. The angular velocity of the cells or vortices around their axis represents the intensity of the magnetic force, or the strength of the magnetic field. The angular momentum of the vortices represents the magnetic induction, hence the mass of each cell, or the *density* of the medium, is the analogue of the magnetic permeability. This is greater in paramagnetic substances than in air or vacuum, and greatest of all in iron; in fact, so exceptional is it in iron that Maxwell supposed the particles of the iron themselves to take part in the vortex action. Hence, the energy of a magnetic field is greater if that field contain iron, and accordingly the presence of iron in a core immensely increases the vortex energy for a given vortex velocity, that is, it increases the inductance of the circuit. The energy associated with any revolving cell or vortex is proportional to the product of its velocity and momentum, or the product of the magnetic force, and the magnetic induction estimated in the same direction is a measure of the energy per unit of volume existing in that portion of the field. The "number of lines of force" passing through any circuit is on this theory to be identified with the whole momentum of the molecular vortices linked with that circuit. If any circuit is traversed by lines of force or linked with lines of molecular vortices, and the cause creating this field is removed, say, by withdrawing the magnet or repressing the electric current creating it, the vortices give up their energy gradually to this secondary circuit, and it appears there as energy of motion of the electric particles or as an electric current. When one system of bodies in motion sets another set in motion by mutual action and reaction, and there is no loss of energy by anything like friction or imperfect elasticity, then the momentum gained by one must be equal to that lost by the other, and the rate of gain of momentum of the one system is at any instant equal to the rate of loss of momentum by the other. Hence, if the vortices lose momentum their rate of loss of momentum—that is, the rate of withdrawal of lines of induction from the circuit, must be equal to the rate of gain of momentum of, or to the force acting on, the electric particles which are absorbing the momentum. Hence we see that the impressed electromotive force in the circuit must be equal to the rate of withdrawal of lines of induction, and the theory conducts us to Faraday's law of induction, as a necessary dynamical consequence of our fundamental assumption. Maxwell has extended the theory of molecular vortices to the explanation of electrostatic phenomena, with which we are not, however, here directly concerned. We have seen that the theory is capable of affording an explanation on mechanical principles, of self-induction, mutual induction, and the law of electro-magnetic induction. In order to complete the theory as far as regards the phenomena of magnetism, it is necessary to suppose that the particles of magnetisable metals, such as iron, are set in rotation by the molecular vortices which traverse them, and that an increase of speed of these vortices does not increase proportionally the rotation of the iron molecules. These last behave like wheels slung loosely on a shaft, between which shaft and the wheel there is friction decreasing as the speed of rotation of the shaft increases. If, then, the wheel experiences a constant frictional resistance from external causes, indefinite increase of speed of the shaft would accelerate the wheel's rotational velocity up to a certain point, and the wheel would then cease to rotate. This supposition would enable us to make our theory agree with the fact that increase of magnetic force does not increase indefinitely the magnetic induction through iron, but brings it up to a point at which, approximately speaking, the induction remains stationary. To sum up, we may say that the hypothesis of molecular vortices is an endeavour to imagine a mechanism capable of accounting for electro-magnetic induction on dynamical principles, and on the assumption that the energy of a magnetic field is energy stored up in a medium in virtue of a particular kind of rotation of its parts.

This medium consists of portions capable of elastic displacement when we consider parts of it lying in dielectrics or capable of progressive movement when in conductors, and these portions constitute what we call electricity. Other portions are capable of rotation round closed axes of rotation, and these constitute what we call "lines of force." The medium possesses, therefore, an elastic resilience, and the reciprocal of this quality, or its freedom of yielding to electromotive force, is recognised as the *specific inductive capacity*. The medium possesses also density, and we call this its magnetic permeability, or magnetic inductance. The mass of unit of length of the vortices is equal for all vortices, whether in vacuum, air, or non-magnetic bodies, but in iron the vortices are *loaded* by the adhesion to them of the molecules of the metal, and the density is increased, and hence the permeability; but for very great angular velocities—that is, for great magnetic forces—the adhesion of the molecules and vortices must be supposed to cease, and the permeability approximates to unity. The magnetic force at any point in a field is the angular velocity of the vortex motion at that point, and the magnetic induction is the angular momentum. Magnetic attraction and repulsion is due to the tension set up along a vortex line by the polar contraction and equatorial expansion of the vortex cells. At places where there is magnetic polarity or free magnetism there is a discontinuity in the angular velocity of the vortices within and without the iron. Self-induction is the result of the *inertia* of the molecular vortices, whereby motion set up in them cannot be generated or checked instantaneously. Mutual induction, or the production of induction currents, is due to the fact that differences in the angular velocity of adjacent vortex filaments or cells cause a displacement of the electric particles or idle wheels. Finally, electromotive force is the force causing displacement of the electric particles, and electric currents consist in continuous or periodic movements of these electric particles. Electric currents always produce magnetic fields because there is nothing of the nature of *slip* between the particles and cells, and, therefore, any progressive movement of the first sets up rotation in the second, and conversely differential rotations or spins of the cells or vortices sets up displacement of the electric particles, causing either electric strain in a dielectric or electric current in a conductor.

§ 4. **Comparison of Theory and Experiment.**—The test of any physical theory is its power to predict new phenomena as well as to interpret ascertained experimental results. The theory of molecular vortices leads to the conclusion that electro-magnetic induction must be propagated through the medium with a finite velocity, and that in dielectrics of unit permeability the velocity of propagation is inversely as the square root of the specific inductive capacity. In the dynamical theory of light it is shown that the ratio of the velocity of light in vacuo to its velocity in any given transparent medium is a constant quantity for each definite wave length, and is called the *index of refraction* of that body for that wave length, and is denoted in physical optics by the symbol $\mu$. Hence, the velocity of light of definite wave-length is inversely as the refractive index for that wave-length. The refractive index for very long wave-lengths can be calculated from observed values of $\mu$ for definite rays, and hence numbers obtained representing the relative velocity of these undulations in various transparent bodies. The values of the *dielectric* *constants*, or reciprocal of the electric elasticities, of various transparent and semi-transparent bodies have also been determined, and it has been found that for a large group of bodies there is a tolerably close agreement between the values of the square root of the dielectric constant and the index of refraction $\mu\infty$ for very long waves, as shown by the selection from the results of some experimental determinations given in Table A.

**Table A.**

| K (Dielectric Constant). | $\sqrt{K}$ | $\mu\infty$ (Refractive Index). | Authority. | Reference. |
|---|---|---|---|---|
| Sulphur .......... 3·84 | 1·96 | 2·04 | Boltzmann | *Pogg. Ann.*, CLI., 1874, p. 482. |
| Colophonium.. .. 2·55 | 1·59 | 1·54 | | |
| Paraffin ............2·32 | 1·52 | 1·54 | | |
| Pure rubber ......2·12 | 1·45 | 1·50 | Schiller | *Pogg. Ann.*, CLII., p. 535. |
| Oil of turpentine 2·21 | 1·49 | 1·46 | Silow | *Pogg. Ann.*, CLVI., 1875, p. 395. |
| Petroleum ... ......2·037 | 1·43 | 1·46 | | |
| Benzine ..... ......2·198 | 1·48 | 1·48 | | |
| Petroleum spirit 1·92 | 1·38 | 1·38 | J. Hopkinson | *Trans.* Roy. Soc. 1877, 1878 and 1881. |
| Petroleum oil ...2·07 | 1·44 | 1·44 | | |
| Ozokerite ........ 2·13 | 1·46 | 1·44 | | |
| Turpentine ......2·23 | 1·49 | 1·46 | | |

For some other dielectrics, such as glass and the vegetable and animal oils, the agreement is not by any means so close but for gases, as determined by Boltzmann (*Pogg. Ann.*, CLI., 1875, p. 403), there is a fair coincidence. (*See* Table B.)

**Table B.**

| Gas. | K | $\sqrt{K}$ | $\mu$ |
|---|---|---|---|
| Air ..... .............................. | 1·00059 | 1·000295 | 1·000294 |
| Carbonic acid ..................... | 1·000946 | 1·000473 | 1·000449 |
| Hydrogen ................. ........ | 1·000264 | 1·000132 | 1·000138 |
| Carbonic oxide ... ....... ......... | 1·000690 | 1·000345 | 1·000340 |
| Nitrous oxide .................... | 1·000994 | 1·000497 | 1·000503 |
| Olefiant gas ........................ | 1·001312 | 1·000656 | 1·000678 |
| Marsh gas ........................... | 1·000944 | 1·000472 | 1·000443 |

The gases are taken at 0°C. and 760 millimetres pressure. Accordingly, we can say that, for a large group of dielectrics, of which the magnetic permeability is unity, and hence the velocity of propagation of an electro-magnetic impulse proportional to the square root of the electric elasticity or to the reciprocal of the square root of the dielectric constant, we do find a fair agreement between these numbers and the numbers representing the refractive indices or the relative velocities of propagation of very long waves or disturbances in the ethereal medium postulated to account for the phenomena of light. The imperfect agreement between the values of the refractive index for long wave-lengths and the square root of the dielectric constant for some other bodies shows that the theory is only approximately in agreement with fact, and that the results obtained by the methods adopted for determining the dielectric constant are perhaps impure, and do not give the true value of the electric elasticity. When we consider that the displacements which constitute the light wave motion of the luminiferous ether are changed some billions of times per second, it is seen to be highly probable that measurements of the specific inductive capacity in which the electric stresses are only reversed tens or hundreds of times in a second may be rendered impure or mixed owing to the presence of effects due to an imperfect electric elasticity introduced by the superposition of electric conduction or of electrolytic transport upon the true or elastic displacement effect. In fact those bodies, such as glass and the vegetable oils, which exhibit the greatest discrepancy, are those in which the chemical composition indicates a possibility of electrolysis. There may be an electro displacement in such electrolisable bodies over and above the true electrostatic displacement which is engendered by a molecular change in the body, which change results in actual decomposition when the electric force reaches a certain limit. Put broadly, it may amount to this, that the true electric displacement is a displacement of electricity within the molecule, but that in electrolisable bodies electric stress sets up a strain of the molecule itself which, within certain limits, is an elastic strain, and disappears with the removal of the stress, but that beyond these limits molecular disruption takes place. In these cases the displacement measured in taking the specific inductive capacity is the true or dielectric displacement *plus* a displacement due to strain of the molecule, and the result would be to make K appear too great, and, in fact, for glass and certain oils the values in Table C have been obtained, which in all cases are such that $\sqrt{K}$ exceeds the value of $\mu_\infty$, or the refractive index, for very long waves of light.*

**Table C.**

| Substance. | K | $\sqrt{K}$ | $\mu_\infty$ (approx.) |
|---|---|---|---|
| Glass, extra dense flint ............ | 9·896 | 3·1 | 1·5 to 1·6 |
| „ light flint ..... ............... | 6·72 | 2·59 | |
| „ crown ...... ................. | 6·96 | 2·63 | |
| „ plate ........... . ........... | 8·45 | 2·90 | |
| Castor oil .......... ................. | 4·78 | 2·18 | 1·46 |
| Sperm oil ............................ | 3·02 | 1·73 | 1·46 |
| Olive oil ...... .... .................. | 3·16 | 1·77 | 1·46 |
| Neatsfoot oil ......... . ............ | 3·07 | 1·75 | 1·45 |

J. Klemencic (abstract in the *Journal of the Society of Telegraph Engineers*, 1886, p. 108) has experimented also on the specific inductive capacity of gases and vapours, and given a table (*see* Table D) in which he compares $\sqrt{K}$ with $\mu$ (refractive index) of these same bodies. It is seen that the agreement of $\sqrt{K}$ and $\mu$ is very close for the simple gases, but that a marked difference exists in the case of more complicated molecules.

**Table D.**

| Gas. | $\sqrt{K}$ Boltzmann. | $\sqrt{K}$ Klemencic. | $\mu$ Refractive index. |
|---|---|---|---|
| Air............. ...... ............. | 1·000295 | 1·000293 | 1·000293 |
| Hydrogen...................... ..... | 1·000132 | 1·000132 | 1·000139 |
| Carbonic acid ..................... | 1·000473 | 1·000492 | 1·000454 |
| Carbonic oxide...................... | 1·000345 | 1·000347 | 1·000335 |
| Nitrous oxide ..................... | 1·000497 | 1·000579 | 1·000516 |
| Olefiant gas ........................ | 1·000656 | 1·000729 | 1·000720 |
| Marsh gas.. ... . .................... | 1·000472 | 1·000476 | 1·000442 |
| Carbonic bisulphide .. ......... | — | 1·001450 | 1·001478 |
| Sulphurous acid .................. | — | 1·00477 | 1·000703 |
| Ether ............................... | — | 1·00372 | 1·00154 |
| Ethyl chloride ...................... | — | 1·00776 | 1·001174 |
| Ethyl bromide ..................... | — | 1·00773 | 1·00122 |

The specific inductive capacity of a vacuum is taken as unity, and Boltzmann's values are given for comparison.

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* *See* Dr. J. Hopkinson, *Phil. Trans.* Royal Society, Vol. CLXXII., 1881, p. 372.

§ 5. **Velocity of Propagation of an Electromagnetic Disturbance.**—There is another line of experimental enquiry which leads to an important relation between electric and optic phenomena. This is the comparison of electrostatic and electromagnetic measurements. If two very small spheres are electrostatically charged and placed with their centres at a unit of distance apart, the stress between them may be mechanically measured. If the conductors are equally charged with opposite kinds of electricity, and the stress when at a unit of distance in air is one unit, the electric quantities are said to be unit electrostatic quantities. If such unit quantities are discharged through a conductor at the rate of one discharge per second, the resulting flow or current is called an electrostatic unit of current.

In the above definition we suppose the dielectric to be a vacuum or some substance such as air, of which the dielectric constant does not differ sensibly from unity. If $q$ and $q^1$ be two quantities measured electrostatically, and then be placed on small conductors separated by a distance $r$ in a dielectric of constant K, the dynamical force between them will be numerically equal to $\frac{q\,q^1}{\mathrm{K}\,r^2}$; and if $q=q^1$, then the force is $\frac{q^2}{\mathrm{K}\,r^2}$. Hence, if $r$ is always taken equal to unity, the real quantity of electricity producing by its action on another equal quantity a unit of force will vary as the square root of K when the experiment is performed in various dielectrics. In other words, the absolute magnitude of the electrostatic unit of quantity, and therefore also of the current, will vary as the square root of the specific inductive capacity of the medium in which the charges exist. There is another mode in which a unit of current may be defined, and this depends on the definition of a *unit magnetic pole*. If two magnetic poles of equal strength, $m$, are placed at a distance $r$ apart in a magnetic medium of permeability $\mu$, the stress or force between them will be numerically equal to $\frac{m^2}{\mu\,r^2}$, in which expression it is seen that $m$ and $\mu$ appear as quantities analogous to $q$ and K in the electrostatic analogue. Hence, when $r$ is unity, we see that to produce a unit stress between the poles $m$ the pole strength must vary as the square root of $\mu$, or the absolute magnitude of the unit magnetic pole varies directly as the square root of the magnetic inductive capacity of the medium in which the experiment is performed, the absolute unit magnetic pole being defined as a pole which at a unit of distance acts on another like pole with a unit of force in a magnetic medium, assumed to be vacuum, or some standard substance of unit permeability.

Since an electric current produces a magnetic force, it may be defined as to magnitude by agreeing that the unit of current is to be one which, when flowing in a circular circuit of unit radius, acts for every unit of length of that circuit with a unit of force on a unit magnetic pole placed at the centre of that circle. The magnitude of the force on the magnetic pole is proportional to the product of the strength of the pole and the strength of the current. Hence, if the magnitude of the unit pole is varied the magnitude of the unit of current will vary inversely as the magnitude of the strength of magnetic pole which is taken as the unit pole. When the medium is varied, the magnitude of the unit magnetic pole, or of the pole which fulfils the condition of acting on another equal pole at a unit of distance with a unit of force varies directly as the square root of the permeability of the medium. It follows, then, that the magnitude of the *electro-magnetic unit of current* varies inversely as the square root of the magnetic permeability of the medium in which the experiment is made.

We have, then, that the electrostatic unit of current is a quantity which varies directly as the square root of the electrostatic inductive capacity of the medium, or as $\sqrt{K}$, and the electromagnetic unit of current is another unit of current which varies inversely as the square root of the magnetic inductive capacity of the medium, or as $\sqrt{\mu}$. The electrostatic unit of current represents a much smaller quantity of electricity per second than the electro-magnetic—in other words the value of the ratio of the magnitude of the unit electro-magnetic current based on the definition of a unit magnetic pole, to the magnitude of the unit electrostatic current, based on the definition of a unit of electrostatic quantity, is an integer number, and a large one. This ratio of the two units of current varies when the fundamental inductive capacities of the medium is changed, but so that the ratio of the electro-magnetic to electrostatic unit varies inversely as the square root of the product of K and $\mu$. If $C_m$ is the magnitude of the electro-magnetic unit of current, and $C_s$ is that of the electrostatic unit for the standard dielectric, in which $K=1$ and $\mu=1$, then, when the dielectric is changed, $\frac{C_m}{C_s}$ is changed in the ratio of $1 : \sqrt{K\mu}$. Let $R_{vac}$ denote the value of the ratio for vacuum or for a standard dielectric, of which $K=1$ and $\mu=1$, and $R_m$ denote its value for any other medium of which the dielectric constant is K and the magnetic constant $\mu$, then

$$R_m = \frac{R_{vac}}{\sqrt{K\mu}}.$$

We have next to consider what is the physical meaning of this *ratio* of the electro-magnetic and electrostatic units.

The degree in which one quantity is greater or less than another, or to put it more precisely, that amount of stretching or squeezing which must be applied to the latter in order to produce the former, is called the *ratio* of the two quantities.* The ratio of two physical quantities is therefore the expression of the operation which must be performed on the one to make it the physical equivalent to the other. What operation must be performed on an electrostatically measured unit of electricity to make it the equivalent in every way of an electro-magnetically measured unit of electricity? The reply is, it must be set in motion with a definite velocity. The electric current produces a magnetic field. The electro-magnetic measure of current is obtained by defining the field by stating its dynamical effect on a defined magnetic pole, and the unit of electric quantity measured electro-magnetically is the quantity conveyed by the unit current so measured in a unit of time. If we imagine a circular or other conductor conveying a unit (electro-magnetic) current to have stretched alongside of it another closely adjacent conductor of like form, each unit of length of which is charged electrostatically with a unit (electrostatic) of electric quantity, we might submit the following question:—The current flowing in the first named conductor transmits a unit (electro-magnetic) quantity of electricity across each section of it per unit of time: with what *velocity* must electricity in the second conductor be set flowing in order that there may be an equality in the quantities flowing past any sections in each of the conductors, as evidenced by equality in the magnetic fields produced by the first-named current and the moving electric charge? This *velocity* is evidently a concrete velocity, which depends on the very nature of the qualities of the medium which determine magnetic and electrostatic attraction, and this velocity may be called the ratio of the magnitude of the electro-magnetic to the electrostatic unit of quantity. This velocity is evidently one which is determined by the nature of the medium, and not by the particular units of length, time, and mass selected for use in the measurements. This comparison assumes that a moving electrostatic charge is in effect the equivalent of an electric current. This has been put to the test of experiment by Prof. Rowland.* A rigid gilt ebonite disc was fixed to an axis, and could be rotated between two gilt glass discs. One member of a very delicate astatic system of magnetic needles was placed near the disc and shielded from electrostatic disturbance. On charging the gilt ebonite disc and setting it in rapid rotation it was found to affect the magnetic needle whilst rotating just as a current of electricity would have done if flowing in a circular conductor coinciding in form with the periphery of the disc. Since 1876 Prof. Rowland has again in the United States repeated the experiment and confirmed the general result. There is, therefore, experimental foundation for the view that a static charge of electricity conveyed on a moving body creates a magnetic field *whilst it is in movement.* This kind of electric current, in which a static charge is bodily moved on a conductor, is called a *convection current.* The experiment of comparing the magnitudes of an electrostatic and an electro-magnetic unit of electric quantity as above defined was first made by Profs. Weber and Kohlrausch, and the value of that ratio for a medium such as air, in which approximately we have K and $\mu$ both equal to unity, gave as a result a velocity very nearly identical with the velocity of light. Since that time very many experimentalists have determined the value of this ratio, which is denoted by the symbol "*v*." The names and the results of the observations made by some of the principal observers are set out in the Table on opposite page.

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* W. K. Clifford, "The Common Sense of the Exact Sciences," p. 99.

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* See *Phil. Mag.*, 1876, Vol. II., Fifth Series, p. 233: Dr. Helmholtz, "On the Electro-Magnetic Action of Electric Convection." These experiments of Prof. Rowland were carried out at Berlin.

One of the best determinations of the velocity of light is that made by Prof. Newcomb, at Washington, in 1882. The method employed was the revolving mirror method of Foucault, the distance between the revolving and fixed mirror being in one portion of the experiments 2,550 metres, and in the other portion 3,720 metres. The resulting velocity of light *in vacuo* is $2{\cdot}99860 \times 10^{10}$ centimetres per second.

The following results of other observations are abstracted from Prof. Everett's book, "Units and Physical Constants," 2nd edition :—

| Observer. | Velocity in centimetres per second. |
|---|---|
| Michelson, at Naval Academy, 1879 | $2{\cdot}99910 \times 10^{10}$ |
| Michelson, at Cleveland, 1882 | $2{\cdot}99853 \times 10^{10}$ |
| Newcomb, at Washington, 1882 (best results) | $2{\cdot}99860 \times 10^{10}$ |
| Newcomb (other results) | $2{\cdot}99810 \times 10^{10}$ |
| Foucault, at Paris, 1862 | $2{\cdot}98000 \times 10^{10}$ |
| Cornu, at Paris, 1874 | $2{\cdot}98500 \times 10^{10}$ |
| Cornu, at Paris, 1878 | $3{\cdot}004 \times 10^{10}$ |
| Last result discussed by Listing | $2{\cdot}9999 \times 10^{10}$ |
| Young and Forbes, 1880-81 | $3{\cdot}01382 \times 10^{10}$ |

Earlier observations gave as follows :—

| | |
|---|---|
| Roemer's method, by Jupiter's satellites | $3{\cdot}000 \times 10^{10}$ |
| Bradley's method, by stellar aberration | $2{\cdot}977 \times 10^{10}$ |
| Fizeau | $3{\cdot}142 \times 10^{10}$ |

The general result of the best determinations is that the velocity of light is very close to $3{\cdot}000 \times 10^{10}$ centimetres per second, or nearly one thousand million *feet* per second.

We have, therefore, the following facts :—The velocity $V_m$ of light of definite wave length in any medium is connected with the velocity $V_v$ of the same ray in vacuo by an equation—

$$V_m = \frac{V_v}{\mu},$$

where $\mu$ is the refractive index of that medium for the particular wave length considered, and also that the velocity V is very nearly $3 \times 10^{10}$ centimetres per second. Also we find that the ratio of the electro-magnetic to the electrostatic unit of electric quantity or current in any dielectric and magnetic

### TABLE OF SOME OBSERVED VALUES OF "$v$" IN CENTIMETRES PER SECOND.

| NAME. | Reference. | Observed value of "$v$." | Corrected value (*). |
|---|---|---|---|
| WEBER and KOHLRAUSCH } 1856 | *Electrodynamische Maasbestimmungen* and *Pogg. Ann.*, XCIX., August 10, 1856. | $3 \cdot 1074 \times 10^{10}$<br>Comparison of electric quantities. | $3 \cdot 1074 \times 10^{10}$ |
| SIR W. THOMSON and W. F. KING } 1867 1868 | *Report of British Assoc.*, 1869, p. 434; and *Reports on Electrical Standards*, F. Jenkin, p. 186. | $2 \cdot 846 \times 10^{10}$<br>Comparison of electromotive forces. | $2 \cdot 81 \times 10^{10}$ |
| SIR W. THOMSON and DUGALD MCKICHAN } 1872 | *Phil. Trans. Royal Soc.*, 1873, p. 409. | $2 \cdot 93 \times 10^{10}$<br>Comparison of electromotive forces. | $2 \cdot 89 \times 10^{10}$ |
| CLERK MAXWELL, 1868 | *Phil. Trans. Royal Soc.*, 1868, p. 643. | $2 \cdot 88 \times 10^{10}$<br>Comparison of electromotive forces. | $2 \cdot 84 \times 10^{10}$ |
| AYRTON and PERRY, 1878 | *Journal of the Society of Telegraph Engineers*, Vol. VIII., p. 126. | $2 \cdot 98 \times 10^{10}$<br>Comparison of electric capacities. | $2 \cdot 94 \times 10^{10}$ |
| SIR W. THOMSON and SHIDA } 1880 | *Phil. Mag.*, Vol. X., p. 431, 1880. | $2 \cdot 995 \times 10^{10}$<br>Comparison of electromotive forces. | $2 \cdot 955 \times 10^{10}$ |
| J. J. THOMSON, 1883 | *Phil. Trans. Royal Soc.*, 1883, p. 707. | $2 \cdot 963 \times 10^{10}$<br>Comparison of electric capacities. | $2 \cdot 963 \times 10^{10}$ |
| HIMSTEDT, 1888 | *Electrician*, March 23, 1888, Vol. XX., p. 530. | $3 \cdot 0074 \times 10^{10}$<br>Comparison of electric capacities. | $3 \cdot 0074 \times 10^{10}$ |
| KLEMENCIC, 1887 | *Proceedings of the Society of Telegraph Engineers*, 1887, p. 162. | $3 \cdot 015 \times 10^{10}$<br>Comparison of electric capacities. | $3 \cdot 015 \times 10^{10}$ |
| SIR W. THOMSON, AYRTON and PERRY } 1888 | *British Association, Bath*, and *Electrician*, September 28, 1888. | $2 \cdot 920 \times 10^{10}$<br>Comparison of electromotive forces. | $2 \cdot 920 \times 10^{10}$ |
| FISON, 1888 | *Electrician*, Vol. XXI., p. 215, and *Proc. Phys. Soc.*, Lond., June 9, 1888. | $2 \cdot 965 \times 10^{10}$<br>Comparison of electric capacities. | $2 \cdot 965 \times 10^{10}$ |
| SIR W. THOMSON, 1889 | *Royal Institution Lecture*, February 8, 1889. | $3 \cdot 004 \times 10^{10}$<br>Comparison of electromotive forces. | $3 \cdot 004 \times 10^{10}$ |

* Corrected value of "$v$" in centimetres per second for value of B.A. unit in terms of the true ohm—1 B.A.U. $= \cdot 9868 \times 10^9$ centimetres per second.

medium $R_m$ is connected with the same ratio measured in vacuo $R_v$ by an equation—

$$R_m = \frac{R_v}{\sqrt{K\mu}},$$

where K is the dielectric constant and $\mu$ the magnetic permeability.* Experiment has also indicated that within narrow limits, taking best results, $R_v$ and $V_v$ have the same value, namely, $3 \times 10^{10}$ centimetres per second, and that $\sqrt{K}$ has the same value as $\mu$ (refractive index) for media, for which $\mu$ (permeability) has the value unity. We are led, therefore, to infer that this close relationship is not a matter of accident, but that it indicates a very intimate connection between electricity and light, and that the hypothesis that light is a disturbance propagated through an elastic medium may be supplemented with some considerable show of reason by the hypothesis that electro-magnetic phenomena are the result of actions taking place in identically the *same medium* or ether. There are no transparent media for which the magnetic permeability differs by more than a very small quantity from unity, and hence the approximate identity of the values of the ratio of the units compared in air with the value of the velocity of light waves of very long wave-length; and the approximate identity for true dielectrics of the value of the refractive index and of the square root of the dielectric constant furnishes a test of the probability of the truth of the electro-magnetic theory of light. Maxwell's mathematical method of arriving at this theory consisted in forming certain equations expressing the velocity of propagation of *vector potential*, and noticing that these equations were mathematically of the same form as those which determine the velocity of propagation of a disturbance through an elastic medium. The physical meaning of this term, vector potential, may be arrived at as follows:—

Suppose a regiment of soldiers to set off marching down a street, the ranks being well spaced out. At any place in the street let two lines be drawn across the street parallel to each other and a few yards apart. Let two observers take note of how many soldiers cross each line. At any instant the total number of soldiers which are contained between the two lines is equal to the difference between the numbers which have crossed each line respectively. However irregular the movement may be, the total number of soldiers at any instant in the area or the product of the area, and the number of soldiers per unit of area within the boundary, will be equal to the number obtained by reckoning the algebraic sum of the soldiers which have from the beginning of the time crossed the whole boundary line, calling those numbers *positive* when soldiers have stepped *into* the area and *negative* when they have stepped *out* of it. We have here a simple example of the way in which a *line integral* may be the equivalent of a *surface integral.* If the area be irregular in shape and contain A square yards, and if the perimeter be $l$ linear yards, then if $n_1$ $n_2$, &c., are the number of men which have stepped across each yard length of the boundary, and if $N_1$ $N_2$, &c., are the number of men in respective square yards within the area at any instant, then $N_1 + N_2 +$, &c., to A terms or $\Sigma N$ is called a surface integral and will be equal to $n_1 + n_2 +$, &c., to $l$ terms, which is a line integral, provided that each $n$ is reckoned positive when men step in, and negative when men step out of the area over each yard of the boundary. The algebraic sum of all the stepping over the boundary all the way *round* the area is equal to the sum of the men per square yard all *over* the area. We have here given an illustration of an important proposition in mathematical physics, viz., that a surface integral, or the summation of a certain quantity *over* an area, can be replaced by a line integral, or the summation of another relative quantity all *along* the boundary line of that area. We proceed to illustrate it from an electrical point of view.

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* It is unfortunate that usage has consecrated the same Greek letter $\mu$ for refractivity in optics and magnetic inductivity in electro-magnetics. In some respects it would be an advantage in electro-optics if these quantities were differently symbolised.

Let C (Fig. 132) be the circular cross-section of an infinite straight wire conveying a current C. Round C describe a circle of radius $r$. The magnetic force at $p$ is known to be equal to $\frac{2\,C}{r}$ units, and is directed along the circumference of the circle; the line integral of the magnetic force along the dotted line is equal to $\frac{2\,C}{r} \times 2\pi\, r = 4\pi\, C$, and the surface integral of the current through the area enclosed by the dotted circle is C. Hence we have generally that the line integral of the magnetic force is equal to $4\pi$ times the surface integral of the current. This proposition is generally true, and it is easy to show that if A be any area (*see* Fig. 133) traversed normally by a current, such that the current density is $u$ over any element of area $ds$, then the integral of $u\,ds$ all over the area, or $\int u\,ds$, is equal to the line integral of the magnetic force taken along the boundary line. The mathematical operation of taking a line integral has been called by Maxwell *curling*, and we express the above proposition by saying that $4\pi$ times the total current through the area is equal to the *curl* of the magnetic force round it. On the theory that lines of magnetic force do not spring suddenly into existence in a field, but are propagated onwards from point to point in the field, it is possible to show

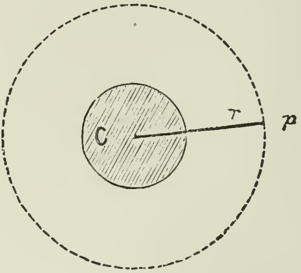

Fig. 132.

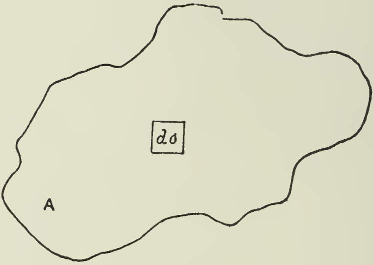

Fig. 133.

that just as the current is the curl of the magnetic force so the magnetic force is the curl of another quantity called the *vector potential.*

Let A B (Fig. 134) be a portion of a straight conductor in which a current can be started. Let $x\,x'$, $y\,y'$ be two lines drawn a unit of distance apart, parallel to each other and at right angles to the conductor. These lines bound a strip of plane space taken in the plane of the current. Draw any two transverse lines $a\,b$, $c\,d$, parallel to the conductor and separated by a small distance. We know that when a current is started in the conductor the lines of magnetic force F will be circles formed round A B as axis, and having their planes perpendicular to the plane $x\,x'$, $y\,y'$. Let us now assume that if a current is suddenly started in the conductor A B the magnetic force is propagated outwards from the conductor with a finite velocity $v$. In other words, each circular line of force must be considered to expand outwards like a circular ripple on the surface of water. When once the field has arrived everywhere at its normal value the magnetic force at a distance $r$ from the wire is $\frac{2\,C}{r}$, where C is the value of the current, and we shall suppose, as usual, that the magnetic field is indicated as to value by the density of the lines of force, or that the number per square centimetre traversing normally the plane $x\,x'$, $y\,y'$ is at any point proportional or numerically equal to the magnetic force at that point. If, then, we neglect for the moment all effect of self-induction, and suppose the current *in* the wire to rise up instantaneously to its full value, we may yet regard the

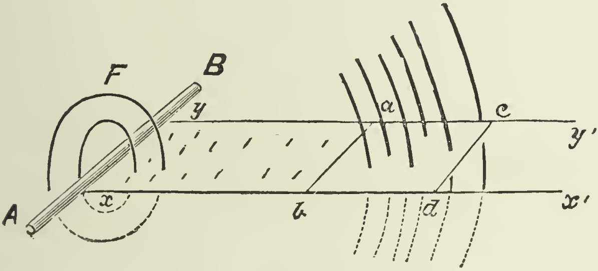

FIG. 134.

circular lines of force as expanding outwards with a certain velocity of enlargement, and attaining or taking up their final positions after a short interval of time. If we represent the intersections of these rings of force on the plane of $x\,x'\,y\,y'$ by dots, these dots will march forward like the soldiers in the previous illustration. The total number of lines of force which at any instant are found traversing the area $a\,b\,d\,c$ is equal numerically to the difference in the number between those which from the beginning of the epoch have intersected or cut through the line $a\,b$ and those which have cut through $c\,d$. In other words, the surface integral of the magnetic force over $a\,b\,c\,d$ may be represented by, or is equal to, the line integral round $a\,b\,c\,d$ of a certain quantity called the *vector potential*, which, physically interpreted, is the total number of lines of force which have cut through a unit element of the boundary in the process of expansion or propagation outwards. This term vector potential is justified as follows:—If F be the total number of lines of force per unit of length of $a\,b$ which have cut through $a\,b$ from the instant of beginning the current, and if the small distance $b\,d$ is called $\delta x$, the length $x\,b$ being called $x$, then by Taylor's theorem (Diff. Calc.), the number which have cut through unit of length of $c\,d$ is $\mathrm{F} - \frac{d\,\mathrm{F}}{d\,x}\,\delta\,x$, and hence the difference between F and this last quantity is $\frac{d\,\mathrm{F}}{d\,x}\,\delta x$, and this last when multiplied by $\delta y$, which we may take for the length of $a\,b$ or $c\,d$—that is $\frac{d\,\mathrm{F}}{d\,x}\,\delta\,x\,\delta\,y$—is the total number of lines of force included in the area $a\,b\,c\,d$. If we call the induction through this area B—that is to say, the number of lines of force per square centimetre is B—it follows that the number through $a\,b\,c\,d$ is $\mathrm{B}\,\delta\,x\,\delta\,y$. Hence, equating the two values,

we have
$$\frac{d\,\mathrm{F}}{d\,x}\,\delta\,x\,\delta\,y = \mathrm{B}\,\delta\,x\,\delta\,y,$$
or
$$\frac{d\,\mathrm{F}}{d\,x} = \mathrm{B}.$$

Hence, the mean magnetic force over the small area is numerically equal to the space variation of a certain quantity F. In electrostatics the electric force X at any point in the electric field is the space variation of a certain quantity V, called the electrostatic or scalar potential—that is to say,

$$-\frac{dV}{dx} = X;$$

and accordingly by anology that quantity F whose space variation gives the magnetic force under the circumstances considered above is called the *vector potential of the current.* From Ampère's investigations it is known that the magnetic force due to an element of a current C of length $\delta s$ at a distance $r$ from this element, has the value $\frac{C\,\delta s}{r^2}$, and is along a line at right angles to the plane containing $\delta s$ and $r$. The space variation of $-\frac{C\,\delta s}{r}$ is $\frac{C\,\delta s}{r^2}$; hence the vector potential of an element of current at any point is proportional to the length of that element divided by its distance from that point.

In electrostatic phenomena we obtain the static potential at any point due to any charge Q by taking each element $q$ of the charge, and dividing the magnitude of this element of charge by its distance from the point at which the potential is required, and taking the sum $\Sigma\frac{q}{r}$ of all such quotients. In electrostatics the potential at a point is a *scalar* or directionless quantity, and the summation is merely an algebraic sum; but in dealing with currents the quotients $\frac{C\,\delta s}{r}$ are vectors, or directed quantities, and have to be added together according to the laws for the addition of vector quantities just as forces and velocities are added. Hence the potential of a current at any point is a vector or directed quantity. The lines of vector potential of a straight current are lines described in space parallel to the current, and the lines of vector potential of a circular current are circles described on planes parallel to the plane of the current. Returning to the simple case of a straight current, let us suppose that a unit of length is described somewhere parallel to the current, and that on starting the current suddenly circular lines of magnetic force are propagated outwards with a velocity V; these lines will, as they expand, cut perpendicularly through the element of length just as the expanding ripples on water due to a stone dropped into it would "cut through" a stick held perpendicularly in the water a little way from the place where the "splash" was made. Suppose that after N lines of force have cut through the element of length this little line is made to move forward parallel to itself, so that there is no further increase in the number of lines of force which afterwards cut through it, it is evident that it must move with the velocity of propagation of the expanding rings of force. But the number expressing the number of lines of force which have cut through the element of length already is the value of the vector potential at that point where the element is at that instant; hence the velocity of propagation of the vector potential *is* the velocity of propagation of an electro-magnetic disturbance. Maxwell's general mathematical method of investigating the propagation of an electro-magnetic disturbance consisted in forming equations expressing the change of the value of the vector potential of a current or system of currents at any point in the field, and deducing equations which mathematically are of the same type as those which express the propagation of a disturbance through an elastic solid or fluid, and his result was that the velocity of propagation of the vector potential through a medium of electrostatic and magnetic inductivities K and $\mu$ was equal to $\frac{1}{\sqrt{K\mu}}$, or to $(K\mu)^{-\frac{1}{2}}$.

The complete proof of the above proposition as given by Maxwell in all its generality requires some elaborate analysis, but is is not difficult to give a simple illustration by treating a reduced case, and which will exhibit the principles of the more complete problem.

Let an infinite straight conductor be supposed situated in a dielectric medium of specific inductive capacity (electrostatic inductivity) K and of permeability (magnetic inductivity) $\mu$. We proceed to investigate the velocity of lateral propagation of electro-magnetic induction on the supposition that if a current is instantaneously started at its full value in the conductor, supposing this possible, the magnetic force travels outwards laterally from the conductor in all directions with a velocity $v$. This amounts to the supposition that the circular lines of magnetic force surrounding the conductor swell out or expand outwards from the surface of the conductor, so that the radius of any determinate circular line of force increases or grows with a velocity $v$. It must be borne in mind that the magnetic force at any point in the field at any instant is defined by the density or concentration of the lines of force—that is, by the number passing normally through a unit of area. If we complicate the problem by supposing the strength of the current in the conductor to gradually increase, then the concentration of the lines at any point must be supposed to increase gradually, but the rate of increase of concentration—that is, of the force—is a different thing from the rate of outward movement of the lines of force.

We might in imagination suppose each line of force to be labelled so as to recognise it. All the lines travel outward from the conductor at the same rate, but some go out farther than others. The first ones shed off expand out to reach positions in the most distant portions of the field, and the succeeding ones reach intermediate positions, and as the current strength grows up fresh arrivals or deliveries of lines of force happen which pack the space fuller, and increase the concentration at all points of the field, at a rate depending on the rate of growth of the current.

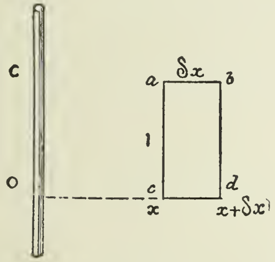

FIG. 135.

Let O C (Fig. 135) be a portion of the straight conductor. In the plane of O C take any little rectangular area $a\,b\,c\,d$, with side $a\,c$ equal to unit of length, and side $a\,b$ equal to $\delta x$, $\delta x$ being a very small quantity compared with the distance between O C and $a\,c$, that is, let the distance $\text{O}\,c = x$ and $\text{O}\,d = x + \delta x$, and let the distance $\delta x$ be the distance by which the radius of any circular line of force of the conductor O C increases in a small time $\delta t$. At any instant the number of lines of force which pass normally through the small area $a\,b\,c\,d$ is equal to the difference between the number which have "cut" across $a\,c$ and those which have cut across $b\,d$ in consequence of our supposition as to the outward growth or expansion of the circular lines of force. Let F be the total number of lines of force due to the current in O C which have from the beginning of the current flow "cut across" $a\,c$, then, by the principles of the Differential Calculus, the number which have cut across $b\,d$ is represented by the quantity $\mathrm{F} - \frac{d\,\mathrm{F}}{d\,x}\,\delta x$, and the number existing in, or perforating through, the area $a\,b\,c\,d$ is the difference between F and $\mathrm{F} - \frac{d\,\mathrm{F}}{d\,x}\,\delta x$, or equal to $\frac{d\,\mathrm{F}}{d\,x}\,\delta x$. Let **B** stand for the induction through unit of area of the rectangle $a\,b\,c\,d$, or to the number of lines of force per unit of area, then the total number of lines of force through $a\,b\,c\,d$ is represented also by $\mathbf{B}\,\delta x$, since the area of $a\,b\,c\,d$ is $\delta x$ square units, $a\,c = b\,d$ being unity.

Hence, $$\frac{d\,\mathrm{F}}{d\,x} = \mathbf{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

or the induction is represented by the space rate of change of the vector potential of the current at that point in the direction of $x$. In this case let it be borne in mind that the vector potential signifies the number of lines of force which have from the beginning of the epoch cut through unit length taken parallel to the current. Again, since by supposition each line of force moves outwards parallel to itself through a distance $\delta x$ in a time $\delta t$, $\frac{\delta x}{\delta t}$ is the velocity of propagation $v$ of the electro-magnetic disturbance or of the vector potential. The rate of "cutting across" $a\,c$ at any instant is represented by $\frac{d\,\mathrm{F}}{d\,t}$; hence the number of lines of force added to the area in a time $\delta t$ must be $\frac{d\,\mathrm{F}}{d\,t}\,\delta t$, and this must be equal to the accumulation of the lines in $a\,b\,c\,d$ in the same time in the area $a\,b\,c\,d$.

If in a small time interval the rate of cutting across $a\,c$ is $\frac{d\,\mathrm{F}}{d\,t}$, then the rate at which "cutting" is taking place across a length $b\,d$, removed by a distance $\delta\,x$, is

$$\frac{d\,\mathrm{F}}{d\,t} + \frac{d}{d\,x}\left(\frac{d\,\mathrm{F}}{d\,t}\right)\delta\,x,$$

and the rate at which accumulation of lines of induction is going on in the area is

$$-\frac{d}{d\,x}\left(\frac{d\,\mathrm{F}}{d\,t}\right)\delta\,x.$$

Hence, since **B** is the induction per unit of area and the area of $a\,b\,c\,d$ is $\delta\,x$ square units, the rate of increase of induction through $a\,b\,c\,d$ is

$$\frac{d}{d\,t}(\mathbf{B}\,\delta\,x).$$

Accordingly we have

$$\frac{d}{d\,t}(\mathbf{B}\,\delta\,x) = -\frac{d}{d\,x}\left(\frac{d\,\mathrm{F}}{d\,t}\right)\delta\,x,$$

or since $\delta\,x$ is constant,

$$\frac{d\,\mathbf{B}}{d\,t} = -\frac{d}{d\,x}\left(\frac{d\,\mathrm{F}}{d\,t}\right),$$

$$= -\frac{d}{d\,t}\left(\frac{d\,\mathrm{F}}{d\,t}\right)\frac{d\,t}{d\,x},$$

or,

$$\frac{d\,\mathbf{B}}{d\,x}\,\frac{d\,x}{d\,t} = -\frac{d^2\,\mathrm{F}}{d\,t^2}\,\frac{d\,t}{d\,x};$$

but $\frac{d\,x}{d\,t} = v =$ velocity of propagation of the impulse. Hence,

$$v^2 = \frac{-\dfrac{d^2\,\mathrm{F}}{d\,t^2}}{\dfrac{d\,\mathbf{B}}{d\,x}}; \quad . \quad . \quad . \quad . \quad . \quad (110)$$

or, generally, since

$$\mathrm{B} = \frac{d\,\mathrm{F}}{d\,x},$$

we have

$$\frac{d^2\,\mathrm{F}}{d\,t^2} + v^2\,\frac{d^2\,\mathrm{F}}{d\,x^2} = 0 \quad . \quad . \quad . \quad . \quad (111)$$

as the equation of motion of the vector potential. This equation, which is a reduced case of the general one, is of the same type as that obtained in the theory of sound for the propagation of an impulse along a tube or canal. In the case of sound the symbol F would be the *velocity potential.** In the electro-magnetic problem the F is the *vector potential.* It might perhaps be more expressively called the *induction potential.*

The rate of cutting, or the value of $\frac{dF}{dt}$, also expresses the electromotive force acting along the unit of length $ac$ in the dielectric. On Maxwell's hypothesis this electromotive force *in* the dielectric acting parallel to the current in the conductor produces a displacement in the dielectric, such that if E is the electromotive force we have as above

$$\frac{dF}{dt}=E=\frac{4\pi}{K}D,$$

where D is the displacement through unit of area ; hence,

$$\frac{d^2F}{dt^2}=\frac{4\pi}{K}\frac{dD}{dt}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

and $\frac{dD}{dt}$ is the rate of displacement or the displacement current flowing through unit of area taken perpendicularly to the current in O C at the point considered. Let this displacement current be denoted by $u$. We have then that $\frac{d^2F}{dt^2}=\frac{4\pi u}{K}$, K being the dielectric constant of the medium.

Consider now a small parallelopipedon (Fig. 136) or solid rectangle described in the dielectric, of which the sides are respectively $ac=1$, $cd=\delta x$, $ce=\delta y$.

The effect of the cutting across of this solid rectangle by expanding lines of induction will be to generate in it a displacement current such that the total displacement current parallel to $ac$ and through $cdfe$ will be $u\,dx\,dy$. By a previous theorem the line integral of magnetic force round any line is equal to $4\pi$ times the surface integral of the current through the area bounded by that line, and this is true whether the magnetic force be produced by that current, or whether it is a current produced by a certain changing magnetic force. Apply the theorem to the small rectangle bounded by the lines $cefd$. The surface integral of the current through $cefd$ is $u\,dx\,dy$. The magnetic force along $c\,e$ is $\frac{\mathbf{B}}{\mu}$, where $\mathbf{B}$ is the induction at $c$ and $\mu$ is the magnetic permeability of the medium, since by a fundamental theorem the magnetic induction $\mathbf{B}$ at any place is equal to $\mu$ times the magnetic force at that point. The magnetic force along $d\,f$ removed by a distance $\delta\,x$ from $c\,e$ is $\frac{1}{\mu}\left(\mathbf{B} - \frac{d\,\mathbf{B}}{d\,x}\,\delta\,x\right)$, and there is no magnetic force along $c\,d$ and $e\,f$, for these sides are perpendicular to the direction of the magnetic force of the current in O C. Hence, the line integral of magnetic force round $c\,e\,f\,d$ is

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* *See* Besant's "Hydromechanics," p. 251 (Third Edition).

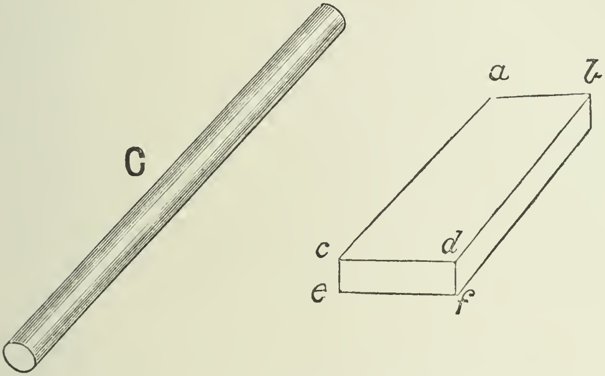

FIG. 136.

$$\frac{1}{\mu}\left(\mathbf{B}\,\delta\,y - (\mathbf{B}\,d\,y - \frac{d\mathbf{B}}{d\,x}\,\delta x\,\delta\,y)\right),$$

or

$$\frac{1}{\mu}\,\frac{d\,\mathbf{B}}{d\,x}\,\delta\,x\,\delta\,y;$$

hence,

$$4\pi\,u\,\delta\,x\,\delta\,y = \frac{1}{\mu}\,\frac{d\,\mathbf{B}}{d\,x}\,\delta\,x\,\delta\,y,$$

or

$$4\pi\,\mu\,u = \frac{d\,\mathbf{B}}{d\,x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113)$$

Accordingly, in the equations (112) and (113) above, we have obtained values for the quantities $\frac{d^2\,\mathrm{F}}{d\,t^2}$ and $\frac{d\,\mathbf{B}}{d\,x}$ in terms of the permanent constants of the medium ; and by substitution of these values in equation (111) above, we see that the square of the velocity of propagation of the vector potential is

$$v^2 = \frac{\frac{d^2 \mathrm{F}}{d\,t^2}}{\frac{d\,\mathbf{B}}{d\,x}} = \frac{\frac{4\pi\,u}{\mathrm{K}}}{4\pi\,\mu\,u} = \frac{1}{\mathrm{K}\,\mu},$$

or

$$v = \sqrt{\frac{1}{\mathrm{K}\,\mu}}, \quad . \quad . \quad . \quad . \quad . \quad (114)$$

that is, the velocity of propagation of the magnetic force is the square root of the reciprocal of the product of the magnetic and electrostatic inductive constants of the medium. We have above proved that the ratio of the electro-magnetic to the electrostatic units of electric current is expressed by the same quantity, and indicated that accurate experiment shows this ratio to be numerically the same as the velocity of light.

Hence, the velocity of an electro-magnetic disturbance or magnetic force is the same as the velocity of light, and the conclusion is urged upon us with great force that the medium concerned in both phenomena is the same.

§ 6. **Electrical Oscillations.**—A survey of the phenomena of electric current induction would be very incomplete if it did not contain some reference to the subject of electrical oscillations. Recent researches have endowed this department of electrical investigation with fresh interest. We proceed to consider the manner in which electrical oscillations may arise. If a material body is subjected to elastic constraint, and is disturbed from a position of equilibrium, it returns when set free to its original position. If that body is endowed with *mass*, and hence possesses the quality of inertia, its motion of return to its position of equilibrium will, under certain circumstances, carry it beyond that point and set up *oscillations*, which decay gradually away. Two illustrations of this readily present themselves, one a mechanical and the other a pneumatical example. The first case is that of a pendulum or straight spring. Let this pendulum or spring be deflected from its position or condition of equilibrium and held in constraint. Next let it be set free—the elastic or restoring forces urge it back again to its first position. In virtue of its mass it will acquire a certain momentum, and on reaching the position of equilibrium this momentum may carry it past this point, and the acquired kinetic energy will then be expended in making a displacement against the elastic forces. If there is nothing of the nature of friction present to fritter away the work expended on the body in making the first displacement, then the energy would remain associated with it for ever, being alternately potential and kinetic, and the oscillations continue with undiminished amplitude. If the spring or pendulum vibrates in a viscous fluid, then a frictional retardation will be experienced, and in so far as this is present the energy is gradually dissipated, and the oscillations decay away, becoming gradually less and less in amplitude. It may so happen that the work done against frictional resistance during the first quarter of a complete oscillation in starting to return from the position of greatest displacement is just equal to the work done in originally making the displacement. When this is the case the whole energy is dissipated by the time the deflected or displaced body reaches its original position of rest, and there are then no oscillations. Accordingly a pendulum or spring may be set in a viscous fluid of such a kind that the frictional resistance is just sufficient to secure that when the body is disturbed and then set free it returns to its original position without ever passing it; in other words, there are no oscillations. Another illustration of oscillatory and non-oscillatory establishment of equilibrium is as follows: Suppose there be two large vessels, or reservoirs, connected by a pipe, closed or able to be closed in the middle by a stopcock. Let one of these vessels, A, be exhausted of its air, and let the other, B, have air in it at the atmospheric, or a greater than the atmospheric pressure. *First*, let the connecting pipe be supposed to be long and narrow; on opening the stopcock air will rush over from B into A, and the flow of air will continue uniformly in the pipe in one direction until the pressure in A and B is equalised. *Second*, let the connecting pipe be very short and large, so that little tubular friction is offered to the flow of air. Under these circumstances the result of opening the tap would be that a rush of air would take place, which would be succeeded by a series of oscillations of the air in the tube. The air, in fact, rebounds from side to side, and the equilibrium is only finally established after a series of gradually diminishing oscillations or backward or forward currents of air in the tube. This establishment of equilibrium or pressure by oscillatory movement takes place when the resistance to the flow is small. That this is no fanciful description is proved by the experience of MM. Clément and Désormes in their experiments to determine the ratio of the specific heats of gases. In these experiments a large glass vessel had a partial vacuum made in it. A stopcock was then quickly opened and closed, and the pressure of the air determined after a short time. These experiments were repeated by MM. Gay Lussac and Welter. See *Journal de Physique*, LXXXIX., 1819, 428, and *Ann. de Ch. et de Phys.* [1], XIX., 1821, 436.

M. Cazin (*Ann. de Ch. et de Phys.* [3] LXVI., 1862, 206) first pointed out a source of error which resulted from these air oscillations, and showed that the final pressure depended upon the phase of the oscillation at which the stopcock is closed.

These examples are sufficient to indicate that when a material system of bodies having *inertia* is displaced against elastic forces which compel it to return, if free, to a definite position, whilst at the same time its motion is resisted by actions of the nature of frictional resistance which dissipate its energy, we have a resulting motion which may be oscillatory or non-oscillatory, according to the relation of the constants of the system. Under certain conditions as to mass, or inertia and friction, we have oscillations dying gradually away. Under other conditions we have a gradual return to the original position without ever passing it. The motion is then said to be perfectly *dead-beat.* We shall investigate presently the conditions which must hold good, and the relation between the *inertia factor*, in virtue of which the moving system possesses kinetic energy, and the *resistance factor*, in virtue of which the energy bestowed upon the system at its first displacement is frittered away into heat, in order that the motion may be vibratory or dead-beat.

When a condenser or Leyden jar is discharged through a conductor, the potential energy runs down in the form of an electric current. In this case we have a similar state of things to that existing when a bent spring is released. This transformation of the potential energy may take place either by a vibratory current, that is, by a series of electrical oscillations—or by a uni-directional discharge. It is highly probable that Prof. Joseph Henry, as far back as 1842, was the first to recognise that the discharge of a condenser might be of an oscillatory character. It is remarked by him* that "The discharge, whatever may be its nature, is not correctly represented by a single transfer of imponderable fluid from one side of the jar to the other; the phenomena require us to admit the existence of a principal discharge in one direction and then several reflex actions backward and forward, each more feeble than the preceding, until equilibrium is attained. All the facts are shown to be in accordance with this hypothesis, and a ready explanation is afforded by it of a number of phenomena which are to be found described in the older works on electricity, but which have until this time remained unexplained." A little later on in the Paper he gives an explanation of the reversal of polarity of the needles by the oscillatory discharge. In his celebrated Essay, "Erhaltung der Kraft" (Berlin, 1847), Helmholtz alluded also to such a possible form of electric discharge in the following words: "We assume that the discharge (of a jar) is not a simple motion of the electricity in one direction, but a backward and forward motion between the coatings in oscillation, which become continually smaller until the entire *vis viva* is destroyed by the sum of the resistances." He adds: "The notion that the discharge consists of alternately opposed currents is also favoured by the phenomena observed by Wollaston while attempting to decompose water by electric shocks, that *both* descriptions of gases are evolved at both electrodes." The investigation which, however, marks an epoch in this subject is the Paper by Lord Kelvin (then Sir William Thomson) in the June number of the *Philosophical Magazine* for 1853, on "Transient Electric Currents." In this Paper the author discusses, first, the equations which determine these currents at any instant when a condenser or Leyden jar is discharged through a conductor. The discharging conductor is supposed to have self-induction, or as

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* "The Scientific Writings of the late Prof. Joseph Henry." Washington: 1886. Vol. I. This statement of Prof. Henry had attention directed to it by Mr. A. D. Raine in *The Electrician* of November 2, 1888, p. 831. It had been previously mentioned, however, in the sketch of the life of Prof. Joseph Henry, given in the *Encyclopædia Brittanica*, Ninth Edition.

Lord Kelvin then called it, "electro-dynamic capacity," and also to have ohmic resistance, which is constant, and independent of the rate of discharge. On these two assumptions he builds up an equation which mathematically contains the whole theory, as follows:—

If C is the electrostatic capacity of the jar or condenser, and R the ohmic resistance, and L the constant inductance of the discharging conductor; and if $q$ is the electric quantity in the jar, and $v$ the potential difference of its coatings at any instant $t$, then by the definition of electric capacity we have

$$q = \mathrm{C}\, v,$$

and $\dfrac{d\,q}{d\,t} = i =$ the current at that instant in the conductor, which is equal by Ohm's law to $\dfrac{v}{\mathrm{R}}$. By the principle of conservation of energy the rate at which electro-magnetic energy is being taken up by the conductor, viz., $\dfrac{d}{d\,t}\,(\tfrac{1}{2}\,\mathrm{L}\,i^2)$, together with the rate at which energy is being dissipated as heat in the conductor, viz., $\mathrm{R}\,i^2$ (by Joule's law), must be equal to the rate of decay of the energy contained in the jar, or to

$$-\frac{d}{d\,t}\,(\tfrac{1}{2}\,q\,v) = -\frac{d}{d\,t}\left(\frac{1}{2}\,\frac{q^2}{\mathrm{C}}\right).$$

Hence
$$-\frac{d}{d\,t}\left(\frac{1}{2}\,\frac{q^2}{\mathrm{C}}\right) = \frac{d}{d\,t}\,(\tfrac{1}{2}\,\mathrm{L}\,i^2) + \mathrm{R}\,i^2,$$

or
$$-\frac{q\,dq}{\mathrm{C}\,d\,t} = \mathrm{L}\,i\,\frac{d\,i}{d\,t} + \mathrm{R}\,i^2\,;$$

but $i = \dfrac{d\,q}{d\,t}$, or the current is the rate of loss of charge, therefore

$$\frac{d^2\,q}{d\,t^2} + \frac{\mathrm{R}}{\mathrm{L}}\,\frac{d\,q}{d\,t} + \frac{1}{\mathrm{L\,C}}\,q = 0. \quad . \quad . \quad . \quad (115)$$

The value of $q$, or the charge in the jar at any instant, is given by the solution of this equation.

Let us write the equation in the form

$$\frac{d^2\,q}{d\,t^2} + a\,\frac{d\,q}{d\,t} + b\,q = 0.$$

In order to solve this equation we may proceed as follows: The charge $q$ in the jar begins by possessing a certain initial value, and ends by being zero. Let us assume that $q$ can be expressed as a function of the time $t$ in the form $q = \mathrm{A}\, e^{mt}$, where A is a constant and $e$ is the base of the Naperian logarithms, and $m$ is also a certain function determined by the capacity, resistance, and inductance of the system. For it is clear that by a suitable value for A and $m$ the function $\mathrm{A}\, e^{mt}$ may be made to express the mode in which the charge $q$ dies away with increase of the time $t$. The problem is reduced, then, to finding A and $m$. The solution of nearly every differential equation is by a process of happy guessing; there is generally no systematic or direct method of obtaining the required result. Take, then, the expression $q = \mathrm{A}\, e^{mt}$, form the first and second differential coefficients, and substitute these results in the original equation, and we arrive at the expression

$$\mathrm{A}\, e^{mt}\,(m^2 + a\,m + b) = 0.$$

Hence, the value $\mathrm{A}\, e^{mt}$ assumed for $q$ will satisfy the equation (115); that is, when substituted for $q$ in the original expression, render it zero, provided that $m$ is such a quantity that $m^2 + a\,m + b = 0$. The two roots of this last quadratic equation are obtained by a simple solution, and they are

$$m = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}.$$

Two cases then arise, *first*, when $\frac{a^2}{4}$ is greater than $b$—that is, when $\frac{\mathrm{R}^2}{4\,\mathrm{L}^2}$ is greater than $\frac{1}{\mathrm{L\,C}}$, or $\frac{\mathrm{R}^2}{4\,\mathrm{L}}$ greater than $\frac{1}{\mathrm{C}}$. In this case the roots of the quadratic are *real*, and if we call them $m_1$ and $m_2$ we can say that the solution of the differential equation is

$$q = \mathrm{A}\, e^{m_1 t} + \mathrm{B}\, e^{m_2 t}. \quad . \quad . \quad . \quad . \quad (116)$$

where A and B are constants determined by the initial circumstances of the discharge, and $m_1$ and $m_2$ are equal respectively to $-\frac{a}{2} + \sqrt{\frac{a^2}{4} - b}$ and $-\frac{a}{2} - \sqrt{\frac{a^2}{4} - b}$. This solution for the value of $q$ is called an exponential solution, and it indicates that under these circumstances when the inductance, resistance and capacity are of such magnitudes that R is *greater* than $\sqrt{\frac{4\,\mathrm{L}}{\mathrm{C}}}$, the quantity $q$ dies away regularly, diminishing with the time in a continuous manner. In this case the discharge of the jar is always in one direction, and the current or rate of decay $\left(-\frac{d\,q}{d\,t}\right)$ of the charge is also always in one direction. If, however, R is *less* than $\sqrt{\frac{4\,\mathrm{L}}{\mathrm{C}}}$, then $\left(\frac{a^2}{4}-b\right)$ is a negative quantity, and the square root of it is an imaginary one, and the roots of the quadratic $m^2+a\,m+b=0$ are unreal. It is shown in treatises on algebra that a quadratic equation has either two real or two imaginary roots, and when this last is the case the roots of the quadratic can always be expressed in the form $\alpha+\beta\sqrt{-1}$.

Accordingly, the solution of the original equation (115) under these circumstances is of the form

$$q=\mathrm{A}\,e^{(\alpha+\beta\sqrt{-1})\,t}+\mathrm{B}\,e^{(\alpha-\beta\sqrt{-1})\,t}. \quad . \quad (117)$$

By a simple transformation, based on the employment of the exponential values of the sine and cosine, as given on page 106, this solution can be thrown into the form

$$q=e^{\alpha t}\,(\mathrm{P}\cos\beta\,t+\mathrm{P}^1\sin\beta\,t) \quad . \quad . \quad . \quad (118)$$

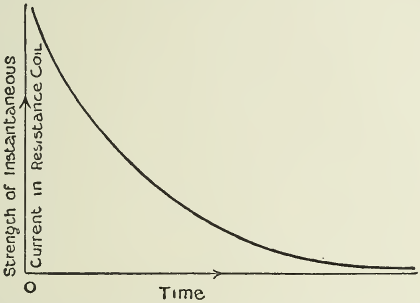
where P and $\mathrm{P}^1$ are constants, and

$$\alpha=-\frac{a}{2}=-\frac{\mathrm{R}}{2\,\mathrm{L}}, \text{ and } \beta=\sqrt{b-\frac{a^2}{4}}=\sqrt{\frac{1}{\mathrm{L\,C}}-\frac{\mathrm{R}^2}{4\,\mathrm{L}^2}}.$$

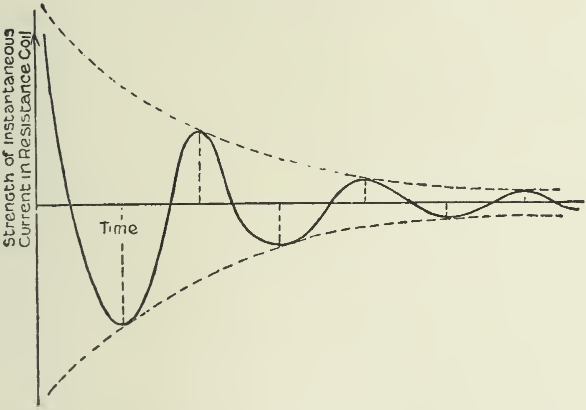
The general result is then that the equation

$$\frac{d^2q}{d\,t^2}+\frac{\mathrm{R}}{\mathrm{L}}\frac{d\,q}{d\,t}+\frac{1}{\mathrm{L\,C}}\,q=0.$$

has two solutions—one, called the *dead beat* case which applies when R is greater than $\sqrt{\frac{4\,\mathrm{L}}{\mathrm{C}}}$, and is of an exponential form, and indicates that the charge $q$ dies away regularly with lapse of time, and the discharge current is uni-directional; the other, called the *oscillatory* case, which applies when R is less than $\sqrt{\frac{4\,\mathrm{L}}{\mathrm{C}}}$, contains sine and cosine terms, and indicates a periodically changing discharge decreasing by a series of oscillations, in which case the charge on each plate of the condenser is first positive and then negative, but at the same time always decreasing; or, in other words, is a periodic variation superimposed on a steadily decreasing variation, the currents or rates of discharge following the same distinction. These two modes of discharge, or solutions of the differential equation, are best indicated graphically by the two curves in Fig. 137, in which the upper curve represents the gradual decrease, according to an exponential law, which is indicated as the proper solution of the equation, when the value of R or the resistance of the



**Curve representing the Discharge of a Condenser through a Large Resistance. Discharge Uni-directional and Continuous.**



**Curve representing the Discharge of a Condenser through a Small Resistance. Discharge is Periodic and Alternate. Maxima gradually diminishing in Geometric Progression.**

Fig. 137.

discharging circuit is greater than $\sqrt{\frac{4L}{C}}$, and the lower one the oscillatory discharge, which is indicated by the trigonometrical solution of the differential equation, when the resistance R is less than $\sqrt{\frac{4L}{C}}$. When R has such a value that $R=\sqrt{\frac{4L}{C}}$, the discharge is just non-oscillatory. We find, then, that according to Lord Kelvin, analysis indicates that for a certain relation between the resistance and inductance of the discharge circuit and of the capacity of the jar the discharge is a simple current in one direction or an oscillatory but decreasing current, according as R is greater or less than $\sqrt{\frac{4L}{C}}$. If the discharge is oscillatory, then the electrical oscillations are isochronous, and the periodic time of a complete oscillation is

$$T=\frac{2\pi}{\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}},$$

for in the second solution (118),

$$q=\epsilon^{at}(P\cos\beta t+Q\sin\beta t),$$

we see that at intervals of time equal to $\frac{\pi}{\beta}$ the sine and cosine terms have the same values, since $\sin\beta t=\sin\beta\left(t+\frac{\pi}{\beta}\right)$, and the same for the cosine. Hence, the trigonometrical factor in the value for $q$ periodically repeats itself in value at intervals of time equal to $\frac{\pi}{\beta}$, and is zero at times when $\tan\beta t=-\frac{P}{Q}$. Hence the complete periodic time of the oscillation is

$$\frac{2\pi}{\beta} \quad \text{or} \quad \frac{2\pi}{\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}},$$

and the frequency of the oscillations, or number in one second, is

$$\frac{1}{2\pi}\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}.$$

Accordingly, when $R = \sqrt{\frac{4L}{C}}$ there are no oscillations in one second, or the motion is just non-oscillatory, or *dead beat.* In the case of the uni-directional discharge the values of the instantaneous current in the discharge circuit can be represented as we have seen by the ordinates of an exponential curve, and in the case of the oscillatory discharge by those of a periodic curve whose successive maxima descend in geometric progression as the time increases in arithmetic progression. During equal intervals of time the whole quantities which pass decrease also in geometric progression, and the zero points, or instants of reversals of sign of current, are uniformly separated.

The foregoing predictions of analysis have been confirmed by the experiments of Feddersen, Paalzow, Bernstein, Blaserna, Helmholtz, Schiller and Rood. Lord Kelvin in his original Paper pointed out and suggested the application of Wheatstone's mirror in the examination of the discharge. In Feddersen's experiments the spark from a Leyden jar battery was taken between two brass balls placed in front of a revolving mirror. The discharge was passed through a high resistance. The image of the spark was viewed by a telescope. Under these circumstances the image of the spark was drawn out when the mirror revolved into a continuous band of light in a direction perpendicular to that of the discharge.* When the resistance was gradually reduced a point was reached at which the image was broken up into a series of separated strips, each strip corresponding to a discharge. This showed that the discharge was intermittent.

In Paalzow's experiments a similar discharge from a Leyden battery was passed through a resistance coil and through a vacuum tube, and the image of the discharge in the vacuum tube viewed in a revolving mirror. As before, with a small resistance the image consisted of a number of separate images, each of which corresponded to a discharge, and a bluish light showed itself at *both* poles of the vacuum tube. When the resistance was increased the bluish light showed itself only at one pole. In the former case a magnet held outside the tube split the discharge into *two* lines of light, showing that it consisted of currents travelling in both directions; but in the last case the magnet did not divide the discharge. This sufficiently indicated that with a low resistance the discharge was oscillatory and alternate, and not uniform or uni-directional.

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* An experimental research of a very complete character on the duration and nature of the discharge of a Leyden jar is described by Prof. Ogden Rood in the *American Journal of Science and Arts* for September, 1869; January, 1871; September, 1871; October, 1872; November, 1872; March, 1873.

Feddersen found that the critical resistance at which the discharge just becomes oscillatory varies inversely as the square root of the capacity of the battery, which is in agreement with the predictions of theory.

A good account of the researches of these experimentalists is given in Wiedemann's *Galvanismus*, Part II, § 800, *et seq.**

We can cast the expressions for the charge at any instant left in the condenser into more convenient forms. *First*, consider the *dead-beat* case (equation 116) is

$$q = \mathrm{A}\, e^{m_1 t} + \mathrm{B}\, e^{m_2 t},$$

where $m_1$ and $m_2$ are the real roots of the quadratic equation

$$m^2 + a\, m + b = 0;$$

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* For the sake of readers wishing to pursue the subject we give here a few references, to original Papers, in which are included some collected by Mr. Tunzelmann in a series of articles on Electrical Oscillations in *The Electrician* of September 14, 1888, and succeeding numbers.

Feddersen, Poggendorff's *Annalen*, Vol. CIII., p. 69, 1858; Vol. CVIII., p. 497, 1859; Vol. CXII., p. 452, 1861; Vol. CXIII., p. 437, 1861; Vol. CXV., p. 336, 1862; Vol. CXVI., p. 132, 1862.

Paalzow, Pogg. *Ann.*, Vol. CXII., p. 537, 1861; Vol. CXVIII., p. 178, 1863.

Bernstein, Pogg. *Ann.*, Vol. CXLII., p. 54, 1871.

Helmholtz, *Monatsberichte der Berl. Akad.*, 1874.

Kirchoff *Gesammelte Abhandlungen*, p. 168, containing remarks and criticisms of Feddersen's results.

Von Oettingen, Pogg. *Ann.*, Vol. CXV., p. 115, 1862; also Jubelbaud, p. 269, 1874.

L. Lorenz, Wiedemann's *Annalen*, Vol. VII., p. 161, 1879.

Schiller, Pogg. *Ann.* Vol. CLII., p. 535, 1872.

Mouton, Thèse, Paris, 1876, *Journal de Physique*, Vol. VI., pp. 5 and 46, 1876.

Kolacek, Beiblätter en Wiedemann's *Annalen*, Vol. VII., p. 541, 1883.

Olearsky, *Verhandlungen der Academie von Krakau*, Vol. VII., p. 141, 1882.

Oberbeck, Wiedemann's *Annalen*, Vol. XVII., pp. 816—1,040, 1882; Vol. XIX., pp. 213 and 265, 1883.

Bichat et Blondlot, *Comptes Rendus*, Vol. XCIV., p. 1,590, 1882.

and as $a = \frac{R}{L}$ and $b = \frac{1}{CL}$, we have $m_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}$, which we will write as $-\alpha + \beta$, and similarly, $m_2$ is $-\alpha - \beta$.

The constants A and B are determined by the condition that when $t = 0$ the charge $q$ is the original charge Q;

hence
$$Q = A + B, \quad . \quad . \quad . \quad . \quad . \quad (119)$$

and since the current $i$ at any instant is the rate of loss of charge, or $-\frac{dq}{dt}$, we have $i = -\frac{dq}{dt} = -A m_1 e^{m_1 t} - B m_2 e^{m_2 t}$, when $t = 0$, $i = 0$.

Hence
$$A m_1 + B m_2 = 0. \quad . \quad . \quad . \quad . \quad . \quad (120)$$

From these two equations (119) and (120) A and B are determined in terms of $m_1$ and $m_2$, or of $\alpha$ and $\beta$, and we find

$$A = \frac{\alpha + \beta}{2\beta} Q.$$

$$B = -\frac{\alpha - \beta}{2\beta} Q.$$

Let the quantity $\frac{1}{\alpha - \beta}$ be called $T_1$ and let $\frac{1}{\alpha + \beta}$ be called $T_2$, then it is easily seen that $A = \frac{T_1}{T_1 - T_2} Q$, and $B = -\frac{T_2}{T_1 - T_2} Q$, and the equation for $q$ may be written

$$q = \frac{Q}{T_1 - T_2} \left\{ T_1 e^{-\frac{t}{T_1}} - T_2 e^{-\frac{t}{T_2}} \right\}. \quad . \quad . \quad (121)$$

The ratio of the potential $v$ of the condenser at any instant to its original potential V is the same as that of $q$ to Q.

The two quantities $T_1$ and $T_2$ are such that their sum is equal to C R and their product to C L—statements easily verified by taking the values of $T_1$ and $T_2$ in terms of $\alpha$ and $\beta$, and recollecting that $\alpha$ stands for $\frac{R}{2L}$, and $\beta$ for $\sqrt{\frac{R^2}{4L^2} - \frac{1}{CL}}$.

Hence also the current $i$ at any instant is given by the equation

$$i = \frac{Q}{T_1 - T_2} \left\{ e^{-\frac{t}{T_2}} - e^{-\frac{t}{T_1}} \right\} \quad . \quad . \quad . \quad . \quad . \quad (122)$$

These two equations (121) and (122) contain the complete solution of the discharge in the *dead-beat* case, giving the current, potential and quantity at any instant reckoned from the moment of closing the circuit of the condenser.

Suppose that the discharging circuit possesses no inductance, then $L = 0$, and the equation reduces to

$$q = Q\, e^{-\frac{t}{RC}}.$$

In the above expression the product R C, or the product of the resistance of the discharging circuit and the capacity of condenser, is a quantity of the *dimensions* of a time, and is called the *time constant* of the condenser. It represents the time in which the charge of the condenser falls to $\frac{1}{e}$th part of its original value ($e$ being 2·71828). Let R C be denoted by T. Then if we begin with a charge Q, in a time T the charge left is $\frac{Q}{e}$. In a time 2 T it is $\frac{Q}{e^2}$, and in a time $n$ T it is $\frac{Q}{e^n}$. Now, since $e^3 = (2{\cdot}71828)^3$, or nearly 20, and $e^4$ is nearly 54, it follows that in time 7 T only one-thousandth of the original charge remains, and in a time 21 T only one thousand millionth; so that in a period of time equal to 5 or 6 times the length of the time constant the condenser is practically discharged. If the discharging circuit possesses inductance then in the dead-beat case there are two time constants of unequal importance. These are the quantities we have called $T_1$ and $T_2$ above. $T_1$ is the larger of the two. The rapidity of decay of the charge with an inductive discharger depends chiefly on $T_1$. For if we refer again to equation 121, we see that $q$ will become zero when the quantity in the bracket, viz., the function $\{T_1 e^{-\frac{t}{T_1}} - T_2 e^{-\frac{t}{T_2}}\}$, becomes zero.

Starting with given values of $T_1$ and $T_2$ depending on the values of L, C, and R, and knowing that $T_1$ is greater than $T_2$, the function starts with a value equal to $T_1 - T_2$ when $t = 0$, and as $t$ increases without limit both exponentials tail away down to zero; but since $T_1$ is greater than $T_2$, the first exponential, viz., $e^{-\frac{t}{T_1}}$, is longer getting down to practical zero than the other. Hence, the evanescence of $e^{-\frac{t}{T_1}}$ practically determines the time of discharge of the condenser, and we may call $T_1$ the principal time constant of the system.

If we call the expression $\frac{L}{C R^2}$ $\lambda$, then bearing in mind that $T_1 = \frac{1}{\alpha - \beta}$ and $T_2 = \frac{1}{\alpha + \beta}$ where $\alpha = \frac{R}{2 L}$ and $\beta = \sqrt{\frac{R^2}{4 L^2} - \frac{1}{C L}}$, we can express $T_1$ and $T_2$ in terms of $\lambda$ and $C R$ or $T$, and we have by simple substitution

$$T_1 = \frac{2 T \lambda}{1 - \sqrt{1 - 4 \lambda}},$$

and

$$T_2 = \frac{2 T \lambda}{1 + \sqrt{1 - 4 \lambda}},$$

and the product $T_1 T_2 = T^2 \lambda$.

Hence, if a horizontal line is taken, on which the values of $\lambda$ are set off (*see* Fig. 138), and values for $T_1$ and $T_2$ plotted off vertically, the locus of the extremities of these ordinates is a parabola. In the figure, lengths along O 1 represent values of $\lambda$, and the corresponding values of $T_1$ and $T_2$ define a parabola P M O, such that $O P = T = C R$, and the ordinates of the upper portion P M of the curve are the values of $T_1$, and those of O M are those of $T_2$. The value of $\lambda = \frac{1}{4}$ is the abscissa O A, for which $T_1 = T_2$, for when $\frac{L}{C R^2} = \frac{1}{4}$ then $\beta = 0$, and in

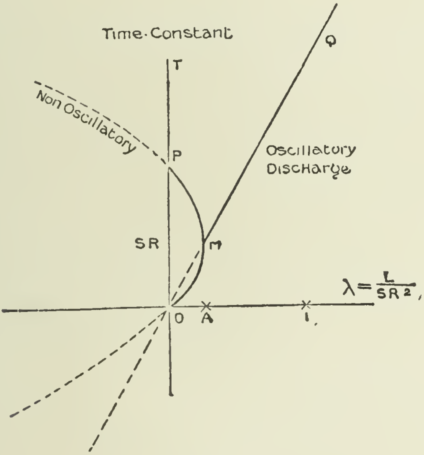

FIG. 138.

this case $T_1 = T_2$, and $T_1$ has its minimum value. For this particular value of $\lambda$, which is just the value when the discharge ceases to be dead-beat, and becomes oscillatory—that is, when $\frac{R^2}{4 L^2} = \frac{1}{C L}$ or $\frac{L}{C R^2} = \frac{1}{4}$—the time constants have equal values, and $T_1$ becomes a minimum. Hence, for this particular value of the inductance the time of discharge of the condenser is a minimum, and *less*, therefore, than the time of discharge when the discharge circuit has no inductance.*

Turning next to the case when the inductance of the discharge circuit is such that $\lambda$ is greater than $\frac{1}{4}$, or when $\frac{R^2}{4 L^2}$ is less than $\frac{1}{C L}$, we have to consider the periodic function which then applies.

Referring to equation 118 for the value of $q$ in terms of $t$ we have

$$q = e^{a t} (P \cos \beta t + P^1 \sin \beta t),$$

where $\alpha = -\frac{R}{2 L}$ as before, but $\beta$ now stands for $\sqrt{\frac{1}{C L} - \frac{R^2}{4 L^2}}$.

From the conditions that $q = Q$ when $t = 0$, and that when $t = 0$, $i = \frac{d q}{d t} = 0$, we find that $P = Q$ and $P^1 = Q \frac{a}{\beta}$.

Hence, $$q = Q e^{-\frac{R t}{2 L}} \left\{ \cos \beta t + \frac{R}{2 \beta L} \sin \beta t \right\}.$$

On the convention that $\gamma$ is such an angle that

$$\tan \gamma = \frac{2 L \beta}{R},$$

we can write the above expression

$$q = Q e^{-\frac{R t}{2 L}} \left\{ \frac{\sin (\beta t + \gamma)}{\sin \gamma} \right\},$$

and $$i = \frac{d q}{d t} = \frac{Q}{\beta L C} e^{-\frac{R t}{2 L}} \sin \beta t.$$

Hence, we see that the expression for the currents and for the remanent quantity of electricity at any time $t$ consists of a periodic part, which is a sine function, and a decreasing part, which is an exponential function, and that the rate of decay of the maxima of the waves is determined by the value of $\frac{R}{2L}$; in other words, $\frac{2L}{R}$ is the time constant for the oscillatory form of discharge.

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* This appears to have been first noticed by Dr. W. E. Sumpner (*Phil. Mag.*, June, 1877), and discussed by Prof. Oliver Lodge in an interesting paper in *The Electrician* for May 18, 1888, p. 39, from which article some portion of the above paragraph and figures have been taken.

This is expressible as $2T\lambda$ in our notation, and is, hence, simply proportional to $\lambda$. In Fig. 138 the variation of the time constant $T_3$, or $\frac{2L}{R}$, for oscillatory discharge is represented by the straight line M Q.

The really important part of the time constant curve is the part P M Q, consisting of a bit of a parabola and a straight line, and having a minimum ordinate corresponding to $\lambda = \frac{1}{4}$.

The current at different times for the two cases $\lambda = 0$ and $\lambda = \frac{1}{4}$ are plotted in Figs. 139 and 140.

For $\lambda = \frac{1}{2}$ we have $T_3 = T$, since $T_3 = 2T\lambda$. In other words, the time of discharge of the condenser when $\frac{L}{CR^2} = \frac{1}{2}$ is the same as when $L = 0$, and just *double* that when $\lambda = \frac{1}{4}$; and in this last case the rate of discharge is a maximum. Hence, so far from reducing the rate of discharge, a *little* self-induction in the discharge circuit is a positive help to the condenser in getting rid of its charge. Dr. Sumpner* has pointed out that since a lightning discharge resembles that of a condenser, a little inductance in a lightning rod may assist matters instead of blocking the way of the discharge.

A pendulum swinging in treacle was long ago suggested by Lord Rayleigh as a mechanical analogue to the Leyden jar discharge. Dr. Lodge† has pointed out that we may make the analogy exact by considering a loaded spring bent aside or compressed in a resisting medium in such way that gravity is not concerned in the motion and then let go.

The pliability of the spring corresponds to the capacity of the condenser, its displacement to the electric charge. The load or inertia corresponds to the self-induction of the circuit; the viscosity of the fluid to its resistance. If the viscosity friction be supposed to vary accurately as the speed, then the equation of motion is

$$m\frac{dv}{dt} - Rv = Rx,$$

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* *Loc. cit.*

† See *The Electrician*, May 18, 1888, p. 41.

where $x$ is the displacement and $v$ the velocity $= -\frac{dx}{dt}$. Writing L for $m$, and $\frac{1}{R}$ for C, and $x$ for Q, we have the condenser equation (115); the two are seen to be the same, and everything we have said of the electrical problem applies to the mechanical one.

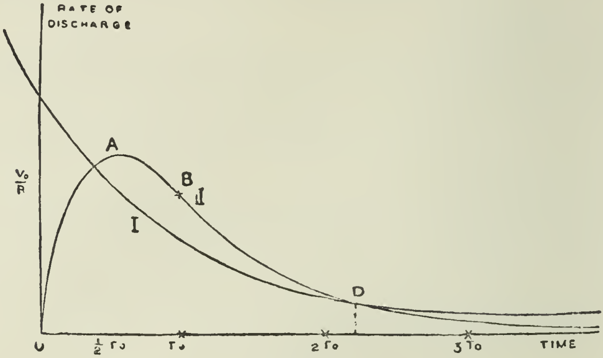

FIG. 139.

Curve I. represents the strength of the discharge current of a condenser in a circuit of no self-induction. $T_0 = SR$. This curve corresponds to the point P in Fig. 138.

Curve II. represents the strength of the discharge current of the same condenser in a circuit of the same resistance, but with self-induction enough just to bring the discharge to the verge of oscillation, this being the condition which effects complete discharge in the shortest time possible. This curve corresponds to the point M in Fig. 138.

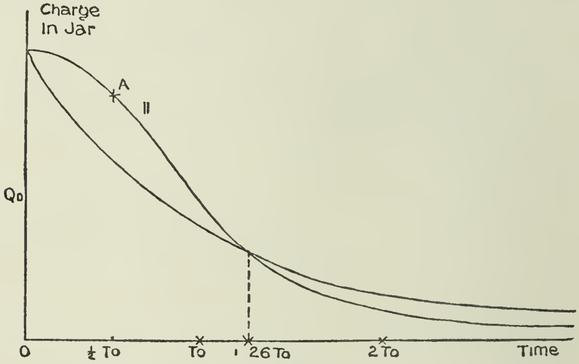

FIG. 140.

Curve I. shows the charge remaining in the jar at any time, the circuit being practically devoid of self-induction.

Curve II. shows the same thing for $L = \frac{1}{4} S R^2$—that is, for the quickest discharge possible. At first Curve I has the advantage, but at a time 1·26 RS the second curve overtakes it and discharges the jar more rapidly.

It is obvious mechanically that if the resistance is moderate and the mass considerable, the recoil of the spring will be accompanied by oscillations, and that with great resistance and small inertia the motion will be a slow sliding back without oscillation; and there must exist between the strength of the spring, the mass of its load, and the viscosity resistance of the medium some definite relation which shall constrain the recoil to be dead beat, just returning to the original position of equilibrium without overshooting the mark. This relation is now seen to be

$$\mathrm{R}^2 = 4\,\mathrm{R}\,m,$$

and under these circumstances the recovery of the spring is effected in the shortest possible time.

In addition to the experimental researches of Blaserna, to which reference has been made at page 246 *et seq.*, very extensive experiments have been made by Bernstein* and by Mouton† on the subject of electrical oscillations in the case of induced currents. Bernstein's experiments were made with a revolving wheel interrupter, which closed a primary circuit, and for a very short time, at a determinable period after the closure of the primary, put the secondary circuit in series with a delicate ballistic galvanometer. In this way the state of the secondary circuit could be investigated at various instants of time after closing or opening the primary circuit, and the general results of Blaserna were confirmed. In Mouton's experiments a rather different form of commutator (*see* Jamin's "Cours de Physique," Vol. IV., p. 201, third edition) was employed to break a primary circuit and to examine with a quadrant electrometer the electrical state of the terminals of an open secondary circuit at various instants afterwards. Mouton found that a potential difference declared itself at less than one four-millionth of a second after rupture of the primary, and that this potential difference died away with decreasing amplitude by rapidly reversing sign, thus indicating the existence of electrical oscillations set up in the open secondary circuit. The duration of the first semi-oscillation was greater than that of succeeding ones. In the case of a secondary circuit of 13,860 turns he found that the first semi-oscillation had a duration of 110 millionths of a second, and the succeeding ones about 77 millionths of a second, and he was able to count about 30 complete oscillations.

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* *Pogg. Ann.*, Vol. CXLII., p. 54, 1871.

† "Etude Expérimental sur les Phénomènes d'Induction Electrodynamique." Thése de Doctorat, 1876.

§ 7. **The Function of the Condenser in an Induction Coil.**—Fizeau appears to have been the first* to suggest that the action of an induction coil employed for raising the electromotive force of a current would be increased by the employment of a condenser. Its mode of use is as follows :—Let P be a primary circuit which takes current from a few cells of a battery, and let I be an interrupter in the primary circuit, either automatically worked by the magnetisation and demagnetisation of the iron core or by any other means. Let S be a secondary circuit of many more turns and high resistance. Under these circumstances each break of the primary current is accompanied by the production of an electromotive force in the secondary circuit capable of producing a discharge across an air space in the secondary circuit. This electromotive force in the secondary is increased by any action tending to increase the suddenness of the stoppage of the primary current, and decreased by anything promoting a spark at the points of rupture of the primary circuit. Fizeau found that if a condenser, formed of alternate sheets of tinfoil and mica or paraffined paper in such fashion as to form a Leyden jar, has its two opposite coatings connected with the two extremities between which the rupture of the primary circuit takes place, then the electromotive force in the secondary circuit under these circumstances is increased. In most current text-books this action is explained by saying that the extra current in the primary circuit, instead of being expended in making a spark at the contact points, darts into the condenser and hastens the decay of the primary current. This explanation as generally given is, however, very imperfect. A more complete examination of the nature of the condenser action has been given by Lord Rayleigh (*Phil. Mag.*, Vol. XXXIX., 1870, p. 428, *et seq.*). In the experiments there detailed a sewing needle was submitted to the magnetising action of an induced secondary current produced by the "break" of the current in a primary circuit. In some previous experiments by the same writer (*Phil. Mag.*, July, 1869, p. 9) it had been shown that the magnetising effect of the secondary current was, *cet. par.*, proportional to the *initial* strength of the induced current, and that this initial strength was proportional to the quotient of M by N, or to the value of the ratio of the coefficient of mutual induction to the coefficient of self-induction of the secondary circuit. It was then found that the magnetising effect of the secondary current was greatly increased by connecting the plates of a condenser respectively to the two points between which the break of the primary circuit occurred. The complete investigation of the values of the induced and primary currents would under these conditions be a good deal more complicated than the investigation of the more simple case of the discharge of a condenser through a single inductive circuit. We are here, however, only concerned with the first part of the electrical motion, the manner in which the currents wear down under the action of the resistances being of subordinate importance. It appears that when the electrical motion is decidedly of the oscillatory type the first few oscillations will take place almost uninfluenced by resistance, and on this supposition the calculation (following Lord Rayleigh) becomes remarkably simple.

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* *Comptes Rendus*, Vol. XXXVI., p. 418, 1853.

Let L, M and N be the primary, mutual and secondary inductance, and let $i$ and $i'$ be the primary and secondary current strengths at any instant, and $q$ and $q'$ the quantities of electricity which have flowed through these circuits from the instant of beginning to reckon the time $t$,

then
$$\frac{dq}{dt}=i \text{ and } \frac{dq'}{dt}=i';$$

and if we neglect resistance effects, as we can do at the instant after "breaking" the primary circuit, and call C the capacity of the condenser bridging across the "break" of the primary circuit, the equations giving the values of the primary and secondary current $i$ and $i'$ at the instant after breaking the primary circuit are—

$$\mathrm{L}\frac{di}{dt}+\mathrm{M}\frac{di'}{dt}+\frac{q}{\mathrm{C}}=0. \quad . \quad . \quad . \quad (123)$$

$$\mathrm{M}\frac{di}{dt}+\mathrm{N}\frac{di'}{dt}=0. \quad . \quad . \quad . \quad (124)$$

Eliminating $i'$ we have

$$\left(L-\frac{M^2}{N}\right)\frac{di}{dt}+\frac{q}{C}=0. \quad . \quad . \quad . \quad (125)$$

(125) may be written

$$\left(L-\frac{M^2}{N}\right)\frac{d^2q}{dt^2}+\frac{q}{C}=0. \quad . \quad . \quad . \quad (126)$$

A differential equation of this type always indicates an oscillatory motion. For, consider the simple periodic function $x=A\sin pt$, where $p=\frac{2\pi}{T}$, T being the periodic time of the motion, we have $\frac{dx}{dt}=pA\cos pt$, and $\frac{d^2x}{dt^2}=-p^2A\sin pt$; hence, $\frac{d^2x}{dt^2}+p^2x=0$, and therefore $x=A\sin pt$ is a particular solution of this equation.

In the above differential equation $p$ is seen to be $2\pi$ times the frequency of the oscillation.

Accordingly, equation (126) indicates an oscillation of the primary current, of which the periodic time is equal to

$$2\pi\sqrt{C\left(L-\frac{M^2}{N}\right)},$$

and this is the periodicity of the electric oscillation set up in the primary at the first instant after "break."

Equation (124) gives by integration the connection between $i$ and $i'$, and it is

$$Mi+Ni'=\text{constant}, \quad . \quad . \quad . \quad (127)$$

which shows that the currents in the primary and secondary oscillate synchronously, the maximum of the one coinciding with the minimum of the other. Since $i'$ is zero at the instant of "break," the constant in equation (127) must be equal to MI, where I is the current strength in the primary just before "breaking" primary circuit.

Accordingly, we have

$$i'=\frac{M}{N}(I-i),$$

so that when, after half an oscillation of the primary, $i$ becomes equal to $-I$, we have

$$i'=2\frac{M}{N}I. \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

This equation gives us the *initial* value of the secondary current $i'$ in terms of the value of the primary current just before the "break" when the condenser is used. Comparing equation (128) with the results on page 236, where it is shown that, if the condenser is not used across the "break" of the primary, the initial value of the secondary current under the assumption of a perfectly sudden break is equal to $\frac{M}{N}I$, we see that the value of the secondary current just immediately after the break of the primary, is *double* that which is there deduced as the value when the primary is simply suddenly stopped without the intervention of the condenser. Stripped of symbolism, what the above amounts to is this: if a condenser is inserted across the "break points" of a primary circuit, then on breaking the primary circuit, the primary current continues to run on into the condenser for a short time; it then *rebounds*, and is reversed in sign, retaining initially its full strength. The electromotive force set up in the secondary circuit is then the result of a stoppage of a primary current and its *immediate reversal* in direction, and this is equivalent to the removal of a certain number of lines of induction from the secondary circuit, and their immediate insertion into it in the opposite direction. Hence, when a condenser is so employed, the inductive electromotive force in the secondary must be just double that which it would be if there were no such rebound of the primary. The condenser acts by setting up electrical oscillations, and it does away with the spark, or largely diminishes it, in virtue of the fact that the condenser acts at the moment of "break" as if it were a shunt circuit of negative self-induction, only with this difference—that instead of dissipating energy like a conducting circuit it returns it again to the primary circuit in the form of a reversed current, and increases the total change of induction through the secondary circuit in the short interval of time immediately succeeding the "break."

Since the sparking distance of the secondary current depends on the initial electromotive force in the secondary—that is, on the maximum of the electromotive force—we see that the condenser so applied can greatly increase the sparking distance of the secondary discharge.

The action is essentially a phenomenon of *resonance.* The condenser causes an elastic recoil in the current and enables the electro-kinetic energy of the steady primary current to be utilised in producing secondary electromotive force rather than suffer dissipation in the form of a contact spark. In order to be efficient in quenching spark the capacity of the condenser must be great enough to take the full primary current, or to receive charge at a rate equal to the delivery of the full primary current for a time during which the contact or break points are separating to a distance too great to permit of much sparking jumping across. There is a certain capacity of condenser suitable for any given coil which produces the most beneficial result in quenching contact spark and lengthening secondary spark. The required capacity is best determined by trial, since the experimental data necessary to furnish the means to calculate it would be probably more difficult to obtain, owing to the fact that it will be determined by several variables, viz., the effective resistance and inductance of the primary circuit, the rate of breaking, and probably also by the amplitude of movement of the "break points." If the primary coil of an induction coil is traversed by an alternating current then the condenser as ordinarily used becomes superfluous. It will be remembered that the late Mr. Spottiswoode obtained secondary sparks of great magnitude from his large coil by so using the alternating current of a De Meritens machine.

If a condenser is discharged through a circuit of which the resistance is so small that it may be neglected in numerical comparisons, then the equation of discharge is

$$L\frac{d^2q}{dt^2}+\frac{q}{C}=0,$$

where the symbols have the same signification as before. As above explained, this indicates that the discharge is oscillatory, and that the time of a complete oscillation is $2\pi\sqrt{LC}$.

In describing the experiments of Blaserna we saw that the frequency of the electrical oscillation set up in circuits on starting and stopping currents in them could be reckoned by tens of thousands per second. In the case of Leyden jars discharged through very short circuits, the frequency may rise to numbers reckoned by millions per second. Since the frequency of luminous vibrations falls between 400 and 700 billions per second, these condenser oscillations fall in frequency in the gap between the acoustic and luminous vibrations.

It is of interest here to note that since these electrical oscillations in a circuit are creating pulsatory electrical disturbances, which spread out from the wire laterally, the wire in which the electrical oscillations are going on is virtually emitting "light," although not such light as can affect our eyes. The ether waves in the case of these electrical disturbances are too long to be eye-affecting. If the velocity of a wave disturbance is V, and the wave length is $\lambda$, and the frequency of the oscillations corresponding to this wave length is $n$, then $V = n\lambda$, for the wave motion travels over the length of one wave in the time of one complete oscillation. In the case of ether disturbances we have seen that V is $3 \times 10^{10}$ centimetres per second, or 186,000 miles per second. Hence when the frequency of the electrical oscillations is known, the wave length of the lateral disturbance emitted can be found. According to Dr. Lodge, a microfarad condenser discharging through a good conducting coil having an inductance of one henry gives a current alternating 160 times in a second, and emits ether waves about 1,200 miles long. A gallon Leyden jar (capacity about 0·003 microfarad) discharging through a stout wire suspended round an ordinary sized room emits ether waves between three and four hundred yards in length, its current alternating at the rate of about one million per second. A pint Leyden jar sparking through an ordinary pair of discharging tongs gives a current of 15 million alternations per second, with ether waves some 20 yards in length. An ordinary electrostatic charge on a sphere two feet in diameter, if disturbed in any way, will surge to and fro at the rate of 300 million vibrations per second, emitting ether waves a yard long. Electric charges on bodies of atomic dimensions, if able to oscillate at all, would vibrate thousands of billions of times a second, and produce ultra-violet light.

The ordinary use of a condenser with an induction coil shows how it can be employed to neutralise the effect of self-induction in a circuit. We have considered on page 187, § 31, of Chapter IV., the case of a condenser having its terminals shunted by a resistance, and the combination placed in series with an inductive circuit and there shown that capacity can neutralize self-induction. We may also consider the case of a condenser in parallel with an inductive circuit as another similar problem. Let L R (Fig. 140) be an inductive circuit, and let the terminals $a\,b$ be closed by a condenser C of capacity C. Let L be the inductance and R the resistance of the coil. Let $i$ be the value at any instant of a simple periodic current sent through the relay and condenser in parallel, and let $i_1, i_2$ be the simultaneous current strengths at that instant in the condenser circuit and the coil circuit. As the potential difference of the points $a$ and $b$ oscillates, an ebb and flow of current is produced in the condenser circuit; the condenser, in fact, is charged and discharged by the periodic current; also a periodic current is produced in the inductive circuit L R. The current in L R lags in phase behind the impressed electromotive force or potential difference of the points $a\,b$, and the current flowing into the condenser lags 90deg. in phase behind the same impressed electromotive force. From this it results that the mean current through the inductive circuit may, under some circumstances, be *greater* when the condenser is joined up to its ends than when it is not so joined; its effective self-induction is thereby lessened, and it acts as if it had experienced a diminution of self-induction. The condition most favourable for producing this result may be investigated as follows:—

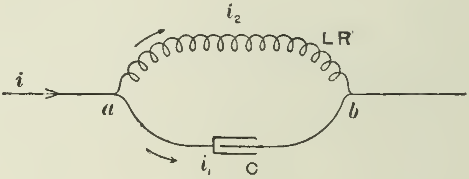

FIG. 140.

Let $v$ be the potential difference of the points $a$ and $b$ at the instant when the current in the undivided circuit is $i$ and that in the branches is $i_2$ and $i_1$. We then have, by the principle of continuity,

$$i = i_1 + i_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

also
$$i_1 = C\frac{d\,v}{d\,t}, \qquad (130)$$

and
$$L\frac{d\,i_2}{d\,t} + R\,i_2 = v, \qquad (131)$$

and we may take the original current before division to be simply periodic, and to be expressed by

$$i = I \sin p\,t \qquad (132)$$

where I is its maximum value.

Then by elimination of $v$ and $i_1$ and $i$ from the above four equations we arrive easily at the equation—

$$C\,L\frac{d^2\,i_2}{d\,t^2} + C\,R\frac{d\,i_2}{d\,t} + i_2 = I \sin p\,t. \qquad (133)$$

Now, since $i_2$ must be a simple periodic current lagging in phase behind that of the undivided current $i$, we may take $i_2$ to be of the form

$$i_2 = I_2 \sin(p\,t - \theta), \qquad (134)$$

$I_2$ being the maximum value of $i_2$, and $\theta$ its phase lag behind $I_1$.

Hence, by differentiation of (134) and substitution in (133) we arrive at

$$(1 - C\,L\,p^2)\,I_2 \sin(p\,t - \theta) + C\,R\,p\,I_2 \cos(p\,t - \theta) = I \sin p\,t, \quad (135)$$

which by the lemma on page 161 may be written—

$$I^2 \sqrt{(1 - C\,L\,p^2)^2 + C^2\,R^2\,p^2}\,(\sin p\,t - \theta + \phi) = I \sin p\,t. \quad (136)$$

Both sides of this last equation are the expressions for the same thing, viz., the value of $i$, and hence, equating the coefficients, we have

$$\left(\frac{I}{I_2}\right)^2 = (1 - C\,L\,p^2)^2 + C^2\,R^2\,p^2. \qquad (137)$$

This gives us the value of the ratio of the maximum or mean values of the strengths of the undivided current and the current in the inductive circuit. If we differentiate the right-hand side of (137) with respect to C, and apply the usual criterion to ascertain whether we have a maximum or minimum value, we find that the expression on the right-hand side of (137) has a minimum value when

$$C = \frac{L}{R^2 + p^2\,L^2}. \qquad (138)$$

In other words, if the capacity of the condenser is so chosen as to have a capacity equal numerically to the quotient of the inductance by the impedance of the coil, then, under these circumstances, the mean strength of the current in the coil circuit will be *greater* than the mean strength of the current before subdivision; and it is easily seen, by substituting in equation (137) the value of C given by (138), which makes the ratio of current strength a minimum, that with this value of the capacity the strength of the current in the inductive coil is to the strength of the current before division in the ratio of the impedance to the resistance of the inductive circuit.

The expression (138) gives the value of the condenser capacity which will produce the required result of minimising the self-induction of a relay of resistance R and inductance L when applied to it. Another problem of a like kind, but not so practically useful, is the investigation of the behaviour of a condenser when joined in *series* with an inductive coil and traversed by a simple periodic current. Let a condenser of capacity C be joined in series with an inductive circuit of resistance R and inductance L, and let a simple periodic current of frequency $n$ be sent through the two in series. It is not difficult to show that, if we take $p$ for $2\pi n$, as usual, and if the capacity and inductance are so related to the frequency of oscillation that $p=\frac{1}{\sqrt{LC}}$, then, under these circumstances, the condenser just annuls the self-induction of the coil, and the two together permit the passage of the same current which would traverse the coil in virtue of its resistance R, assuming it to have no inductance. This is easily proved as follows:—Let L be the inductance and R the resistance of the inductive circuit, and C the capacity of the condenser in series with it. Let $v = V \sin p t$ be the potential difference at the instant $t$, measured over the condenser and inductive resistance, and let $v_1$ and $v_2$ be the fall of potential down the inductive circuit and condenser respectively. Then

$$v=v_1+v_2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (139)$$

also,

$$L\frac{di}{dt}+Ri=v_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

and

$$\frac{1}{C}\int i\,dt=v_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

where $i$ is the value of the current flowing in the circuit at the instant when the potential difference between the ends of the whole circuit is $v$.

Hence by substitution we have

$$L\frac{di}{dt}+Ri+\frac{1}{C}\int i\,dt=V\sin pt. \quad . \quad . \quad (142)$$

The current being also simply periodic must have a value $i$ expressed by the equation,

$$i=I\sin(pt-\theta), \quad . \quad . \quad . \quad . \quad (143)$$

since it will differ in phase by an angle $\theta$ from the potential difference $v$.

Accordingly, we find that by differentiating (143) and substituting the values in (142) we arrive at the equation

$$(1-CLp^2)\sin(pt-\theta)+RCp\cos(pt-\theta)=\frac{CVp}{I}\cos pt.$$

The maximum value of the current, viz., I, is therefore given by the expression

$$I=\frac{CVp}{\sqrt{(1-CLp^2)^2+R^2C^2p^2}}.$$

If then $1=CLp^2$ or $p=\frac{1}{\sqrt{CL}}$, we see that the above equation reduces to

$$I=\frac{V}{R},$$

and the whole circuit of coil and condenser is equivalent to a simple non-inductive circuit of resistance R, in other words the inductance is annuled. Hence a certain relation between the inductance, capacity, and frequency, causes the inductance to be neutralised by the capacity, and the whole circuit to be effectively non-inductive.

§ 8. **Impulsive Discharges and Relation of Inductance thereto.**—If between the ends of a conductor a difference of potential is created which is brought about slowly, the result shows itself in a current in the conductor, and the resulting current is determined as to strength by the mode of variation of the potential and by the capacity as well as by the inductance and ohmic resistance of the conductor. If, however, the difference of potential is created with great suddenness, the resulting electric flow is less determined by the true resistance, and more by the inductance of the conductor. In this case we have the phenomena of *impulsive discharges.* We have a mechanical analogy in the case of impulses or sudden blows given to heavy bodies, which well illustrates how strikingly force phenomena may be altered when for steady or slowly varying forces we substitute exceedingly brief impulses or blows. If an explosive, such as gun-cotton, is laid on a stone slab in open air, and simply ignited, it burns away with comparative slowness ; the slab is uninjured, and the evolved gases simply push the air away to make room for themselves. But it is well known that by means of detonators the same explosive can be fired with enormously greater rapidity, and in this case the blow or impulse given to the air is so sudden that it has not time to be pushed away, and in virtue of its inertia its incapacity of receiving a finite velocity in an infinitely small time bestows on it an *inertia resistance*, which causes nearly the whole of the effect of the explosion to take effect downwards on the slab, and this last is shattered. The inductance of conductors introduces a series of phenomena which are the electrical analogues of the above mechanical experiment. We have seen that the counter electromotive force of self-induction is proportional to the rate of change of another quantity, called the electro-kinetic momentum, and this quantity physically interpreted is the total flux of induction or number of lines of induction enclosed by the conducting circuit at that instant. A conductor of sensible inductance can no more have a current of finite magnitude created in it instantaneously than a body of sensible mass can have a finite velocity instantaneously given to it. In both cases there is an immense resistance to very sudden change of condition. A very loose plug of snow or earth stuffed into the muzzle of a loaded gun will cause it to burst when fired, since the inertia resistance of the plug to very sudden motion is exceedingly large, though the frictional resistance may be small. Accordingly, the study of the behaviour of conductors under exceedingly sudden electric blows or electromotive impulses leads us to consider some very interesting effects. We shall best elucidate these effects by describing some interesting and suggestive experiments due to Dr. Oliver Lodge.*

His first experiment is called the experiment of the *alternative path.* The two terminal knobs of a Voss or Wimshurst electrical machine (*see* Fig. 141) are connected to the two inside coatings of a pair of Leyden jars. The two outside coatings are connected to the balls of another discharger, B, and the terminals of this discharger are short-circuited by a metal wire, indicated by the dotted line. The Leyden jars stand on a badly insulating wooden base. On turning the handle of the electrical machine the inside coatings receive equal and opposite electrical charges, and there is an induced charge on the outer coating of each, which, in the language of the old school of electricians, was called the "bound" charge. When the difference of potentials of the inner coatings reaches a certain value the air space at A is cracked, and a spark passes, discharging the inner coatings of the jars. At that instant the charges of the outer coatings are set "free," or, in modern language, the potential of one rises and that of the other falls. The effect of this is that whereas before the spark passed at A the balls at B

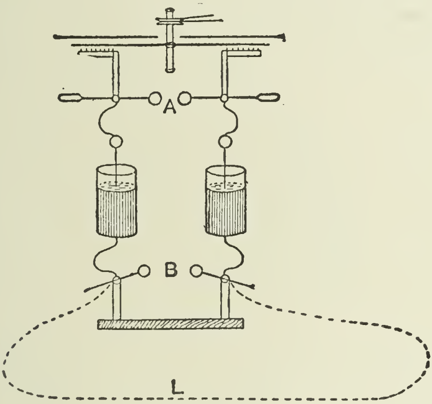

FIG. 141.

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* The account of these experiments is taken from the report of Dr. Lodge's Mann Lectures before the Society of Arts. These suggestive lectures were reprinted in *The Electrician*, entitled "Protection of Buildings from Lightning," Vol. XXI., pp. 234, 273, 302.

were at equal potential, on a spark at A happening the balls at B are instantaneously brought to a very great difference of potential. It might be thought that since the balls are short-circuited by a metallic wire this difference of potential will expend itself on making a current in the wire. On the contrary, very little of the discharge may take place through the wire. A spark passes at B, or, in other words, the discharge passes in great part across the exceedingly highly resisting air space at B, rather than take the circuit of the metallic wire of very low resistance, so that although there is a divided circuit open to the discharge, one branch of which measures hundreds of thousands of ohms or megohms and the other only a small fraction of an ohm, it nearly all goes by the route of higher resistance. The explanation of this is that when the balls at B are thrown with great suddenness into opposite electrical states the counter electromotive force of self-induction of the circuit of metal L makes it virtually non-conducting. The electromotive impulse meets with such resistance owing to the electro-magnetic inertia of the circuit that it rebounds and cracks through the air. In order that it shall do this, however, the distance of this air gap at B has to be less than a certain amount. There is a certain critical distance of the knobs B for less than which the discharge always jumps across B, and for greater than which the discharge keeps mainly to the metallic circuit. Even if the short-circuiting metal is a thick rod, still when B is not great the discharge chooses the air-gap path. The phenomenon here presented has had fresh interest and attention called to it by Dr. Lodge, but it has really been long a familiar one, though its explanation has not stood out hitherto so sharply as it does now.

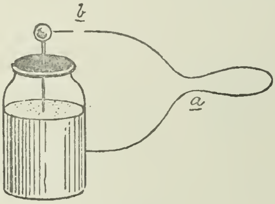

Fig. 142.

Faraday was acquainted with it, and showed that if a charged Leyden jar is discharged by means of a wire crossed or bent so that there was a loop (Fig. 142), the wires at *a* nearly but not quite touching, then when the spark happened at *b*, a spark took place also at *a*, showing that some at least of the discharge jumped across *a* instead of pursuing the course of the metal loop.

The same fact lies at the base of the action of all lightning arresters placed on telegraph or telephone instruments. Mr. C. F. Varley, we believe, first suggested that the coils of the single-needle instrument might be protected from damage by lightning by twisting together the earth and line wires where they leave the case, the theory being that although ordinary currents were not short-circuited by reason of the cotton covering of the wires, yet lightning discharges would meet with such resistance in the inductive coils that they would jump across the knot from wire to wire rather than pass round and damage the coils. In the same way the ordinary *comb protector* is supposed to act. Between the line wire L (*see* Fig. 143) and the electromagnetic instrument, relay, telephone, &c., is placed a metal comb, which has its points in opposition to another comb in connection with the earth, and the other terminal of the electromagnetic instrument is also "to earth." An incoming current has then two paths open to it to get to earth, one of comparatively low resistance through the instrument, and one of enormously high resistance across the

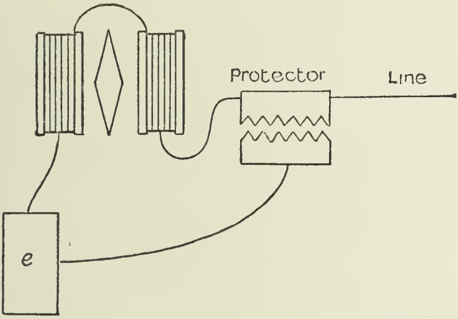

FIG. 143.

air-gap between the comb points. Ordinary currents, steady or periodic, pass entirely through the metallic circuit. Very violent electric impulses, such as a lightning discharge, meet with an enormous inductive opposition in the electromagnetic instrument owing to the inability of an inductive circuit to respond to an electromotive impulse instantaneously. Hence the air-gap is cracked, and the discharge passes across the combs and the instrument may be saved. Evidence exists, however, pointing to the fact that the protection afforded by these contrivances is very far from complete. We are not here concerned with their efficiency as practical devices, but only with them as illustrating the principle of the alternative path and the behaviour of inductive circuits to impulsive discharges. That these devices are insufficient has been fully demonstrated by many experimentalists.*

In these experiments of the alternative path it was found by Dr. Lodge that the critical distance at which the discharge just prefers to jump the air-gap was greater for a thick copper rod 40 feet long (No. 1 B.W.G.) than for an iron wire (No. 27 B.W.G.) of 33·3 ohms resistance, indicating a less inductive inertia on the part of the iron; but this fact is only true for the particular circumstances of the experiment. A very clear difference was established between copper rod and tape, using conductors of the same length and weight. The tape has an advantage in permitting more easily the passage of sudden electric discharges. A controversy on the relative suitability of rod and tape for lightning conductors dates from the time of Faraday and Sir W. Snow Harris, and a possible explanation of the reasons for preferring one rather than the other presents itself when we consider the matter in the light of those considerations which induce us to think that an electric current begins always at the surface of conductor, and takes a certain time to diffuse or soak into the mass of the metal. It is not cross-section but surface which is here concerned; and, other things being equal, the conductor which offers the greatest surface to the dielectric is able to drain the energy out of the dielectric most quickly and dissipate it as heat in the conductor. We have referred to this on a previous page (see *ante*, p. 252), and it will be mentioned again in connection with some views of Prof. Poynting.* With respect to the apparent superiority of iron, it would naturally have been supposed that, since the magnetic permeability of iron bestows upon it greater inductance, it would form a less suitable conductor for discharging electric energy with great suddenness. Owing to the fact that the current only penetrates just into the skin of the conductor, there is but little of the mass of the iron magnetised, even if these instantaneous discharges are capable of magnetising iron. This last fact has been thought to be due to an actual time lag of magnetisation, viz., that magnetising force required to endure for a sensible time in order to produce magnetisation, but recent views tend in the direction of considering the apparent lag as a consequence of the fact that the eddy currents produced in the surface layers of the metal by the discharge shield the inner and deeper layers from inductive influence, as described under the head of Magnetic Screening. In any event the final result is the same; the electromotive impulses, or sudden rushes of electricity, do not magnetise the iron, and hence do not find in it any greater self-inductive opposition than they would find in a non-magnetic but otherwise similar conductor. Dr. Lodge's further researches seem to show that there is a real advantage in using iron for lightning conductors over copper, and that its greater specific resistance and higher fusing point enable an iron rod or tape to get rid safely of an amount of electric energy stored up in a dielectric which would not be the case if it were copper. This point is further elucidated by some other experiments of Dr. Lodge. Two tinfoil conductors were prepared of approximately equal resistance and length. One of these was formed into a spiral, each layer being insulated with paraffin paper, and wound on a glass tube. The other was made into a zig-zag or non-inductive resistance. These conductors were then employed as alternative paths, as in the former experiment with the copper wire. In the case when the tin-foil zig-zag was employed to short-circuit the jars it was not possible to get a B spark (*see* Fig. 141) until the distance of the A balls was shortened to 0·6 (tenths of inch). When the tinfoil spiral was used the critical spark distance at B rose to 6·4. When the iron wire bundle was inserted in the tube it did not in any perceptible degree increase this distance. The length of the sparking distance at A was 7·3, and when no alternative path was used at all to connect the jars the critical distance of the B balls, at which sparks sometimes passed and sometimes failed, was 11·1. Here, then, we have the non-magnetisability of iron by sudden discharges illustrated. Dr. Lodge has called attention to the fact that a "choking" coil having a core of divided iron and wound over with many turns of wire does not add to the apparent self-induction of a circuit discharging a Leyden jar. It may even diminish it when the discharge is oscillatory and of sufficient frequency, although the oscillations may be as few as 500 per second. This experiment shows, as we know from other facts, that eddy currents are set up even in a core of finely-divided iron, and that these eddy currents, under sufficiently rapid alternations, are confined to the surface of the core, and moreover, since they are as regards phase nearly in opposition to that of the current in the coil, they actually tend to diminish the total flux of induction through the coil, and hence diminish the self-induction of the circuit.

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* For an account of some interesting experiments by Prof. Hughes and Prof. Guillemin on "Lightning Protectors" see *The Electrician*, Vol. XXI., p. 304, July 13, 1888. It was found that a protector consisting of two opposed flat plates was better than a comb or opposed point.

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* For some special remarks on the self-induction of wires of various cross-sections see Mr. Oliver Heaviside in the *Phil. Mag.*, January, 1887, p. 11:—"The magnetic energy per unit of length of a circuit is $\frac{1}{2} L i^2$, where $i$ is the current in the wire and L the inductance per unit of length. As regards the diminution of L in general by spreading out the current in a strip instead of concentrating it in a wire, that is a matter of elementary reasoning founded on the general structure of L. If we draw apart currents, keeping the currents constant, thus doing work against their mutual attraction, we diminish their energy at the same time by the amount of work done against their attraction. Thus the quantity $\frac{1}{2} L i^2$ of a circuit is the amount of work that must be done to take the current to pieces, so to speak—that is, to separate all its filamentary elements of currents to an infinite distance. If wires are taken, each of a unit of length and of the same total cross-sectional area, but of different forms of cross-section, round, square, elliptical, equilateral triangle, narrow rectangle, &c., the ratio of their inductances is the same as the ratio of their torsional rigidities. Thus the narrow strip has the least torsional rigidity, and the circular-sectioned wire the greatest, and this is true also for their relative self-inductions."

The inductive opposition to electric discharge presented by even a short length of conductor, when the difference of potential between the ends is made very suddenly, is seen in the tendency under such circumstances to *side flash*. If a conductor, say, a straight rod of copper, has one end to earth, and somewhere very near its side is the end of another conductor also "to earth," then if the free end of the first conductor is suddenly exalted in potential the impulsive rush of electricity, meeting with such an obstacle in the inductance of the conductor, spits or flashes out laterally and sparks to the other conductor. No conductor is able to prevent side flash altogether unless it has practically no inductance. As long as a conductor must be straight (like a lightning conductor) so long will there be a tendency to side flash. This is illustrated by the following experiment. A massive conductor has (Fig. 144) a very fine wire stretched alongside and air gaps in this byepath left by bringing the ends of the fine wire very near to the sides of the large conductor. On sending an impulsive rush of electricity through the large conductor little sparks are seen at *a* and *b*, showing that some of the discharge has left the thick conductor and travelled along the fine wire, even although it had to leap across an air gap. If the bare hands are applied to the ends of an open spiral of very stout copper wire, one end of which is connected to a "good earth," shocks will be felt when a Leyden jar is discharged through the copper. In this case the human body forms the bye-path, and the experiment indicates that the law of division of steady currents or slow discharges between conductors in parallel, viz., a division in the ratio of their conductivities, does not hold good for impulsive discharge, and that the relative inductance of the circuits has more influence in the latter case in determining what happens.

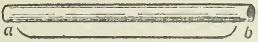

FIG. 144.

The distinction between the resulting discharges due to a steady electromotive force or strain and that due to an electromotive impulse or impulsive rush of electricity has been illustrated by some further experiments by Dr. Lodge on the behaviour of model lightning conductors when subjected to the action of these two modes of discharge. Two tin plates are placed horizontally and insulated, and these are supposed to represent the earth and a thunder-cloud. These plates are connected, as in Fig. 145, to an electrical machine, and by working the handle are brought up to a steady potential difference. On the lower plate are placed little rods of various heights, sharp, or having knobs, and these represent lightning conductors. At a certain potential difference the electric strain set up in the air exceeds the limit which the dielectric can sustain, and it breaks down, giving rise to a spark. A discharge then takes place towards one or other of the mimic lightning conductors. In one experiment three conductors were used—one with a large knob, 0·9in. less in height than the distance between the plates, the second with a small knob, 2in. less in height, and a sharp short point. The point even when very low prevents discharge altogether. It may be too low to be effective, or it may be insufficient to cope with the supply of electricity if that is supplied very fast, but it acts to prevent discharge. If the point is removed or covered up we then find that the discharge takes place, when the potential difference of the plates is made great enough, to the small knob by preference, and it does so even when the stem of the short knob is lower than that of the large knob. In other words, when the stems are the same height the small knob protects the large one, and it does this until lowered in height to about two inches less than the other; when this is the case both knobs are struck indifferently. And it does this even when a resistance of one megohm is interposed in the stem of the smaller knob. The state of things is, however, very much altered if in place of bringing up the two plates gradually to a sparking potential difference they are very suddenly thrown

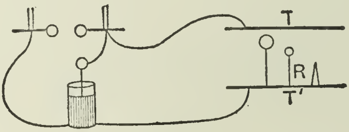

Fig. 145.

into opposite electrical conditions by connecting them to the electrical machine as shown in Fig. 146.

The jars charge up as they stand on the same wooden table, and when the potential rises to sparking amount they discharge at A, and a violent electric rush then takes place between the two plates, and the conductors between are struck. If the same three kinds of conductors are used, and they be adjusted until they are all about equally struck, we find that the smaller and shorter-stemmed knob no longer protects the larger one, and the sharp point no longer protects either; all three, large ball, small ball and point, are liable to be struck equally if at the same height, and if they differ in height the highest is most likely to be struck, no matter what it is. Points are, then, no protection against these impulsive rushes of electricity. The special virtue of a point in the case of the slower-timed discharges is that it prepares the path of the discharge to itself, for in this case the path is pre-arranged by induction. If one of the conductors has a large resistance—say a liquid megohm inserted in it—then this one is no longer struck; it ceases to protect the other conductors even if higher than them, and even if it be so raised in height that it touches the top plate, thus connecting the plates by a bad conductor, the two other conductors get struck with apparently the same ease as before. This indicates that a lightning conductor with a bad earth cannot protect well against discharges of the nature of a sudden rush. Mr. Wimshurst has, however, shown reason for considering that in this experiment the electrical state of plates, as regards sign of electrification, may be of importance. The question how far the point protects from the impulsive rush is not altogether cleared up. It is still *sub judice.*

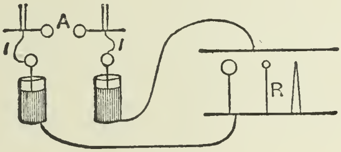

FIG. 146.

In performing the first experiment of the alternative path (Fig. 141) it was noticed that the B spark was longer than the A spark. Plainly this indicates that the discharge at A sets up electrical oscillations. The manner in which this is brought about is as follows :—On the commencement of the discharge the air-space is intensely heated, and its conductivity so far increased that the conditions as to the relation of inductance, resistance and capacity of the discharger and condenser are fulfilled, and the discharge takes the oscillatory form. If a couple of long leads are attached to the A discharger (Fig. 147), the farther ends being insulated, and a discharger B bridged across at $B_1$, $B_2$ or $B_3$, then it is found that at every discharge at A a spark can be obtained at B, and for a certain length of A spark the B spark will be longer at $B_3$ than at the nearer positions. Evidently what happens is that the electrical oscillation across the A discharge intervals sets up violent surgings to and fro in the open circuit wires, just like water in a long trough when it is tilted, and the recoil at the insulated ends, combined with the inductance of these leads, produces a cross flash at B. It is, in fact, a case of resonance; the long open circuit leads act like resonators to the oscillating discharge across A, and the nearer the length of the leads approaches to half a wave length or to some multiple of half a wave length the more perfect will be the resonance and the greater the recoil at the open ends, and hence the greater the spark at $B_3$.

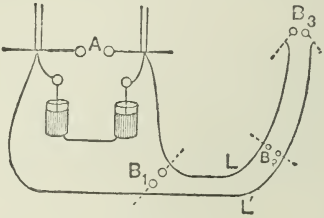

FIG. 147.

If the experiment is tried in the dark, the B discharger being removed, it is seen that the leads glow at the ends with a vivid brush light at the moment when the jars are discharged. When the proper length of open circuit lead has been found which resonates best in accord with the jars used as dischargers, then the whole of the effects described can be made to disappear by connecting a very small Leyden jar to the ends of the wires. The increase of static capacity thus given to the leads reduces their potential below sparking point. Arranging the jar so as to leave an air-space between it and one of the wires, a spark passes into it at each A spark; but the jar is not in the least charged afterwards, proving that the spark is a double one, first in and then out of the jar, a real recoil of the reflected pulse. Hence, also, we see that the brush visible in the dark is the same on each wire, and one is not able to say that one brush is positive and the other negative, for each is both.

A curious experiment illustrating the electrical surgings or oscillations set up in a conductor which is suddenly discharged at one end is as follows: Attach one end of a long wire to one knob of a Wimshurst machine, and connect the other pole to earth. The wire is otherwise insulated, and now forms one coating of a condenser of which the other is the walls of the room. The wire is bent round so that its free end nearly touches its initial end (*see* Fig. 148). Under these circumstances one would naturally say that a spark at B was absurd, and yet it is found that even if the wire is a stout copper wire a spark happens at B when one is produced at A. This B spark is caused by an electrical oscillation in the wire. The wire is, as it were, pumped full of electricity by the machine, and when the spark happens at A a release is given at that end for one brief instant. Then ensues a rebound of the electricity, and the pressure rises at the free end to sparking amount. The whole effect is just analogous to the effect of suddenly opening and closing a tap on a high-pressure water service—a concussion is heard in the tap on shutting, and if one could see the water it would be found that it rebounds, and a reflected

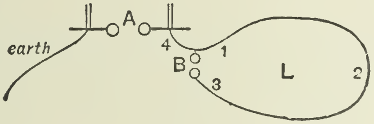

FIG. 148.

wave is set up in the pipe, which, if the pipe is not strong enough, will burst it at some weak point. The practical moral of this is that any large conductor suddenly discharged has set up in it violent electrical surgings, which may cause it to spit off discharges at other points, and these sparks may be as long as the principal spark.

Another way of making these electrical surgings conspicuous is by their effect in causing a Leyden jar to overflow, *i.e.*, to spark round its edge. A jar does this when its coatings are very suddenly raised to a great potential difference. Fig. 149 shows the arrangement. The inside of the jar is made to communicate direct to one machine pole, and the outer coating, through the intervention of a long wire, to the other pole.

When a spark happens at A, and the length of the wire L is sufficiently great, the jar sparks over its edge. The explanation of this is as follows:—Whilst the handle of the machine is being turned the potential difference of the jar coatings increases. At a certain limit the air in the A space breaks down, and, being heated, becomes for a moment a very good conductor; there is, therefore, a rush of electricity out of the inner coating and into the outer coating, but the spark at A ceasing, this outflow from the jar is suddenly stopped and rebounds, whilst at the same time the inductance of the wire L causes a rush to continue into the jar. The rebound of the flow when the rush through the air space is suddenly stopped causes the potential difference of the coatings to rise to a point at which they spark over the edge of the glass. In an example given by Dr. Lodge the jar was a one gallon jar, with glass fully three inches above the tinfoil. L was a thick No. 1 copper wire circuit round a room. The jar

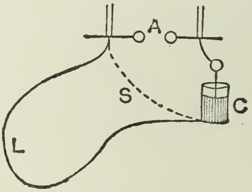

Fig. 149.

overflows every time a spark happens at A, even though the length of this spark is only 0·64in. If the long lead L is short-circuited, then the jar refuses to overflow until the A spark has been increased to 1·7in. The higher potential difference needed to cause overflow or rebound in the case with a short circuit is illustrative of the fact that a little self-induction in the discharging circuit bestows momentum on the flow and assists in making a back splash.

A hydraulic analogue to the above might be found in considering the case of a liquid flowing steadily along a trough or canal. If an obstruction was suddenly created, as by closing a valve or sluice, the liquid would rebound and a wave would be created; and, as in the case of the hydraulic ram, the rebound of the liquid against a closed valve might be made to lift some of it to a higher level than that from which it originally fell. In the electrical case, the rebound is made to raise the jar coatings to a greater potential difference than that which existed at the instant when the jar commenced to discharge.

### § 9. Theory of Experiments on the Alternative Path.

—We may proceed, following Dr. Lodge,* and quoting freely from him in what follows, to examine a little more in detail the electrical oscillations set up in an open circuit by Leyden jar discharges. These stationary electrical oscillations in linear conductors resemble those which can be set up in a cord fixed at one end, or in a trough of liquid, by suitably-timed impulses. As we have seen, if a jar discharges at A (*see* Fig. 150) in the ordinary way, simultaneously an even longer spark may be obtained at B, at the far end of two long open circuit leads. Or if the B ends of the wire are too far apart to allow of a spark, the wires glow and spit off brushes every time a discharge occurs at A. The theory of the effect seems to be that oscillations occur in the A circuit with a period $T = 2\pi\sqrt{LC}$, where L is the inductance of the A circuit and C the capacity of the jar. These oscillations disturb the surrounding medium, and send out radiations of the precise nature of light, only too long in wave length to affect the retina of our eyes. The velocity of these electro-magnetic impulses is, as we have seen, equal to $v$, where

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* See *Phil. Mag.*, August, 1888; also *The Electrician*, August 10, 1888, p. 435.

$$v = \frac{1}{\sqrt{\mu \mathrm{K}}};$$

so the wave length of the oscillations is

$$\lambda = v\mathrm{T} = 2\pi\sqrt{\frac{\mathrm{L}}{\mu} \cdot \frac{\mathrm{C}}{\mathrm{K}}}.$$

Now $\frac{\mathrm{L}}{\mu}$ is the electro-magnetic measure of inductance, and $\frac{\mathrm{C}}{\mathrm{K}}$ is the electrostatic measure of capacity, $\mu$ being the magnetic permeability, and K the electrostatic inductivity of the medium surrounding the wire.

Each of these quantities is of the dimensions of a *length*, and the wave length of the radiation is $2\pi$ times their geometric mean. We may look upon it, then, that the magnetic field due to the oscillatory current in the A circuit, which circuit consists partly of metal wires, partly of the dielectric of the jar, and partly of the heated air in the spark space, acts inductively upon the other or B circuit which is adjacent to it, and has, in fact, the jar dielectric as a common boundary. The pulsating field induces oscillatory currents in the open B circuit. These electric pulses rush along the surface of the wires with a certain amount of dissipation, and are reflected at the distant end, producing a recoil kick or impulse tending to break down the dielectric in the air gap B with production of a spark. These currents continue to oscillate to and fro unti damped out of existence by the resistance of the wires. The best effect in the way of spark at B is observed when the length of each wire is such that the time occupied by an electric pulse in travelling along the wires and back again is equal to the time of a complete oscillation in the A circuit; that is,

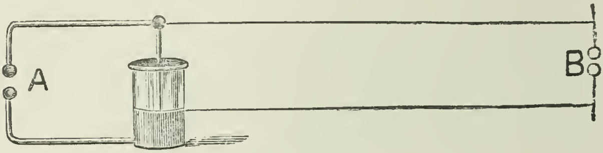

Fig. 150.

when the length of the open circuit wires is equal to half a wave length or to some multiple of half a wave length. The natural period of oscillation in the long wires will then agree with the oscillation period of the discharging circuit and the oscillations in the open circuit wires, and the field due to the oscillations in the A circuit will vibrate in unison like a column of air in a pipe resonating a tuning fork, or like a string vibrating when attached to the tongue of a reed.

The elementary theory of the open circuit oscillations is as follows:—

Let $l_1$ and $r_1$ be the inductance and resistance of the straight wires per unit of length, *as affected by the periodicity*, and let $c_1$ be the capacity per unit of length. It has been shown by Lord Rayleigh (*Phil. Mag.*, May, 1886) that with very rapid oscillations owing to the circumferential distribution of the current the inductance and resistance have values different from the steady current values, and when the frequency of the oscillations is very great the resistance $r_1$ per unit of length is the geometric mean of its ordinary value $r$ and $\frac{1}{2} p \mu_0$, where $\mu_0$ is the magnetic permeability of the material of the conductor, or $r_1 = \sqrt{\frac{1}{2} p \mu_0 r}$, $p$ being, as usual, $2\pi n$, $n$ being the number of complete oscillations per second.

And again, when $n$ is very great, the inductance $l_1$ per unit of length is equal to a constant *plus* $\frac{r_1}{p}$, or

$$l_1 = l + \frac{r_1}{p},$$

$l$ being the induction for slowly fluctuating currents.

In the case of the two parallel wires we have for the slope of the potential $-\frac{dV}{dx}$ along them the usual equations,

$$l_1 \frac{di}{dt} + r_1 i = -\frac{dV}{dx}, \quad . \quad . \quad . \quad . \quad (144)$$

$i$ being the instantaneous current in the section of the length lying at a distance $x$ from the origin; and also for the accumulation of charge in this element $dx$ of the length we have the equation

$$-\frac{dV}{dt} = \frac{1}{C_1} \frac{di}{dx}. \quad . \quad . \quad . \quad . \quad (145)$$

The elimination of $i$ between these equations gives us a differential equation for V, and shows that stationary waves of current are set up in finite wires of suitable length under the action of an alternating electromotive force. The solution of the equation for a long wire when $r_1$ is small and $p$ is very large is

$$V = V_0 e^{-\frac{m_1}{n_1}x} \cos p\left(t - \frac{x}{n_1}\right),$$

where $$m_1 = \frac{r_1}{2\, l_1} \text{ and } n_1 = \frac{1}{\sqrt{L_1 C_1}}.$$

The velocity of propagation of the wave is therefore $n_1$ and the wave length is $\frac{2\pi}{p} n_1$.

For two parallel wires, as in the Leyden jar case, we have each wire

$$r_1 = \sqrt{\tfrac{1}{2} p\, \mu_0\, r},$$

$r$ being the ordinary resistance. And again, as Lord Rayleigh has shown (*Phil. Mag.*, May, 1886), we have

$$l_1 = 4\, \mu \log \frac{b}{a} + \frac{r_1}{p},$$

$b$ being the distance between the parallel wires and $a$ the radius of either, and $\mu$ the magnetic permeability of the material of the conductors.

For immensely quick oscillations the second term is zero. Also, the capacity $C_1$ of the wires per unit of length is, by a known theorem,

$$C_1 = \frac{K}{4 \log \frac{b}{a}};$$

hence $$\frac{1}{\sqrt{L_1 C_1}} = \frac{1}{\sqrt{\mu K}},$$

and the velocity of the pulse along the wires is the same as in the dielectric round them. In other words, the electric pulses set up in the wires rush to and fro with a velocity equal to that with which the electro-magnetic impulse is propagated through the dielectric round them. Hence, we have here a means of determining experimentally the wave length of a given discharging circuit. Either vary the size of the A circuit or adjust the length of the B wires until the recoil spark B is as long as possible. Then measure, and see whether the length of each wire is not equal to

$$\pi\sqrt{\frac{L}{\mu}\cdot\frac{C}{\kappa}}.$$

A small condenser can be made having an electrostatic capacity of, say, two or three centimetres, and if such a coated pane be made to discharge over its edge, the discharged circuit will have an electro-magnetic inductance of a few centimetres. Under these circumstances the electrical oscillations would be at the rate of a thousand million a second, and the wave length of the electro-magnetic disturbance radiated would be about 20 to 30 centimetres.

If a conductor as small as an atom could have its electrical charge disturbed in the same way, oscillations would be set up of the frequency of light waves and electro-magnetic disturbances of light wave length radiated; and it seems probable that this is just what light waves are, viz., electro-magnetic disturbances propagated through the ether and due to electric oscillatious set up in the atomic charge.

§ **10. Impulsive Impedance.**—In the experiments of the "alternative path," as described by Dr. Lodge, the main result is very briefly summed up by saying that when a sudden discharge had to pass through a conductor it was found that iron and copper acted about equally well, and indeed iron sometimes exhibited a little superiority, and that the thickness of the conductor and its ordinary conductivity mattered very little indeed. We are led by this to see that the impedance which a conductor offers to a sudden discharge, and which may be called its impulsive impedance, is something quite different from its ordinary or ohmic resistance, or even its impedance, defined as $\sqrt{R^2+p^2L^2}$, to slowly periodic or oscillatory currents. As already mentioned, the resistance of a conductor to very rapidly changing currents is expressed by $R_1$, where

$$R_1 = \sqrt{\tfrac{1}{2} p\, l\, \mu_0\, R},$$

R being the resistance to steady current, $\mu_0$ the permeability of the material of the conductor and $l$ its length, and $p = 2\pi$

times the frequency of the oscillation. Also the corresponding inductance $L_1$ is

$$L_1 = L + \frac{R_1}{p},$$

where L is a constant depending on the size and form of the circuit, but only in a small degree upon its thickness. Hence, forming the function $\sqrt{R_1^2 + p^2 L_1^2}$, and calling this $\text{Im}_1$, we have

$$\begin{aligned} \text{Im}_1 &= \sqrt{(pL + R_1)^2 + R_1^2} \\ &= pL\sqrt{1 + \frac{2m}{\sqrt{p}} + \frac{2m^2}{p}}, \end{aligned}$$

where $$m = \frac{1}{L}\sqrt{\tfrac{1}{2} l \mu_0 R}.$$

In the case of enormously rapid oscillations the value of $\text{Im}_1$ practically reduces to $p$ L, and hence the impulsive impedance varies in simple proportion to the frequency, and depends on the form and size of the circuit, but not at all on its specific resistance, magnetic permeability, or diameter.

All this is borne out by experiment. In some of his experiments Dr. Lodge found the impedance of a No. 2 wire of two and a-half metres length bent into a circle to be 180 ohms at twelve million oscillations per second, and for a No. 40 wire the impedance was only 300 ohms, although the ohmic resistances of these wires were respectively ·004 ohms and 2·6 ohms. At three million oscillations per second, or at one-fourth the frequency, the impedances of the same circuits were 43 ohms and 78 ohms. At one-quarter million oscillations per second the impedances are reduced to four and six ohms respectively for the thick rod and fine wire. Hence, for frequencies of a million per second and upwards, such as occur in jar discharges, and perhaps in lightning, the impedance of all reasonably conducting circuits is the same, and independent of conductivity and permeability, and hardly affected greatly by enormous changes in diameter.

§ **11. Hertz's Researches on the Propagation of Electro-magnetic Induction.**—The classical researches of Hertz on electrical oscillations and the propagation of electro-magnetic induction through space form an epoch in the history of electrical science. These investigations have been well described by Dr. Lodge in his book on "The Work of Hertz,"* and the reader is referred to this for an account of the chief work of Hertz and his followers. There is therefore no need to enter here at very great length into an account of these discoveries; but a very excellent abstract of Hertz's work has been given by Mr. G. W. de Tunzelmann.†

*Preliminary Experiments.*—It is known that if in the secondary circuit of an induction coil there be inserted, in addition to the ordinary air space across which sparks pass, a Riess spark micrometer, with its poles joined by a long wire, the discharge will pass across the air space of the micrometer in preference to following the path of least resistance through the wire, provided this air space does not exceed a certain limit; and it is upon this principle that lightning protectors for telegraph lines are constructed. It might be expected that the sparks could be made to disappear by diminishing the length and resistance of the connecting wire; but Hertz found that though the length of the sparks could be diminished in this way, it is almost impossible to get rid of them entirely, and they can still be observed when the balls of the micrometer are connected by a thick copper wire only a few centimetres in length.

This shows that there must be variations in the potential measurable in hundreds of volts in a portion of the circuit only a few centimetres in length, and it also gives an indirect proof of the enormous rapidity of the discharge; for the difference of potential between the micrometer knobs can only be due to self-induction in the connecting wire. Now the time occupied by variations in the potential of one of the knobs must be of the same order as that in which these variations can be transmitted through a short length of a good conductor to the second knob. The resistance of the wire connecting the knobs is found to be without sensible effect on the results.

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* Published by "The Electrician" Printing and Publishing Company, Limited.

† This section originally appeared as a series of articles in the pages of *The Electrician*, in Vol. XXI., pp. 587, 625, 663, 696, 725, 757, 788 (1888). The writer felt it would be difficult to make a more complete digest of Hertz's work than is contained in these excellent articles, and, by the kind permission of their author, he is allowed to reproduce them in these pages.

In Fig. 151, A is an induction coil and B a discharger. The wire connecting the knobs 1 and 2 of the spark micrometer M consists of a rectangle, half a metre in length, of copper wire two millimetres in diameter. This rectangle is connected with the secondary circuit of the coil in the manner shown in the diagram, and, when the coil is in action, sparks, sometimes several millimetres in length, are seen to pass between the knobs 1 and 2, showing that there are violent electrical oscillations not only in the secondary circuit itself, but in any conductor in contact with it. This experiment shows even more clearly than the previous one that the rapidity of the oscillations is comparable with the velocity of transmission of electrical disturbances through the copper wire, which, according to all the evidence at our disposal, is nearly equal to the velocity of light.

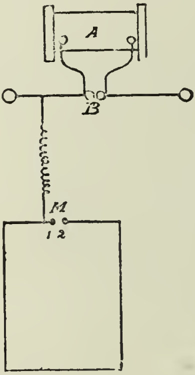

Fig. 151.

In order to obtain micrometer sparks some millimetres in length a powerful induction coil is required, and the one used by Hertz was 52 centimetres in length and 20 centimetres in diameter, provided with a mercury contact breaker, and excited by six large Bunsen cells. The discharger terminals consisted of brass knobs three centimetres in diameter. The experiments showed that the phenomenon depends to a very great extent on the nature of the sparks at the discharger, the micrometer sparks being found to be much weaker when the discharge in the secondary circuit took place between two points or between a point and a plate than when knobs were used. The micrometers sparks were also found to be greatly enfeebled when the secondary discharge took place in a rarefied gas, and also when the sparks in the secondary were less than half a centimetre in length; while, on the other hand, if they exceeded 1½ centimetres the sparks could no longer be observed between the micrometer knobs. The length of secondary spark which was found to give the best results, and which was therefore employed in the further observations, was about three-quarters of a centimetre.

Very slight differences in the nature of the secondary sparks were found to have great effect on those of the micrometer, and Hertz states that after some practice he was able to determine at once from the sound and appearance of the secondary spark whether it was of a kind to give the most powerful effects at the micrometer. The sparks which gave the best results were of a brilliant white colour, only slightly jagged, and accompanied by a sharp crack.

The influence of the spark is readily shown by increasing the distance between the discharger knobs beyond the striking distance, when the micrometer sparks disappear entirely, although the variations of potential are now greater than before. The length of the micrometer circuit has naturally an important influence on the length of the spark, as the greater its length the greater will be the retardation of the electrical wave in its passage through it from one knob of the micrometer to the other.

The material, the resistance, and the diameter of the wire of which the micrometer circuit is formed have very little influence on the spark. The potential variations cannot, therefore, be due to the resistance; and this was to be expected, for the rate of propagation of an electrical disturbance along a conductor depends mainly on its capacity and coefficient of self-induction, and only to a very small extent on its resistance. The length of the wire connecting the micrometer circuit with the secondary circuit of the coil is also found to have very little influence, provided it does not exceed a few metres in length. The electrical disturbances must therefore traverse it without undergoing any appreciable change. The position of the point of the micrometer circuit which is joined to the secondary circuit is, on the other hand, of the greatest importance, as would be expected, for, if the point is placed symmetrically with respect to the two micrometer knobs, the variations of potential will reach the latter in the same phase, and there will be no effect, as is verified by observation. If the two branches of the micrometer circuit on each side of the point of contact of the connection with the secondary are not symmetrical the spark cannot be made to disappear entirely; but a minimum effect is obtained when the point of contact is about half-way between the micrometer knobs. This point may be called the null point.

Fig. 152 shows the arrangement employed, *e* being the null point of the rectangular circuit, which is 125 centimetres long by 80 centimetres broad. When the point of contact is at *a* or *b* sparks of from three to four millimetres in length are observed; when it is at *e* no sparks are seen, but they can be made to reappear by shifting the point of contact a few centimetres to the right or left of the null point. It should be noted that sparks only a few hundredths of a millimetre in length can be observed. If, when the point of contract is at *e*, another conductor is placed in contact with one of the micrometer knobs, the sparks reappear.

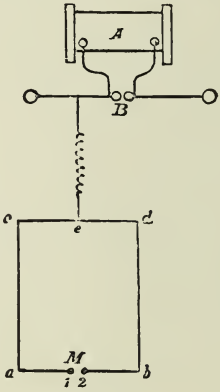

FIG. 152.

Now, the addition of this conductor cannot produce any alteration in the time taken by the disturbances proceeding from *e* to reach the knobs, and therefore the phenomenon cannot be due simply to single waves in the direction *c a* and *d b* respectively, but must be due to repeated reflection of the waves until a condition of stationary vibration is attained, and the addition of the conductor to one of the knobs must diminish or prevent the reflection of the waves from that terminal. It must be assumed, then, that definite oscillations are set up in the micrometer circuit just as an elastic bar is thrown into definite vibrations by blows from a hammer. If this assumption is correct, the condition for the disappearance of the sparks at M will be that the vibration periods of the two branches *e* 1 and *e* 2 shall be equal. These periods are determined by the products of the coefficients of self-induction of these conductors into the capacities of their terminals, and are practically independent of their resistances.

In confirmation of this it is found that if, when the point of contact is at *e* and the sparks have been made to reappear by connecting a conductor with one of the knobs, this conductor is replaced by one of greater capacity, the sparking is greatly increased. If a conductor of equal capacity is connected with the other micrometer knob, the sparks disappear again; the effect of the first conductor can also be counteracted by shifting the point of contact towards it, thereby diminishing the self-induction in that branch. The conclusions were further confirmed by the results obtained when coils of copper wire were inserted into one or other and then into both of the branches of the micrometer circuit.

Hertz supposed that as the self-induction of iron wires is, for slow alternations, from eight to ten times that of copper wires, therefore a short iron wire would balance a long copper one; but this was not found to be the case, and he concludes that, owing to the great rapidity of the alternations, the magnetism of the iron is unable to follow them, and therefore has no effect on the self-induction.*

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* In a note in Wiedemann's *Annalen*, Vol. XXXI., p. 543, Dr. Hertz stated that since the publication of his Paper in the same volume he had found that Von Bezold had published a Paper, in 1870 (Poggendorff's *Annalen*, Vol. CXL., p. 541), in which he had arrived by a different method of experimenting at similar results and conclusions as those given by him under the head of Preliminary Experiments.

*Induction Phenomena in Open Circuits.*—In order to test more fully his conclusion that the sparks obtained in the last experiments described were due to self-induction, Hertz placed a rectangle of copper wire with sides 10 and 20 centimetres in length respectively, broken by a short air space, with one of its sides parallel and close to various portions of the secondary circuit of the coil and of the micrometer circuit, with solid dielectrics interposed to obviate the possibility of sparking across, and he found that sparking in this rectangle invariably accompanied the discharges of the induction coil, the longest sparks being obtained when a side of the rectangle was close to the discharger.

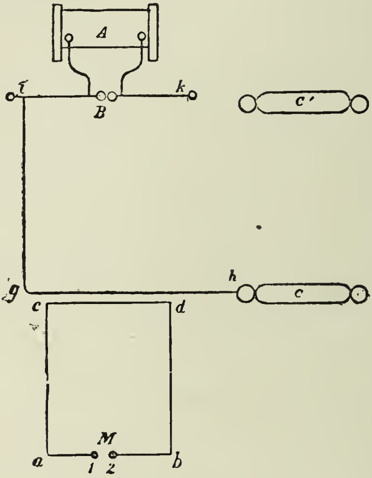

FIG. 153.

A copper wire, $igh$ (Fig. 153), was next attached to the discharger, and a side of the micrometer circuit, which was supported on an insulating stand, was placed parallel to a portion of this wire, as shown in the diagram. The sparks at M were then found to be extremely feeble until a conductor, C, was attached to the free end, $h$, of the copper wire, when they increased to one or two millimetres in length. That the action of C was not an electrostatic one was shown by its producing no effect when attached at $g$ instead of at $h$. When the knobs of the discharger B were so far separated that no sparking took place there, the sparks at M were also found to disappear, showing that these were due to the sudden discharges and not to the charging current. The sparks at the discharger which produced the most effect at the micrometer were of the same character as those described under the head of *Preliminary Experiments.* Sparks were also found to occur between the micrometer circuit and insulated conductors in its vicinity. The sparks became much shorter when conductors of large capacity were attached to the micrometer knobs, or when these were touched by the hand, showing that the quantity of electricity in motion was too small to charge these conductors to a similarly high potential. Joining the micrometer knobs by a wet thread did not perceptibly diminish the strength of the sparks. The effects in the micrometer circuit were not of sufficient strength to produce any sensation when it was touched or the circuit completed through the body.

In order to obtain further confirmation of the oscillatory nature of the current in the circuit $k\ i\ h\ g$ (Fig. 153), the conductor C was again attached to $h$, and the micrometer knobs drawn apart until sparks only passed singly. A second conductor, C′, as nearly as possible similar to C, was then attached to $k$, when a stream of sparks was immediately observed, and it continued when the knobs were drawn still further apart. This effect could not be ascribed to a direct action of the portion of circuit $i\ k$, for in this case the action of the portion of circuit $g\ h$ would be weakened, and it must therefore have consisted in C′ acting on the discharging current of C—a result which would be quite incomprehensible unless the current in $g\ h$ were of an oscillatory character.

Since an oscillatory motion between C and C′ is essential for the production of powerful inductive effects, it will not be sufficient for the spark to occur in an exceedingly short time, but the resistance must at the same time not exceed certain limits. The inductive effects will therefore be excessively small if the induction coil included in the circuit C C′ is replaced by an electrical machine alternately charging and discharging itself, or if too small an induction coil is used, or, again, if the air space between the discharger knobs is too great, as in all these cases the motion ceases to be oscillatory.

The reason that the discharge of a powerful induction coil gives rise to oscillatory motion is that, firstly, it charges the terminals C and C′ to a high potential; secondly, it produces a sudden spark in the intervening circuit; and thirdly, as soon as the discharge begins the resistance of the air space is so much reduced as to allow of oscillatory motion being set up. If the terminal conductors are of a very large capacity—for example, if the terminals are in connection with a battery—the current of discharge may indefinitely reduce the resistance of the air space, but when the terminal conductors are of small capacity this must be done by a separate discharge, and therefore, under the conditions of Hertz's experiments, an induction coil was absolutely essential for the production of the oscillations.

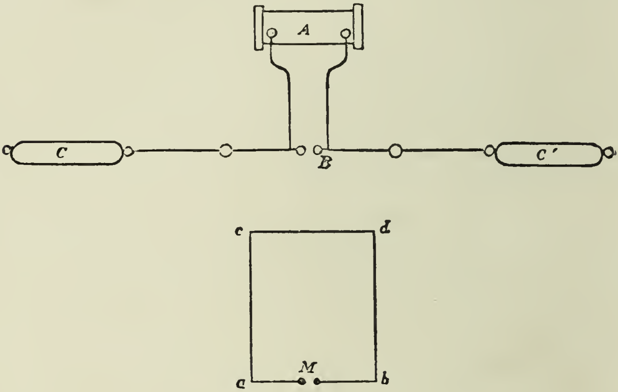

Fig. 154.

As the induced sparks in the experiment last described were several millimetres in length, Hertz modified it by using the arrangement shown in Fig. 154, and greatly increasing the distance between the micrometer circuit and the secondary circuit of the induction coil. The terminal conductors C and C′ were three metres apart, and the wire between them was of copper, 2 millimetres in diameter, with the discharger B at its centre.

The micrometer circuit consisted, as in the preceding experiments, of a rectangle 80 centimetres broad by 120 centimetres long. With the nearest side of the micrometer circuit at a distance of half a millimetre from C B C′, sparks two millimetres in length were obtained at M, and though the length of the sparks decreased rapidly as the distance of the micrometer circuit was increased, a continuous stream of sparks was still obtained at a distance of one and a-half metres. The intervention of the observer's body between the micrometer circuit and the wire C B C′ produced no visible effect on the stream of sparks at M. That the effect was really due to the rectilinear conductor C B C′ was proved by the fact that when one or other, or both, halves of this conductor were removed, the sparks at M ceased. The same effect was produced by drawing the knobs of the discharger B apart until sparks ceased to pass, showing that the effect was not due to the electrostatic potential difference of C and C′, as this would be increased by separating the discharger knobs beyond sparking distance.

The closed micrometer circuit was then replaced by a straight copper wire, slightly shorter than the distance C C′, placed parallel to C B C′ and at a distance of 60 centimetres from it. This wire terminated in knobs, 10 centimetres in diameter, attached to insulating supports, and the spark micrometer divided it into two equal parts. Under these circumstances sparks were obtained at the micrometer as before.

With the rectilinear open micrometer circuit sparks were still observed at the micrometer when the discharger knobs of the secondary coil circuit were separated beyond sparking distance. This was, of course, due simply to electrostatic induction, and shows that the oscillatory current in C C′ was superposed upon the ordinary discharges. The electrostatic action could be got rid of by joining the micrometer knobs by means of a damp thread. The conductivity of this thread was therefore sufficient to afford a passage to the comparatively slow alternations of the coil discharge, but was not sufficient to provide a passage for the immeasurably more rapid alternations of the oscillatory current. Considerable sparking took place at the micrometer when its distance from C B C′ was 1·2 metre, and faint sparks were distinguishable up to 3 metres. At these distances it was not necessary to use the damp thread to get rid of the electrostatic action, as, owing to its diminishing more rapidly with increase of distance than the effect of the current induction, it was no longer able to produce sparks in the micrometer, as was proved by separating the discharger knobs beyond sparking distance, when sparks could no longer be perceived at the micrometer.

*Resonance Phenomena.*—In order to determine whether the oscillations were of the nature of a regular vibration, Hertz availed himself of the principle of resonance. According to this principle, an oscillatory current of definite period would, other conditions being the same, exert a much greater inductive effect upon one of equal period than upon one differing even slightly from it.*

If, then, two circuits are taken having as nearly as possible equal vibration periods, the effect of one upon the other will be diminished by altering either the capacity or the coefficient of self-induction of one of them, as a change in either of them would alter the period of vibration of the circuit.

This was carried out by means of an arrangement very similar to that of Fig. 154. The conductor C C′ was replaced by a straight copper wire 2·6 metres in length and 5 millimetres in diameter, divided into two equal parts as before by a discharger. The discharger knobs were attached directly to the secondary terminals of the induction coil. Two hollow zinc spheres, 30 centimetres in diameter, were made to slide on the wire, one on each side of the discharger, and since, electrically speaking, these formed the terminals of the conductor, its length could be varied by altering their position. The micrometer circuit was chosen of such dimensions as to have, if the author's hypothesis were correct, a slightly shorter vibration period than that of C C′. It was formed of a square, with sides 75 centimetres in length, of copper wire 2 millimetres in diameter, and it was placed with its nearest side parallel to C B C′ and at a distance of 30 centimetres from it. The sparking distance at the micrometer was then found to be 0·9 millimetre. When the terminals of the micrometer circuit was placed in contact with two metal spheres 8 centimetres in diameter, supported on insulating stands, the sparking distance could be increased up to 2·5 millimetres. When these were replaced by much larger spheres the sparking distance was diminished to a small fraction of a millimetre. Similar results were obtained on connecting the micrometer terminals with the plates of a

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* *See* Oberbeck, Wiedemann's *Annalen*, Vol. XXVI., p. 245, 1885.

Kohlrausch condenser. When the plates were far apart the increase of capacity increased the sparking distance, but when the plates were brought close together the sparking distances again fell to a very small value.

The simplest method of adjusting the capacity of the micrometer circuit is to suspend to its ends two parallel wires the distance and lengths of which are capable of variation. By this means the author succeeded in increasing the sparking distance up to three millimetres, after which it diminished when the wires were either lengthened or shortened. The decrease of the sparking distance on increasing the capacity was naturally to be expected; but it would be difficult to understand, except on the principle of resonance, why a decrease of the capacity should have the same effect.

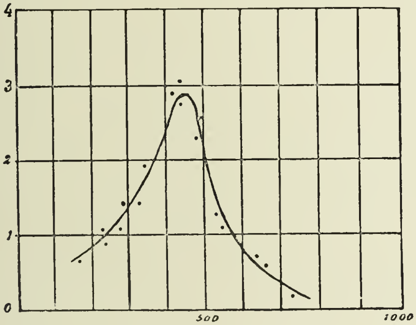

Fig. 155.—Curve showing relation between length of side of rectangle (taken as abscissa) and maximum sparking distance (taken as ordinate), the sides consisting of straight wires of varying lengths.

The experiments were then varied by diminishing the capacity of the circuit C B C′ so as to shorten its period of oscillation, and the results confirmed those previously obtained; and a series of experiments in which the lengths and capacities of the circuits were varied in different ways showed conclusively that the maximum effect does not depend on the conditions of either one of the two circuits, but on the existence of the proper relation between them.

When the two circuits were brought very close together, and the discharger knobs separated by an interval of 7 millimetres, sparks were obtained at the micrometer, which were also

7 millimetres in length, when the two circuits had been carefully adjusted to have the same period. The induced E.M.F's must in this case have attained nearly as high a value as the inducing ones.

To show the effect of varying the coefficient of self-induction, a series of rectangles, *a b c d* (Fig. 154), were taken, having a constant breadth, *a b*, but a length, *a c*, continually increasing from 10 centimetres up to 250 centimetres: it was found that the maximum effect was obtained with a length of 1·8 metre. The quantitative results of these experiments are shown in Fig. 155, in which the abscissæ of the curve are the double lengths of the rectangles, and the ordinates represent the corresponding maximum sparking distances. The sparking distances could not be determined with great exactness, but the errors were not sufficient to mask the general nature of the result.

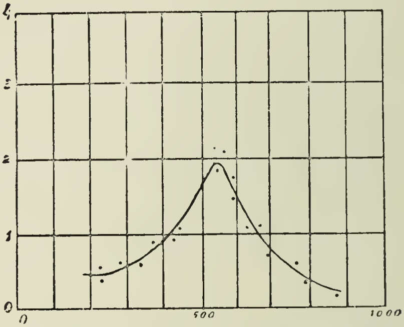

Fig. 156.—Curve showing relation between length of side of rectangle (taken as abscissa) and maximum sparking distance (taken as ordinate), the sides consisting of spirals gradually drawn out.

In a second series of experiments the sides *a c* and *b d* were formed of loose coils of wire which were gradually pulled out, and the result is shown in Fig. 156. It will be seen that the maximum sparking distance was attained for a somewhat greater length of side, which is explained by the fact that in the latter experiments the self-induction only was increased by increase of length, while in the former series the capacity was increased as well. Varying the resistance of the micrometer circuit by using copper and German silver wires of various diameters was found to have no effect on the period of oscillation, and extremely little on the sparking distance.

When the wire *c d* was surrounded by an iron tube, or when it was replaced by an iron wire, no perceptible effect was obtained, confirming the conclusion previously arrived at that the magnetism of the iron is unable to follow such rapid oscillations, and therefore exerts no appreciable effect.

It is only proper, however, to interpolate at this point the remark that other observers do not endorse entirely this statement of Hertz. We may especially draw attention to the work of Prof. J. Trowbridge and of Mr. C. E. St. John* on the propagation of electrical oscillations on iron wires. The experimental results obtained by these investigators may be summed up as follows :—

1. The magnetic permeability of iron wires exercises an important influence upon the decay of electrical oscillations of high frequency. The influence is so great that the oscillations may be reduced to half an oscillation on a circuit of suitable self-induction and capacity for producing them.

2. Currents of high frequency such as are produced in Leyden jar discharges therefore magnetise iron.

3. The self-induction of iron circuits is sensibly greater than that of similar copper circuits under rapid electrical oscillations $115 \times 10^6$ reversals per second.

4. This increase in self-induction produces a shortening of the wave-length.

5. The permeability of annealed iron under the above rate of alternation is about 385.

For full information as to the methods of obtaining these results we must refer the reader to the original Papers.

*Nodes.*—The vibrations in the micrometer circuit which have been considered are the simplest ones possible, but not the only ones. While the potential at the ends alternates between two fixed limits, that at the central portion of the circuit retains a constant mean value. The electrical vibration, therefore, has a node at the centre, and this will be the only nodal point. Its existence may be proved by placing a small insulated sphere close to various portions of the micrometer circuit while sparks are passing at the discharger of the coil, when it will be found that if the sphere is placed close to the centre of the circuit the sparking will be very slight, increasing as the sphere is moved further away. The sparking cannot, however, be entirely got rid of, and there is a better way of determining the existence and position of the node. After adjusting the two circuits to unison, and drawing the micrometer terminals so far apart that sparks can only be made to pass by means of resonant action, let different parts of the circuit be touched by a conductor of some capacity, when it will be found that the sparks disappear, owing to interference with the resonant action, except when the point of contact is at the centre of the circuit. Hertz then endeavoured to produce a vibration with two nodes, and for this purpose he modified the apparatus previously used by adding to the micrometer circuit a second rectangle, *e f g h*, exactly similar to the first (as shown in Fig. 157), and joining the points of the circuit near the terminals by wires 1 3 and 2 4, as shown in the diagram.

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* See *Phil. Mag.*, December, 1891, Mr. J. Trowbridge on "Damping of Electrical Oscillations on Iron Wires ;" and *Phil. Mag.*, November, 1894, Mr. C. E. St. John on "Wave-Lengths of Electricity on Iron Wires.'

The whole system then formed a closed metallic circuit, the fundamental vibration of which would have two nodes. Since the period of this vibration would necessarily agree closely with that of each half of the circuit, and, therefore, with that of the circuit C C′, it was to be expected that the vibration would have a pair of loops at the junctions 1 3 and 2 4, and a pair of nodes at the middle points of *c d* and *g h*. The vibrations were determined by measuring the sparking distance between the micrometer terminals 1 and 2. It was found that, contrary to what was expected, the addition of the second rectangle diminished this sparking distance from about three millimetres to about one millimetre. The existence of resonant action between the circuit C C′ and the micrometer circuit was, however, fully demonstrated, for any alteration in the circuit *e f g h*, whether it consisted in increasing or in decreasing its length, diminished the sparking distance. It was also found that much weaker sparking took place between *c d* or *g h* and an insulated sphere than between *a e* or *b f* and the same sphere, showing that the nodes were in *c d* and *g h*, as expected. Further, when the sphere was made to touch *c d* or *g h* it had no effect on the sparking distance of 1 and 2; but when the point of contact was at any other portion of the circuit the sparking distance was diminished, showing that these nodes did really belong to the vibration, the resonant action of which increased this sparking distance.

The wire joining the points 2 and 4 was then removed. As the strength of the induced oscillatory current should be zero at these points, the removal ought not to disturb the vibrations, and this was shown experimentally to be the case, the resonant effects and the position of the nodes remaining unchanged. The vibration with two nodal points was, of course, not the fundamental vibration of the circuit, which consisted of a vibration with a node between *a* and *e*, and for which the highest values of the potential were at the points 2 and 4.

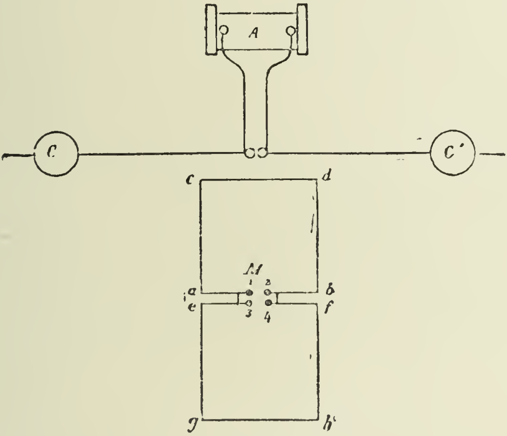

Fig. 157.

When these spheres forming the terminals at these points were brought close together slight sparking was found to take place between them, which was attributed to the excitation, though only to a small extent, of the fundamental vibration. This explanation was confirmed in the following manner:—The sparks between 1 and 2 were broken off, leaving only the sparks between 2 and 4, which measured the intensity of the fundamental vibration. The period of vibration of the circuit C C′ was then increased by drawing it out to its full length, and thereby increasing its capacity, when it was observed that the sparking gradually increased to a maximum, and then began to diminish again. The maximum value must evidently occur when the period of vibration of the circuit C C′ is the same as that of the fundamental vibration of the micrometer circuit, and it was shown that when the sparking distance between 2 and 4 had its maximum value the sparks corresponded to a vibration with only one nodal point, for the sparks ceased when the previously existing nodes were touched by a conductor, and the only point where contact could take place without effect on the sparking was between *a* and *e*. These results show that it is possible to excite at will in the same conductor either the fundamental vibration or its first overtone, to use the language of acoustics.

Hertz appeared to consider it very doubtful whether it was possible to get higher overtones of electrical vibration, the difficulty of obtaining such lying not only in the method of observation, but also in the nature of the oscillations themselves. The intensity of these is found to vary considerably during a series of discharges from the coil even when all the circumstances are maintained as constant as possible, and the comparative feebleness of the resonant effects shows that there must be a considerable amount of damping. There are, moreover, many secondary phenomena which seem to indicate that irregular vibrations are superposed upon the regular ones, as would be expected in complex systems of conductors. If, therefore, we wish to compare electrical oscillations from a mathematical point of view with those of acoustics, we must seek our analogy in the high notes intermixed with irregular vibrations, obtained, say, by striking a wooden rod with a hammer rather than in the comparatively slow harmonic vibration of tuning forks or strings; and in the case of vibrations of the former class we have to be contented even in the study of acoustics with little more than indications of such phenomena as resonance and nodal points.

Referring to the conditions to be fulfilled in order to obtain the best results, Hertz noted a fact of very considerable interest and novelty, namely, that the spark from the discharger should always be visible from the micrometer, as, when this was not the case, though the phenomena observed were of the same character, the sparking distance was invariably diminished. This effect of the light from the spark of an induction coil in increasing the sparking distance in a secondary circuit has been fully described by Dr. Lodge in his book on the work of Hertz, and he has pointed out that the same effect is produced by light from burning magnesium wire or other sources rich in the ultra-violet rays.

*Theory of the Experiments.*—The theories of electrical oscillations which have been developed by Lord Kelvin, von Helmholtz, and Kirchoff have been shown* to hold good for the open circuit oscillations of induction apparatus, as well as for the oscillatory Leyden jar discharge; and it is of interest to inquire whether the observed results are of the same order as those indicated by theory.

Hertz considers, in the first place, the vibration period. Let T be the period of a single or half vibration proper to the conductor exciting the micrometer circuit; L its coefficient of self-induction in absolute electromagnetic measure, expressed, therefore, in centimetres; C the capacity of one of its terminals in electrostatic measure, and therefore also expressed in centimetres; and $v$ the velocity of light in centimetre-seconds; then, if the resistance of the conductor is small,

$$T = \frac{\pi \sqrt{LC}}{v}.$$

In the case of the resonance experiments, the capacity C was approximately the radius of the sphere forming the terminal, so that C = 7·5 centimetres.† The coefficient of self-induction was that of a wire of length $l=150$ centimetres and diameter $d=1/2$ centimetre.

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* Lorentz, Wiedemann's *Annalen*, Vol. VII., p. 161, 1879.

† In Hertz's original Paper the capacity of the spherical terminal ball was taken as 15 units. M. Poincaré first drew attention to the fact that the capacity C in the above formula denotes the amount of electricity which exists at one end of an oscillating conductor when the difference of potential between the two ends is equal to unity. Hence, if the spheres are far apart, the difference of potential between each of them and surrounding space is $\pm\frac{1}{2}$. Therefore the charge on the sphere is formed by dividing its capacity, *i.e.*, its radius in centimetres, by 2. Hence, C in the above formula is $\frac{15}{2} = 7{\cdot}5$.

According to Neumann's formula,

$$\mathrm{L}=\iint\frac{\cos\epsilon}{r}\,ds\,ds',$$

which gives in the case considered

$$\mathrm{L}=2l\left(\log\frac{4l}{d}-0{\cdot}75\right)=1{,}902 \text{ centimetres.}$$

As, however, it is not quite certain that Neumann's formula is applicable to an open circuit, it is better to use von Helmholtz's more general formula, containing an undetermined constant $k$, according to which

$$\mathrm{L}=2l\left(\log\frac{4l}{d}-0{\cdot}75+\frac{1-k}{2}\right).$$

Putting $k=1$, this reduces to Neumann's formula; for $k=0$ it reduces to that of Maxwell, and for $k=-1$ to Weber's. The greatest difference in the values of L obtained by giving these different values to $k$ would not exceed a sixth of its mean value, and therefore, for the purposes of the present approximation, it is enough to assume that $k$ is not a large positive or negative number; for if the number 1,902 does not give the correct value of the coefficient for the wire 150 centimetres in length, it will give the value corresponding to a conductor not differing greatly from it in length.

Taking $\mathrm{L}=1{,}902$ centimetres, we have $\pi\sqrt{\mathrm{C\,L}}=531$ centimetres, which represents the distance traversed by light during the oscillation, or, according to Maxwell's theory, the length of an electromagnetic ether wave. The value of T is then found to be 1·26 hundred-millionths of a second, which is of the same order as the observed results.

The ratio of damping is then considered. In order that oscillations may be possible, the resistance of the open circuit must be less than $2v\sqrt{\mathrm{L/C}}$. For the exciting circuit used this gives 676 ohms as the upper limit of resistance. If the actual resistance, $r$, is sensibly below this limit, the ratio of damping will be $e^{\frac{r\mathrm{T}}{2\mathrm{L}}}$. The amplitude will therefore be reduced in the ratio 1:2·71 in

$$\frac{2\mathrm{L}}{r\mathrm{T}}=\frac{2v}{\pi r}\sqrt{\frac{\mathrm{L}}{\mathrm{C}}}=\frac{676}{\pi r}=\frac{215}{r}$$

oscillations. We have, unfortunately, no means of determining the resistance of the air space traversed by the spark, but as the resistance of a strong electric arc is never less than a few ohms we shall be justified in assuming this as the minimum limit. From this it would follow that the number of oscillations due to a single impulse must be reckoned in tens, and not in hundreds or thousands, which is in accordance with the character of the experimental results, and agrees with results observed in the case of the oscillatory Leyden jar discharge. In the case of closed metallic circuits, on the other hand, theory indicates that the number of oscillations before equilibrium is attained must be reckoned by thousands.

Hertz compares, lastly, the order of the inductive actions of these oscillations according to theory with that of the effects actually observed. To do this it must be noted that the maximum E.M.F. induced by the oscillation in its own circuit is approximately equal to the maximum potential difference at its extremities; for if there were no damping these quantities would be identical, since at any moment the potential difference at the extremities and the E.M.F. of induction would be in equilibrium. In the experiments under consideration the potential difference at the extremities was such as to give a spark 7 to 8 millimetres in length, which must therefore represent the maximum inductive action excited in its own circuit by the oscillation. Again, at any instant the induced E.M.F. in the micrometer circuit must be to that in the exciting conductor in the same ratio as that of the coefficient of mutual induction M of the two circuits to the coefficient of self-induction L of the exciting circuit. The value of M for the case considered is easily calculated from the ordinary formulæ, and it is found to lie between one-ninth and one-twelth of L. This would only give sparks of from $\frac{1}{2}$ to $\frac{2}{3}$ millimetre in length, so that according to theory visible sparks ought in any case to be obtained; but, on the other hand, sparks several millimetres in length, as were obtained in the experiments previously described, can only be explained on the assumption that the successive inductive actions produce an accumulative effect; so that theory indicates the necessity of the existence of the resonant effects actually observed.

Hertz was at first inclined to suppose that as the micrometer circuit was only broken by the extremely short air space limited by the maximum sparking distance under the conditions of the experiment, it might therefore be treated as a closed circuit, and only the total induction considered. The ordinary methods of electro-dynamics give the means of completely determining the total inductive effect of a current element on a closed circuit, and would, therefore, in this case have sufficed for the investigation of the phenomena observed. He found, however, that the treatment of the micrometer circuit as a closed circuit led to incorrect results, so that it, as well as the primary, had to be treated as an open circuit, and therefore a knowledge of the total induction was insufficient, and it became necessary to consider the value both of the E.M.F. of induction and of the electrostatic E.M.F. due to the charged extremities of the exciting circuit at each point of the micrometer circuit.

The investigations to which these considerations led are described by Hertz in a Paper, "On the Action of a Rectilinear Electrical Oscillation upon a Circuit in its Vicinity," published in Wiedemann's *Annalen*, Vol. XXXIV., p. 155, 1888.

In what follows the exciting circuit will be spoken of as the primary and the micrometer circuit as the secondary. Hertz points out that the reason that electrostatic effect cannot be neglected is to be found in the extreme rapidity with which the electrostatic forces change their sign. If the electrostatic alternations in the primary were comparatively slow they might attain a very high intensity without giving rise to a spark in the secondary, since the electrostatic distribution on the secondary would vary so as to remain in equilibrium with the external E.M.F. This, however, is impossible, because the variations in direction follow each other too rapidly for the distribution to follow them.

In the present investigations the primary circuit consisted of a straight copper wire 5 millimetres in diameter, carrying at its extremities hollow zinc spheres 30 centimetres in diameter. The centres of the spheres were one metre apart, and at the middle of the wire was an air space $\frac{3}{4}$ centimetre in length. The wire was placed in a horizontal position, and the observations were all made at points near to the horizontal plane through it, which, however, did not of course affect their generality, as the same effects would necessarily be produced in any plane through the horizontal wire. The secondary circuit consisted of a circle of 35 centimetres radius, of copper wire 2 millimetres in diameter, the circle being broken by an air space capable of variation by means of a micrometer screw.*

The circular form was selected for the secondary circuit because the former investigations had shown that the sparking distance was not the same at all points of the secondary, even when the conductor as a whole remained unchanged in position, and with a circular circuit it was easier to bring the air space to any part than if any other form had been used. To attain this object the circle was made movable about an axis passing through its centre perpendicular to its plane.

The circuits of the dimensions stated were very nearly in unison, and they were further adjusted by means of little strips of metal soldered to the extremities, and varied in length until the maximum sparking distance was obtained.

We shall follow Hertz in first considering the subject theoretically, and then examining how far the experimental results are in accordance with the theoretical conclusions. It will be assumed that the E.M.F. at every point is a simple harmonic function of the time, but that it does not undergo reversal in direction, and it will further be assumed that the oscillations are at any given moment everywhere in the same phase. This will certainly be the case in the immediate neighbourhood of the primary, and for the present we shall confine our attention to such points. Let $s$ be the distance of a point measured along the circuit from the air space of the secondary, and F the component E.M.F. at that point along the circular arc $ds$. Then F is a function of $s$, which assumes its original value after passing once round the circle of circumference S. It may, therefore, be expanded in the form

$$F = A + B\cos\frac{2\pi s}{S} + \ldots\ldots + B'\sin\frac{2\pi s}{S} + \ldots\ldots$$

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* This small circular detector circuit may be called *an electro-magnetic eye*, because it enables us to see the electro-magnetic disturbance and to localise it.

The higher terms of the series may be neglected, as the only result of so doing will be that the approximate theory will give an absolute disappearance of sparks where really the disappearance is not quite complete, and indeed the experiments are not delicate enough to enable us to compare their results with theory beyond a first approximation.

The force A acts in the same direction and is of constant amount at all points of the circle, and therefore it must be independent of the electrostatic E.M.F., as the integral of the latter round the circle is zero. A, then, represents the total E.M.F. of induction, which is measured by the rate of variation of the number of magnetic lines of force which pass through the circle. If the electro-magnetic field containing the circle is assumed to be uniform, A will therefore be proportional to the component of the magnetic induction perpendicular to the plane of the secondary. It will therefore vanish when the direction of the magnetic induction lies in the plane of the secondary. A will consist of an oscillation, the intensity of which is independent of the position of the air space in the circle, and the corresponding sparking distance will be called $a$.

The term $B' \sin \frac{2\pi s}{S}$ can have no effect in exciting the fundamental vibration of the secondary, since it is symmetrical on opposite sides of the air space.

The term $B \cos \frac{2\pi s}{S}$ will give a force acting in the same direction in the two quadrants opposed to the air space, and will excite the fundamental vibration. In the two quadrants adjacent to the air space it will give a force in the opposite direction, but its effect will be less than that of the former one; for the current is zero at the extremities of the circuit, and therefore the electricity cannot move so freely as near the centre. This corresponds to the fact that if a string fastened at each end has its central portion and ends acted on respectively by oppositely-directed forces, its motion will be that due to the force at the central portion, which will excite the fundamental vibration if its oscillations are in unison with the latter. The intensity of the vibration will be proportional to B. Let E be the total E.M.F. in the uniform field of the secondary, $\phi$ the angle between its direction and the plane of the latter, and $\theta$ the angle which its projection on this plane makes with the radius drawn to the air space. Then we shall have, approximately,

$$F = E \cos\phi \sin\left(\frac{2\pi s}{S} - \theta\right),$$

and, therefore, $B = -E \cos\phi \sin\theta$.

B, therefore, is a function simply of the total E.M.F. due both to the electrostatic and electro-dynamic actions. It will vanish when $\phi = 90°$—that is to say, when the total E.M.F. is perpendicular to the plane of the circle, whatever be the position of the air space on the circle. B will also vanish when $\theta = 0$—that is to say, when the projection of the E.M.F. on the plane of the circle coincides with the radius through the air space. If the position of the air space on the circle is varied, the angle $\theta$ will vary, and, therefore, also the intensity of the vibration and the sparking distance. The sparking distance corresponding to the second term of the expansion for F can therefore be represented approximately by a formula of the form $\beta \sin\theta$.

Now, the oscillations giving rise to sparks of lengths $\alpha$ and $\beta \sin\theta$ respectively are in the same phase. The resulting oscillations will therefore be in the same phase, and their amplitudes must be added together. The sparking distance being approximately proportional to the maximum total amplitude may therefore also be obtained by adding the sparking distances due to the two oscillations respectively. The sparking distance will therefore be given as a function of the position of the air space on the secondary circuit by the expression $\alpha + \beta \sin\theta$. Since the direction of the oscillation in the air space does not come into consideration, we are concerned only with the absolute value of this expression and not with its sign. The determination of the absolute values of the quantities $\alpha$ and $\beta$ would involve elaborate theoretical investigations, and is, moreover, unnecessary for the explanation of the experimental results.

*Experiments with the Secondary Circuit in a Vertical Plane.*—When the circle forming the secondary circuit was placed with its plane vertical, anywhere in the neighbourhood of the primary, the following results were obtained.

The sparks disappeared for two positions of the air space, separated by 180deg., namely, those in which it lay in the horizontal plane through the primary; but in every other position sparks of greater or less length were observed.

From this it followed that the value of $\alpha$ must have been constantly zero, and that $\theta$ was zero when the air space was in the horizontal plane through the primary.

The electromagnetic lines of force must therefore have been perpendicular to this horizontal plane, and therefore consisted of circles with their centres on the primary; while the electrostatic lines of force must have been entirely in the horizontal plane, and therefore this system of lines of force consisted of curves lying in planes passing through the primary. Both of these results are in agreement with theory.

When the air space was at its greatest distance from the plane the sparking distance attained a maximum value of from 2 to 3 millimetres. The sparks were shown to be due to the fundamental vibration by slightly varying the secondary so as to throw it out of unison with the primary when the sparking distance was diminished, which would not have been the case if the sparks had been due to overtones. Moreover, the sparks disappeared when the secondary was cut at its points of intersection with the horizontal plane through the primary, though these would be nodal points for the first overtone.

When the air space was kept at its greatest possible distance from the horizontal plane through the primary, and turned about a vertical axis, the sparking distance attained two maxima at the points for which $\phi = 0$, and almost disappeared at the points for which $\phi = 90°$.

The lower half of Fig. 158 shows the different positions of minimum sparking. A A′ is the primary conductor, and the lines $m$ $n$ represent the projections of the secondary circuit on the horizontal plane. The arrows perpendicular to these give the direction of the resultant lines of force. As this did not anywhere vanish in passing from the sphere A to the sphere A′, it could not change its sign.

The diagram brings out the two following points:—

1. The distribution of the resultant E.M.F. in the vicinity of the rectilinear vibration is very similar to that of the electro-

static E.M.F. due to the action of its two extremities. It should be specially noted that near the centre of the primary the direction is that of the electrostatic E.M.F., showing that it is more powerful than the electro-dynamic, as required by theory.

2. The lines of force deviate more rapidly from the line A A′ than the electrostatic lines, though this is not so evident on the reduced scale of the diagram as in the author's original drawings on a much larger scale.

It is due to the components of the electrostatic E.M.F. parallel to A A′ being weakened by the E.M.F. of induction, while the perpendicular components remained unaffected.

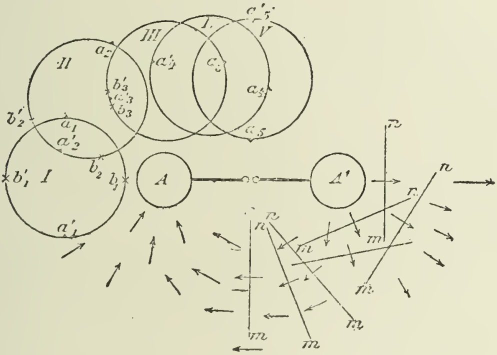

Fig. 158.

*Experiments with the Secondary Circuit in a Horizontal Plane.* The results obtained when the plane of the secondary was horizontal can best be explained by reference to the upper half of the diagram in Fig 158.

In the position I., with the centre of the circle in the line A A′ produced, the sparks disappeared when the air space occupied either of the positions $b_1$ or $b'_1$, while two equal maxima of the sparking distance were obtained at $a_1$ and $a'_1$, the length of the spark in these positions being 2·5 millimetres. Both these results are in accordance with theory.

In the position II. the circle is cut by the electro-magnetic lines of force, and therefore $a$ does not vanish. It will, however, be small, and we should expect that the expression $\alpha+\beta \sin\theta$ would have two unequal maxima, $\beta+\alpha$ and $\beta-\alpha$, both for $\theta=90°$, and having the line joining them perpendicular to the resultant E.M.F., and between these two maxima we should expect two points of no sparking near to the smaller maximum. This was confirmed by the observations.

The maximum sparking distances were 3·5 millimetres at $a_2$ and 2 millimetres at $a'_2$. Now, with the air space at $a_2$, the sphere A being positive, the resultant E.M.F. in the opposite portion of the circle will repel positive electricity from A, and therefore tend to make it flow round the circle clockwise. Between the two spheres the electrostatic E.M.F. acts from A towards A′, and the opposite E.M.F. of induction in the neighbourhood of the primary acts from A′ to A, parallel to the former, and, acting more strongly on the nearer than on the further portion of the secondary, tends to cause a current in same direction as that due to the former, namely, in a clockwise direction. Thus the resultant E.M.F. is the sum of the two as required by theory, and in the same way it is easily seen that when the air space is at $a'_2$ the resultant E.M.F. is equal to their difference.

As the position III. is gradually approached the maximum disappears, and the single maximum sparking distance $a_3$ was found to be four millimetres in length, having opposite to it a point of disappearance $a'_3$. In this case clearly $\alpha=\beta$, and the sparking distance is given by the expression $\alpha(1+\sin\theta)$. The line $a_3\,a'_3$ is again perpendicular to the resultant E.M.F.

As the circle approaches further towards the centre of A A′, $\alpha$ will become greater than $\beta$, and the expression $\alpha+\beta\sin\theta$ will not vanish for any value of $\theta$, but will have a maximum $\alpha+\beta$ and a minimum $\alpha-\beta$; and in the experiments it was found that the sparks never entirely disappeared, but varied between a maximum and a minimum, as indicated by theory.

In the position IV. a maximum sparking distance of 5·5 millimetres was observed at $a_4$, and a minimum of 1·5 millimetre at $a'_4$.

In the position V. there was a maximum sparking distance of 6 millimetres at $a_5$, and a minimum of 2·5 millimetres at $a'_5$. In these experiments the air space should be screened off from the primary in the latter positions as well as in the earlier ones, in which it is unavoidable, as otherwise the results would not be comparable.

In passing from the position III. to the position V. the line $a\,a'$ rapidly turned from its position of parallelism to the primary circuit into a position perpendicular to it. In the latter positions the sparking was essentially due to the inductive action, and therefore Hertz was justified in his previous experiments in assuming the effect in these positions to be due to induction.

Even in these positions, however, the sparking is not totally independent of electrostatic action, except when the air space is half-way between the maximum and minimum positions, and therefore $\beta \sin \theta = 0$.

*Other Positions of the Secondary Circuit.*—Hertz made numerous observations with the secondary circuit in other positions, but in no case were any phenomena observed which were not completely in accordance with theory. As an example of these consider the following experiment:—

The secondary was first placed in the horizontal plane in the position V. (Fig. 158), and the air space was in the position $a_5$ relatively to the primary. The circle was then turned about a horizontal axis through its centre and parallel to the primary, so as to raise the air space above the horizontal plane. During this rotation $\theta$ remained equal to 90deg., and the value of $\beta$ remained nearly constant, but $\alpha$ varied approximately in the same ratio as $\cos \Psi$, $\Psi$ being the angle between the plane of the circle and the horizontal, for $\alpha$ is proportional to the number of magnetic lines of force passing through the circle. Let $\alpha_0$ be the value of $\alpha$ in the initial position, then in the other postions its value would be $\alpha_0 \cos \Psi$, and therefore the sparking distance should be given by the expression $\alpha_0 \cos \Psi + \beta$, in which $\alpha_0$ was known to be greater than $\beta$. This was confirmed by observation, for it was found that as the air space increased its height above the horizontal plane the sparking distance diminished from 6 millimetres down to 2 millimetres, its value when the air space was at its greatest distance above the horizontal plane. During the rotation through the next quadrant the sparking distance diminished almost to zero, and then increased to the smaller maximum of 2·5 millimetres, which it attained when the circle had turned through 180deg., and was therefore again horizontal. Similar results were obtained in the opposite order as the circle was rotated from 108deg. to 360deg. When the circle was kept with the air space at its maximum height above the horizontal plane, and then raised or lowered bodily without rotation, the sparking distance was found to diminish in the former case and to increase in the latter—results completely in accordance with theory.

*Forces at Greater Distances.*—Experiments with the secondary at greater distances from the primary are of great importance, as the distribution of E.M.F. in the field of an open circuit is very different according to different theories of electro-dynamic action, and the results may, therefore, serve to eliminate some of them as untenable. In making these experiments, however, an unexpected difficulty was encountered, as it was found that at distances of from 1 to 1·5 metre from the primary, the maximum and minimum, except in certain positions, became indistinctly defined ; but when the distance was increased to upwards of two metres, though the sparks were then very small, the maximum and minimum were found to be very sharply marked when the sparks were observed in the dark. The positions of maximum and minimum were found to occur with the circle in planes at right angles to each other. At considerable distances the sparking diminished very slowly as the distance was increased. Hertz was not able to determine an upper limit to the distance at which sensible effects took place, but, in a room 14 metres by 12, sparks were distinctly observed when the primary was placed in one corner of the room, wherever the secondary was placed. When, however, the primary was slightly displaced no effects could be observed, even when the secondary was brought considerably nearer. The interposition of solid screens between the two circuits greatly diminished the effect.

Hertz mapped out the distribution of force throughout the room by means of chalk lines on the floor, putting stars at the points where the direction of the E.M.F. became indeterminate. A portion of the diagram obtained in this manner is shown on a reduced scale in Fig. 159, with respect to which the following points are noteworthy :—

1. At distances beyond three metres the E.M.F. is everywhere parallel to the primary oscillation. Within this region,

therefore, the electrostatic E.M.F. is negligible in comparison with the E.M.F. of induction. Now all the theories of the mutual action of current elements agree in giving an E.M.F. of induction inversely proportional to the distance; while the electrostatic E.M.F., being due to the differential action of the two extremities of the primary, is approximately inversely proportional to the cube of the distance. Some of these theories, however, are not in accordance with the experimental result that the effect diminishes much more rapidly in the direction of the primary oscillation than in a direction at right angles to it, induced sparks being observed at a distance exceeding 12 metres in the latter direction, while they disappeared at a distance of about four metres in the former direction.

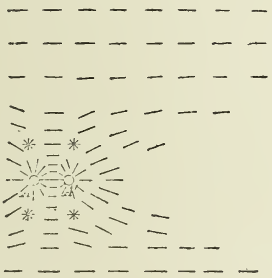

FIG. 159.

2. That, as already proved, for distances less than one metre the distribution of E.M.F. is practically that of the electrostatic E.M.F.

3. There are two straight lines at all points of which the direction of the E.M.F. is determinate, namely, the line in which the primary oscillation takes place, and the perpendicular to the primary through its middle point. Along the latter the E.M.F. does not vanish at any point; the sparking diminishes gradually as the distance is increased. This, again, is inconsistent with some of the theories of mutual action of current elements, according to which it should vanish at a certain definite distance. A very important result of the investigation is the demonstration of the existence of regions within which

the direction of the E.M.F. becomes indeterminate. These regions form two rings encircling the primary circuit. Since the E.M.F. within them acts very nearly equally in every direction, it must assume different directions in succession, for, of course, it cannot act in different directions simultaneously.

The observations, therefore, lead to the conclusion that within these regions the magnitude of the E.M.F. remains very nearly constant, while its direction varies through all the points of the compass at each oscillation. Hertz stated that he was unable to explain this result, as also the existence of overtones, by means of the simplified theory in which the higher terms of the expansion of F are neglected, and he considers that no theory of simple action at a distance is capable of explaining it. If, however, the electrostatic E.M.F. and the E.M.F. of induction are propagated through space with unequal velocities, it admits of very simple explanation. For within these annular regions the two E.M.F.s are at right angles and of the same order of magnitude; they will, therefore, in consequence of the distance traversed, differ in phase, and the direction of the resultant will turn through all the points of the compass at each oscillation.

This phenomenon appeared to him to be the first indication which had been observed of a finite rate of propagation through space of electrical actions, for, if there is a difference in the rate of propagation of the electrostatic and electro-dynamic E.M.F., one at least of them must be definite.

At the end of the Paper in which the preceding experiments are described, Hertz describes some observations which he made on the conditions at the primary sparking point which affect the production of sparks in the secondary circuit. He found that illuminating the primary spark diminished its power of exciting rapid oscillations, the sparks in the secondary being observed to cease when a piece of magnesium wire was burnt or an arc lamp lighted near the primary point. The observed effect on the primary sparks is that they are no longer accompanied by a sharp crackling sound as before. The effect of a second discharge is especially noteworthy, and it was found that the secondary sparks could be made to disappear by bringing an insulated conductor close to the opposed surfaces of the spheres forming the terminals at the primary air space, even when no visible sparking took place between the latter and the insulated conductor. The secondary sparking could also be stopped by placing a fine point close to the primary air space, or by touching one of the opposed surfaces of the terminals with a piece of sealing-wax, glass, or mica. Hertz states that further experiments led him to conclude that, even in these cases, the effect is due to light too feeble to be perceived by the eye, arising from a side discharge. He points out that these effects afford another example of the effects of light on electric discharges, which have been observed by E. Wiedemann, H. Hebert, and W. Hallwachs.

Hertz's next Paper in order of publication in Wiedemann's *Annalen* is "On Some Induction Phenomena Arising from Electrical Actions in Dielectrics" (Vol. XXXIV., p. 273), and contains an account of some researches which were undertaken with a view of obtaining direct experimental confirmation of the assumption involved in the most suggestive theory of electrical actions, viz., that of Faraday and Maxwell, that the well-known electrostatic phenomena observed in dielectrics are accompanied by corresponding electro-dynamic actions. The method of observation consisted in placing a secondary conductor adjusted to unison, as regards electrical oscillations, with the primary, as near as possible to the former, and in such a relative position that the sparks in the primary produced no sparking in the secondary. As the equilibrium could be disturbed and sparking induced in the secondary by the approach of conductors, it formed a kind of induction balance; but the point of special interest in connection with it was that a similar effect was produced when the conductors were replaced by insulators, provided the latter were of comparatively large size. The observed rapidity of the oscillations induced in the dielectrics showed that the quantities of electricity in motion under the influence of dielectric polarisation were of the same order of magnitude as in the case of metallic conductors.

The apparatus employed is shown diagrammatically in Fig. 160, and was supported on a light wooden framework, not shown in the illustration. The primary conductor consisted of two brass plates, A A′, with sides 40 centimetres in length, joined by a copper wire 70 centimetres long and half a centimetre in diameter, containing an air space of three-quarters of a centimetre, with terminals formed of polished brass spheres. When placed in connection with a powerful induction coil, oscillations are set up, the period of which, determined by the dimensions of the primary, can be determined to within a hundred-millionth of a second. The secondary conductor consisted of a circle, 35 centimetres in radius, of copper wire two millimetres in diameter, containing an air space, the length of which could be varied by means of a screw from a few hundredths of a millimetre up to several millimetres. The dimensions stated were such as to bring the two conductors into unison, and secondary sparks up to six or seven millimetres in length could be obtained.

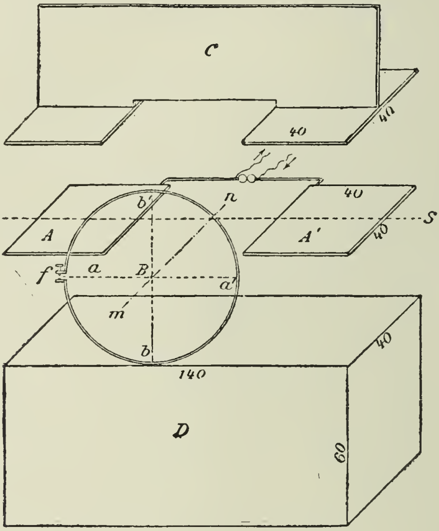

Fig. 160.

The circle was movable about an axis through its centre perpendicular to its plane, to enable the position of the air space to be varied. The axis was fixed in the position *m n* in the plane of A and A′, and half-way between them. The centre of the circle was at a distance of 12 centimetres from the nearest points of A and A′.

When $f$ was in either of the positions $a$ or $a'$ lying in the plane of A A′ no sparking occurred in the secondary, while maximum sparking took place at $b$ and $b'$ 90deg. from the former positions. The E.M.F. giving rise to the secondary sparks is, as in previous experiments, partly electrostatic and partly electro-magnetic, and the former being the greater will determine the sign of the resultant E.M.F. The oscillations must, for the reason previously explained, be considered as produced in the part of the secondary most remote from the air space. Assuming the E.M.F. and the amplitude of the resulting oscillation to be positive when $f$ is in the position $b'$, they will both be negative when $f$ is at $b$.

When the circle was slightly lowered in its own plane the sparking distance was increased at $b'$ and diminished at $b$, and the null points lay at a certain distance below $a$ and $a'$. The electrostatic E.M.F. is scarcely affected by such a displacement, but the integral of the E.M.F. of induction taken round the circle is no longer zero, and therefore gives rise to an oscillation which will be of positive sign whatever be the position of $f$, for the direction of the resultant E.M.F. of induction is opposite to that of the electrostatic E.M.F. in the upper half of the circle, and coincides with it in the lower half, where the electrostatic E.M.F. has been assumed to be positive. Since the new oscillation so produced is in the same phase as the previously existing one, their amplitudes must be added to give the resultant amplitude, which explains the phenomena.

*Effects of the Approach of Conductors.*—In making these observations it was found necessary to remove all conductors to a considerable distance from the apparatus, in order to obtain a complete disappearance of sparking at the points $a$ and $a'$. Even the neighbourhood of the observer was sufficient to set up sparking when the air space $f$ was in either of these positions, and the sparks had therefore to be observed from a distance. The conductors used for the experiments were of the form shown at C (Fig. 160), and consisted of thin metal foil. The objects kept in view in selecting the material and dimensions were to obtain a conductor which would give a moderately large effect and having an oscillation period less than that of the primary.

When the conductor C was brought near to A A′, it was found that the sparking distance decreased at $b$ and increased at $b'$, and the null points were displaced upwards—that is, in the direction of C.

From the results of experiments already described it is evident that the effect of displacing A A′ upwards would be the same, qualitatively, as that of a current in the same direction as that in A A′ directly above it. The effect produced by the approach of C was the reverse of this, and could be explained by an inductive action, supposing there were a current in C in the opposite direction to that in A A′, which is exactly what must occur; for the electrostatic E.M.F. would give rise to such a current, and since the oscillations in C are more rapid than those of the E.M.F., the current must be in the same phase as the inducing E.M.F. The truth of this explanation was confirmed by the following experiments. The horizontal plates of the conductor C being left in the same position as before, the vertical plate was removed, and successively replaced by wires of increasing length and fineness, in order to lengthen the oscillation period of C. The effect of this was to displace the null points more and more in an upward direction, while at the same time they became less sharply defined, a minimum sparking taking the place of the previous absolute disappearance. The sparking distance at the highest point had previously been much less than at the lowest point, but after the disappearance of the null points it began to increase. At a certain stage the sparking distance at the two positions became equal, and then no definite minimum points could be found, but sparking took place freely at all positions of $f$. Beyond this stage the sparking distance at the lowest point diminished, and very soon two minimum points made their appearance close to it, not clearly defined at first, but gradually becoming more distinct, and at the same time approaching the points $a\ a'$, with which they ultimately coincided, when the minimum points again became absolute null points. These results are in agreement with the conclusion drawn from the former observations, for as the oscillation period of C approaches that of A A′ the intensity of the current in the former increases, but a difference of phase arises between it and the existing E.M.F. When the two are in unison the current in C attains its maximum, and, as in other cases of resonance, the difference of phase gives rise to a slightly damped oscillation, having a period of about a quarter that of the original one, which makes any interference between the oscillations excited in the circle B by A A′ and C respectively impossible. These conditions clearly correspond to the stage at which the sparking distances at *b* and *b*′ were equal. When the oscillation period of C becomes decidedly greater than that of A A′, the amplitude of the oscillation in the former will again diminish, so that the difference in phase between it and the exciting E.M.F. will approach half of the original period. The current in C will therefore always be in the same direction as that in A A′, so that interference between the two oscillations excited in B will again become possible, and the effect of C will then be opposite to its original effect. When the conductor C was made to approach A A′ the sparks in B became much smaller, which is explained by the fact that its effect will be to increase the oscillation period of A A′, and therefore to throw it out of unison with B.

*Effects of the Approach of Dielectrics.*—A very rough estimate shows that when a dielectric of large mass is brought near to the apparatus the quantities of electricity set in motion by dielectric polarisation are at least as large as in metallic wires or thin rods. If, therefore, the action of the apparatus were unaffected by the approach of such masses, it would show that, in contradiction to the theories of Faraday and Maxwell, no electro-dynamic actions are called into play by means of dielectric polarisation, or as Maxwell calls it, electric displacement. The experiments, however, showed an effect similar to that which would be produced if the dielectric were replaced by a conductor with a very small oscillation period. In the first experiment made, the mass of dielectric consisted of a pile of books, 1·5 metre long, 0·5 metre broad, and 1 metre high, placed under the plates A A′. Its effect was to displace the null points through about 10deg. towards the pile. A block of asphalte (D, in Fig. 160), weighing 800 kilogrammes, and measuring 1·4 metre in length, 0·4 metre in breadth, and 0·6 metre in height, was then used in place of the books, the plates being allowed to rest upon it.

The following results were then obtained:—

1. The spark at the highest point of the circle was now decidedly stronger than that at the lowest point, which was nearer to the asphalte.

2. The null points were displaced through about 23deg. downwards—that is, in the direction of the block—and at the same time were transformed into mere points of minimum sparking, a complete disappearance being no longer obtainable.

3. When the plates A A′ rested on the asphalte block the oscillation period of the primary was increased, as shown by the fact that the period of B had to be slightly increased in order to obtain the maximum sparking distance.

4. When the apparatus was moved gradually away from the block its action steadily diminished without changing its character.

5. The action of the block could be compensated by bringing the conductor C over the plates A A′ while they rested on the block, the null points being brought back to *a* and *a*′ when C was at a height of 11 centimetres above the plates. When the upper surface of the asphalte was 5 centimetres below the plates, compensation was obtained when C was placed at a height of 17 centimetres above them, showing that the action of the dielectric was of the order of magnitude which had been anticipated.

The asphalte contained about 5 per cent. of aluminium and iron compounds, 40 per cent. of calcium compounds, and 17 per cent. of quartz sand. In order to make sure that the observed effects were not due to the conductivity of some of these substances a number of further experiments were made.

In the first place, the asphalte was replaced by a mass of the same dimensions of the so-called artificial pitch prepared from coal, and effects of a similar kind were observed, but slightly weaker, the greatest displacement of the null points amounting to 19deg. Unfortunately this pitch contains free carbon, the amount of which it is difficult to determine, and this would have some conductivity.

The experiments were then repeated with a conductor, C, of half the linear dimensions of the former one, and smaller blocks of various substances, on account of the great cost of obtaining large blocks of pure materials. The substances used were asphalte, coal-pitch, paper, wood, sandstone, sulphur, paraffin, and also a fluid dielectric, namely, petroleum. With the smaller apparatus it was not possible to obtain quantitative results of the same accuracy as before, but the effects were of an exactly similar character, and left little room for doubt of the reality of the action of the dielectric.

The results might possibly be supposed to be due to a change in the distribution of the electrostatic E.M.F. in the neighbourhood of the dielectric, but, in the first place, Hertz stated that he was unable to explain the details of the observations on this hypothesis, and in the second place it is disproved by the following experiment:—

The smaller apparatus was placed with the line *r s* on the upper near corner of one of the large blocks, in which position the dielectric was bounded by the plane of the plates A A′ and the perpendicular plane through *r s*, both of which are equipotential surfaces, so that if the action were electrostatic no effect should be produced by the dielectric. It was found, however, to produce the same effect as in other positions. It might also be supposed that the effects were due to a slight conductivity, but this could hardly be the case with such good insulators as sulphur and paraffin. Suppose, moreover, that the conductivity of the dielectric is sufficient to discharge the plate A in the ten-thousanth of a second, but not much more rapidly; then, during one oscillation, the plates would loose only the ten-thousandth part of their charge, and the conduction current in the substance experimented on would not exceed the ten-thousandth part of the primary current in A A′, so that the effect would be quite insensible.

It is thus shown in the experiments described above that when variable electrical forces act in the interior of dielectrics of specific inductive capacity not equal to unity the corresponding electric displacements produce electro-dynamic effects. In a Paper, "On the Velocity of Propagation of Electro-Dynamic Actions," in Wiedemann's *Annalen*, Vol. XXXIV., p. 551, Hertz showed that similar actions take place in the air, which proves, as was previously pointed out, that electro-dynamic action must be propagated with a finite velocity.

The method of investigation was to excite electrical oscillations in a rectilinear conductor in the same manner as in former experiments, and then to produce effects in a secondary conductor by exciting electrical oscillations in it by means of those in the rectilinear conductor, and at the same time by the primary conductor acting through the intervening space. This distance was gradually increased, when it was found that the phase of the vibrations at a distance from the primary lagged behind those in its immediate neighbourhood, showing that the action is propagated with a finite velocity which was found to be greater than the velocity of propagation of electrical waves in wires in the ratio of about 45 to 28, so that the former is of the same order as the velocity of light. Hertz was unable to obtain any evidence with respect to the velocity of propagation of electrostatic actions.

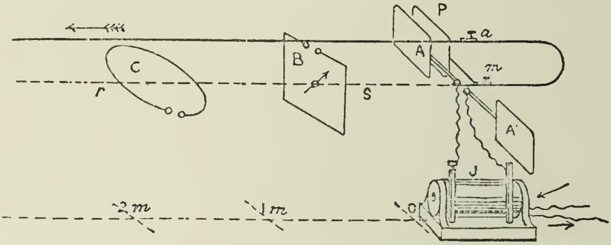

Fig. 161.

The primary conductor A A′ (Fig. 161) consisted of a pair of square brass plates with sides 40 centimetres in length, connected by a copper wire 60 centimetres in length, at the middle point of which was an air space, across which sparks were made to pass by means of powerful discharges from the induction coil J. The conductor was fixed at a height of 1·5 metre above the base-plate of the coil, with its plates vertical, and the connecting wire horizontal. A straight line, *r s*, drawn horizontally through the air space of the primary, and perpendicular to the direction of the primary oscillation, will be called "the base-line;" and a point in this, situated at a distance of 45 centimetres from the air space, will be referred to as "the null point."

The experiments were made in a large lecture-room, with nothing near the base-line for a distance of 12 metres from the primary conductor. The room was darkened during the experiments.

The secondary conductor consisted either of a circular wire, C, of 35 centimetres radius, or of a square of wire, B, with sides 60 centimetres long. The primary and secondary air spaces were both capable of adjustment by means of micrometer screws. Both the secondary conductors were in unison with the primary, the (half) vibration period of each being one hundred-millionth of a second, as calculated from the capacity and coefficient of self-induction. It is doubtful whether the ordinary theory of electrical oscillations would lead to accurate results under the conditions of these experiments, but as it gives correct numerical results in the case of Leyden jar discharges, it may be expected to be correct as far as the order of the results is concerned. When the centre of the secondary lies in the base-line, and its plane coincides with the vertical plane through the base-line, no sparks are observed in the secondary, the E.M.F. being everywhere perpendicular to the direction of the secondary. This will be referred to as "the first principal position" of the secondary. When the plane of the secondary is vertical and perpendicular to the base-line, the centre still lying in the base-line, the secondary will be said to be in its "second principal position." Sparking then occurs in the secondary when its air space is either above or below the horizontal plane through the base-line, but not when it is in this plane. As the distance from the primary was increased, the sparking distance was observed to decrease, rapidly at first, but ultimately very slowly. Sparks were observed throughout the whole distance of 12 metres available for the experiments. The sparking in this position is due essentially to the E.M.F. produced in the portion of the secondary remote from the air space. The total E.M.F. is partly electrostatic and partly electro-dynamic, and the experiments show beyond the possibility of doubt that the former is greater, and therefore determines the direction of the total E.M.F. close to the primary, while at greater distances it is the electro-dynamic E.M.F. which is the greater.

The plane of the secondary was then turned into the horizontal, its centre still lying in the base-line. This may be called "the third principal position." When the centre of the circular secondary conductor was kept fixed at the null point, and the air space was made to travel round the circle, vigorous sparking was observed in all positions. The sparking distance attained its maximum length of about six millimetres when its air space was nearest to that of the primary, and its minimum length of about three millimetres when the distance between the two air spaces was greatest. If the secondary had been influenced by the electrostatic force, sparking would only be expected when the air space was close to the base-line, and a cessation of sparks in the intermediate positions. The direction of the oscillation would, moreover, be determined by the direction of the E.M.F. in the portion of the secondary furthest from the air space. There is, however, superposed upon the electrostatically excited oscillation a second oscillation, due to the E.M.F. of induction, which produces a considerable effect, since its integral round the circle (considered as a closed circuit) does not vanish; and the direction of this integral E.M.F. is independent of the position of the air space, opposing the electrostatic E.M.F. in the portion of the secondary next to A A′, and assisting it in the portion furthest from A A′, as explained previously.

The electrostatic and electro-dynamic E.M.F.s, therefore, act in the same direction when the air space is turned towards the primary conductors, and in opposite directions when the air space is turned away from the primary. In the latter position it is the E.M.F. of induction which is the more powerful, as is shown by the fact that there is no disappearance of sparking in any position of the air space, for when this is 90deg. to the right or left of the base-line it coincides with a node with respect to the electrostatic E.M.F. In these positions the inductive action in the neighbourhood of the primary can be observed independently of the electrostatic action.

*Waves in Rectilinear Wires.*—In order to produce in a wire by means of the primary oscillations a series of advancing waves of the character required for these experiments, the following arrangements were made:—Behind the plate A was placed a plate, P, of equal size. A copper wire one millimetre in diameter connected P to the point M of the base-line. From M the wire was continued in a curve about a metre in length to the point N, situated about 30 centimetres above the air space, and was then further continued in a straight line parallel to the base-line for such a distance as to obviate all danger of disturbance from reflected waves. In the present series of experiments the wire passed through a window, and after being carried to a distance of about 60 metres, was put to earth, and a special series of experiments showed that this length was sufficient. When a wire, bent so as to form a nearly closed circuit with a small air space, was brought near to this straight wire, a series of fine sparks was seen to accompany the discharges of the induction coil. Their intensity could be varied by varying the distance between the plates P and A. The waves in the rectilinear wire were of the same period as that of the primary oscillations, as was proved by their being shown to be in unison with each of the two secondary conductors previously described. The existence of stationary waves showed that the waves in the rectilinear wire were of a steady character in space as well as in time. The nodal points were determined in the following manner :—The further end of the wire was left free, and the secondary conductor was brought near to it in such a position that the wire lay in its plane, and had the air space turned towards it. As the secondary was moved along the wire, points of no sparking were observed to recur periodically. The distance from the point *n* to the first of these was measured, and the length of the wire made equal to a multiple of this distance. The experiments were then repeated, and it was found that the nodal points occurred at approximately equal intervals along the wire.

The nodes could also be distinguished from the loops in other ways. The secondary conductor was brought near to the wire, with its plane perpendicular to it, and with its air space neither directed completely towards the wire nor completely away from it, but in an intermediate position, so as to produce E.M.F.s perpendicular to the wire. Sparks were then observed at the nodes, while they disappeared at the loops. When sparks were taken from the rectilinear wire by means of an insulated conductor, they were found to be stronger at the nodes than at the loops; the difference, however, was small, and was, indeed, scarcely distinguishable unless the position of the nodes and loops was previously known. The reason that this and other similar methods do not give a well-defined result lies in the fact that irregular oscillations are superposed upon the waves considered; the regular waves, however, can be picked out by means of the secondary, just as definite notes are picked out by means of a Helmholtz resonator. If the wire is severed at a node, no effect is produced upon the waves in the portion of wire next to the origin; but if the severed portion of wire is left in its place the waves continue to be propagated through it, though with somewhat diminished strength.

The possibility of measuring the wave-lengths leads to various applications. If the copper wire hitherto used is replaced by one of different diameter, or by a wire of some other metal, the nodal points retain their position unchanged. It follows from this that the velocity of propagation in a wire has a definite value independent of its dimensions and material. Hertz states that even iron wires offer no exception to this, showing that the magnetic susceptibility of iron does not play any part in the case of such rapid motions. This conclusion is not, however, confirmed by the researches of Prof. J. Trowbridge, and investigations, referred to on page 431, show that the magnetisability of the iron does exert an influence sensible though small. It would be interesting to investigate the behaviour of electrolytes in this respect. In their case we should expect a smaller velocity of propagation, because the electrical motions are accompanied by motions of the molecules carrying the electric charges. It was found that no propagation of the waves took place through a tube 10 millimetres in diameter, filled with a solution of sulphate of copper; but this may have been due to the resistance being too high. By the measurement of wave-lengths the relative vibration periods of different primary conductors can be determined, and it therefore becomes possible to compare in this manner the vibration periods of plates, spheres, ellipsoids, &c.

In the experiments made by Hertz, nodes were very distinctly produced when the wire was severed at a distance of either 8 metres or 5·5 metres from the null point of the base-line. In the first case the nodes occurred at distances from the null point of – 0·2 metre, 2·3 metres, 5·1 metres, and 8 metres, and in the latter case at distances of – 0·1 metre, 2·8 metres, and 5·5 metres. It appears, therefore, that the (half) wave-length in a free wire cannot differ much from 2·8 metres. The fact that the wave-lengths nearest to P were somewhat smaller was to be expected from the influence of the plates and of the curvature of the wire. This wave-length, with a period of one hundred-millionth of a second, gives 280,000 kilometres per second for the velocity of propagation of electrical waves in wires. Fizeau and Gounelle (Poggendorff's *Annalen*, Vol. LXXX., p. 158, 1850) obtained for the velocity in iron wires 100,000 kilometres per second, and 180,000 in copper wires. W. Siemens (Poggendorff's *Annalen*, Vol. CLVII., p. 309, 1876), by the aid of Leyden jar discharges, obtained a velocity of from 200,000 to 260,000 kilometres per second in iron wires. Hertz's result is very nearly the same as the velocity of light. Space will not allow us to fully discuss the causes which led to certain discrepancies in Hertz's earlier results. Suffice it to say that he subsequently found that the velocity of propagation of an electromagnetic disturbance along a wire was the same as in free space, viz., the velocity of light. The apparent difference between the velocity of long and short waves was afterwards explained by Hertz himself, and the causes of this were made clear by the experiments conducted in the large hall of the Rhone waterworks by MM. Sarasin and de la Rive. From these experiments it became clear that the interference due to surrounding objects was the cause of the apparent difference between the velocities of long and short waves, but that in a sufficiently large space this difference disappeared, and the velocity of both long and short electromagnetic waves was the same. The reader may consult with advantage on this point the notes and text of the full translation of Hertz's electrical Papers made by Mr. D. E. Jones.*

*Interference of the Direct Actions with those transmitted through the Wire.*—If the square circuit B is placed at the null point in the second principal position, with the air space at its highest point, it will be unaffected by the waves in the wire, but the direct action when in this position was found to produce sparks 2 millimetres in length. B was then turned about a vertical axis into the first principal position, in which there would be no direct action of the primary oscillation, but the waves in the wire gave rise to sparks, and by bringing P near enough to A a sparking distance of 2 millimetres could be obtained. In the intermediate positions sparks were produced in both these ways, and it would therefore be possible to get a difference of phase, such that one should either increase or diminish the effect of the other. Phenomena of this nature were, indeed, observed. When the plane of B was in such a position that the normal drawn towards A A′ was directed away from that side of the primary conductor on which P was placed, there was more sparking than even in the principal position; but if the normal were directed towards P the sparks disappeared, and only reappeared when the air space was made smaller. When the air space was at the lowest point of B, the other conditions remaining the same, the sparks disappeared when the normal was turned away from P. Further variations of the experiment gave results in accordance with these.

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* "Electric Waves." Authorised English translation of Hertz's Papers, by D. E. Jones.

It is easily seen that these phenomena were exactly what were to be expected. To fix the ideas, suppose the air space to be at the highest point, and the normal directed towards P, as in Fig. 161. Consider what happens at the moment that the plate A has its greatest positive charge. The electrostatic, and therefore the total, E.M.F. is directed from A towards A′. The oscillation to which this gives rise in B is determined by the direction of the E.M.F. in the lower portion of B. Therefore positive electricity will flow towards A′ in the lower portion, and away from A′ in the upper portion.

Consider next the action of the waves. As long as A is positively charged, positive electricity will flow from the plate P. This current is at the moment considered at its maximum value at the middle point of the first half wave-length. A quarter of a wave-length further from the origin—that is to say, in the neighbourhood of the null point—it first changes its direction. The E.M.F. of induction will here, therefore, impel positive electricity towards the origin. A current will therefore flow round B towards A′ in the upper portion and away from A′ in the lower portion. The electrostatic and electro-dynamic E.M.F.s are therefore in opposite phases and oppose each other's action. If the secondary circuit is rotated through 90deg., through the first principal position, the direct action changes its sign, but not so the action of the waves, so that they now tend to strengthen each other. The same reasoning holds when the air space is at the lowest point of B.

Greater lengths of wire were then included between *m* and *n*, and it was found that the interference became gradually less marked, until within a length of 2·5 metres it disappeared entirely, the sparks being of equal length whether the normal were directed towards or away from P. When the length of wire between *m* and *n* was further increased, the distinction between the different quadrants reappeared, and with a length of 4 metres the disappearance of the sparks was fairly sharp. The disappearance, however, then took place (with the air space at the highest point) when the normal was directed away from P, the opposite direction to that in which the disappearance previously took place. With a still further increase in the length of the wire the interference reappeared, and returned to its original direction with a length of 6 metres. These phenomena are clearly to be explained by the retardation of the waves in the wire, and show that here again the direction of motion in the advancing waves changes its signs at intervals of about 2·8 metres.

To obtain interference phenomena with the secondary circuit C in the third principal position, the rectilinear wire must be removed from its original position and placed in the horizontal plane through C either on the side of the plate A or of the plate A′. Practically it is sufficient to stretch the wire loosely, and to fix it by means of an insulated clamp on each side of C alternately. It was found that when the wire was on the same side as the plate P the waves in it diminished the previous sparking, and when on the opposite side the sparking was increased, both results being unaffected by the position of the air space in the secondary circuit. Now it has been already pointed out that at the moment when the plate A has its maximum positive charge, and at which, therefore, the primary current begins to flow from A, the current at the first node of the rectilinear wire begins to flow away from the origin. The two currents, therefore, flow round C in the same direction when C lies between the rectilinear wire and A, and in opposite directions when the wire and A are on the same side of C. The fact that the position of the air space is indifferent confirms the conclusion formerly arrived at that the direction of oscillation is that due to the electro-dynamic E.M.F. These interferences are also changed in direction when the wire $m\,n$, 1 metre in length, is replaced by a wire 4 metres in length.

Hertz also succeeded in obtaining interference phenomena when the centre of the secondary circuit was not in the base-line, but these results were of no special importance, except that they confirmed the previous conclusions.

*Interference Phenomena at Various Distances.*—Interference may be produced with the secondary at greater distances than that of the null point; but care must then be taken that the action of the waves in the wire is of about the same magnitude as the direct action of the primary circuit through the air. This can be effected by increasing the distance between P and A.

Now, if the velocity of propagation of the electro-dynamic disturbances through the air is infinite, the interference will change its sign at every half-wave length in the wire—that is to say, at intervals of about 2·8 metres. If the velocities of propagation through the air and through the wire are equal, the interference will be in the same direction at all distances. Finally, if the velocity of propagation through the air is finite, but different from the velocity in the wire, the interference will change in sign at intervals greater than 2·8 metres.

The interferences first investigated were those which occurred when the secondary circuit was rotated from the first into the second principal position, the air space being at the highest point. The distance of the secondary from the null point was increased by half-metre stages from 0 up to 8 metres, and at each of these positions an observation was made of the effects of directing the normal towards and away from P respectively. The points at which no difference in the sparking was observed in the two positions of the normal are marked 0 in Table I. Those in which the sparking was least, showing the existence of interference, when the normal was directed towards P, are marked +, and those in which the sparking was least when the normal was directed away from P are marked −. The experiments were repeated with different lengths of wire *m n*, varying by steps of half a metre from 1 metre up to 6 metres. The first horizontal line in the table gives the distance, in metres, of the centre of the secondary circuit from the null point, while the first vertical line gives the lengths of the wire *m n*, also in metres.

Table I.

| — | 0 | | 1 | | 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100 | + | + | 0 | − | − | − | − | − | 0 | 0 | 0 | 0 | 0 | + | + | + | + |
| 150 | + | 0 | − | − | − | − | 0 | 0 | 0 | 0 | 0 | + | + | + | + | + | 0 |
| 200 | 0 | − | − | − | − | − | 0 | + | + | + | + | + | 0 | 0 | 0 | 0 | 0 |
| 250 | 0 | − | − | − | − | 0 | 0 | + | + | + | + | 0 | 0 | 0 | 0 | 0 | 0 |
| 300 | − | − | − | − | 0 | + | + | + | + | + | 0 | 0 | 0 | 0 | − | − | − |
| 350 | − | − | 0 | + | + | + | + | + | + | 0 | 0 | 0 | − | − | − | − | − |
| 400 | − | − | 0 | + | + | + | + | 0 | 0 | 0 | 0 | − | − | − | − | − | − |
| 450 | − | 0 | + | + | + | + | + | 0 | 0 | 0 | − | − | − | − | − | − | 0 |
| 500 | − | 0 | + | + | + | + | 0 | − | − | − | − | − | 0 | 0 | 0 | 0 | + |
| 550 | 0 | + | + | + | + | 0 | 0 | − | − | − | − | − | 0 | 0 | 0 | 0 | + |
| 600 | + | + | + | + | 0 | 0 | − | − | − | − | − | 0 | 0 | + | + | + | + |

An inspection of the table shows, in the first place, that the changes of sign take place at longer intervals than 2·8 metres; and, in the second place, that the change of phase is more rapid in the neighbourhood of the origin than at a distance from it. As a variation in the velocity of propagation is very unlikely, this is probably due to the fact indicated by theory that the electrostatic E.M.F., which is more powerful than the electro-dynamic E.M.F. in the neighbourhood of the primary oscillation, has a greater velocity of propagation than the latter.

In order to obtain a definite proof of the existence of similar phenomena at greater distances, Hertz continued the observations, in the case of three of the lengths *m n*, up to a distance of 12 metres, and the result is given in Table II.

If we make the assumption that at the greater distance it is only the E.M.F. of induction which produces any effect, the experiments would show that the interference of the waves excited by the E.M.F. of induction with the original waves in the wire changes its sign only at intervals of about 7 metres.

Table II.

| — | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100 | + | 0 | – | – | 0 | 0 | 0 | + | + | + | + | + | 0 |
| 250 | 0 | – | – | 0 | + | + | 0 | 0 | 0 | 0 | – | – | – |
| 400 | – | 0 | + | + | 0 | 0 | – | – | – | – | 0 | 0 | 0 |

Table III.

| — | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 100 | – | – | – | – | 0 |
| 150 | – | – | 0 | 0 | 0 |
| 200 | 0 | 0 | 0 | + | + |
| 250 | 0 | + | + | + | + |
| 300 | + | + | + | + | + |
| 350 | + | + | + | + | 0 |
| 400 | + | + | + | + | 0 |
| 450 | + | + | + | 0 | 0 |
| 500 | + | + | 0 | 0 | 0 |
| 550 | + | 0 | 0 | 0 | – |
| 600 | 0 | – | – | – | – |

In order to investigate the E.M.F. of induction close to the primary oscillation, where the results are of special importance, Hertz made use of the interferences which were obtained when the secondary circuit was in the third principal position, and the air space was rotated through 90deg. from the base-line. The direction of the interference at the null point, which has already been considered, was taken as negative, the interference being considered positive when it was produced by the passage of waves on the side of C remote from P, which make the signs correspond with those of the previous experiments. It must be borne in mind that the direction of the resultant E.M.F. at the null point is opposed to that of the E.M.F. of induction, and therefore the first table would have begun with a negative sign if the electrostatic E.M.F. could have been eliminated. The present experiments showed that up to a distance of 3 metres interference continued to occur, and always of the same sign as at the null point. It was unfortunately impossible to extend these observations to a greater distance than 4 metres on account of the feebleness of the sparks, but the results obtained were sufficient to give distinct evidence of a finite velocity of propagation of the E.M.F. of induction. These observations, like the former ones, were repeated with various lengths of the wire *mn* in order to exhibit the variation in phase, and the results obtained are given in Table III., which shows that, as the distance increases, the phase of the interference changes in such a manner that a reversal of sign takes place at intervals of from 7 to 8 metres. This result is further confirmed by comparing the results of Table III. with the results for greater distances given in Table II., for in the former series the effect of the electrostatic E.M.F. is eliminated, owing to the special position of the secondary circuit, while in the latter it becomes insensible at the greater distances owing to its rapid decrease with increasing distance. We should therefore expect the results given in the first table for distances beyond 4 metres to follow without a break the results given in Table III. for distances up to 4 metres. This was found to be the case, as is evident from inspection of Tables II. and III.

To show this more clearly, the signs of the interference of the waves, due to the electro-dynamic E.M.F., with the waves in the wire are collected together in Table IV., the first four columns of which are taken from Table III., and the remaining columns from Table II.

Table IV.

| — | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 100 | – | – | – | – | 0 | 0 | 0 | + | + | + | + | + | 0 |
| 250 | 0 | + | + | + | + | + | 0 | 0 | 0 | 0 | – | – | – |
| 400 | + | + | + | + | 0 | 0 | – | – | – | – | 0 | 0 | 0 |

From the results given in this table Hertz drew the following conclusions:—

1. The interference does not change its sign at intervals of 2·8 metres. The electro-dynamic actions are therefore not propagated with an infinite velocity.

2. The interference is not in the same phase at all points. Therefore the electro-dynamic actions are not propagated through air with the same velocity as electric waves in wires.

3. A gradual retardation of the waves in the wire has the effect of displacing a given phase of the interference towards the origin of the waves. The velocity of propagation through the air is therefore greater than through a wire.

4. The sign of the interference is reversed at intervals of 7·5 metres, and therefore in traversing this distance an electrodynamic wave gains one length of the waves in the wire.

Thus, while the former travels 7·5 metres, the latter travels $7{\cdot}5 - 2{\cdot}8 = 4{\cdot}7$ metres, and therefore the ratio of the velocities is 75 : 47, which gives for the half wave-length of the electrodynamic action $2{\cdot}8 \times 75/47 = 4{\cdot}5$ metres. Since this distance is traversed in 1·4 hundred-millionth of a second, the absolute velocity of propagation through the air must be 320,000 kilometres per second. This result can only be considered reliable as far as its order is concerned; but its true value can hardly exceed half as much again, or be less than two-thirds of this amount. In order to obtain a more accurate determination of the true value it will be necessary to determine the velocity of electric waves in wires with greater exactness.

It does not necessarily follow from the fact that in the immediate neighbourhood of the primary oscillation the interference changes its sign after an interval of 2·8 metres, that the velocity of propagation of the electrostatic action is infinite, for such a conclusion would rest upon a single change of sign, which might, moreover, be explained independently of any change of phase, by a change in the sign of the amplitude of the resultant force at a certain distance from the primary oscillation. Quite independently, however, of any knowledge of the velocity of propagation of electrostatic actions, there exist definite proofs that the rates of propagation of electrostatic and electro-dynamic E.M.F.s are unequal.

In the first place, the total force does not vanish at any point on the base line. Now, near the primary the electrostatic E.M.F. is the greater, while the electro-dynamic E.M.F. is the greater at greater distances. There must, therefore, be some point at which they are equal, and since they do not balance they must take different times to reach this point.

In the second place, the existence of points at which the direction of the resultant E.M.F. becomes indeterminate does not seem capable of explanation, except on the supposition that the electrostatic and electro-dynamic components perpendicular to each other are in appreciably different phases, and, therefore, do not compound into a rectilinear oscillation in a fixed direction. The fact that the two components of the resultant are propagated with different velocities is of considerable importance, in that it gives an independent proof that one of them at any rate must have a finite velocity of propagation.

Further researches of Hertz on electrical oscillations, of which accounts have been published, are to be found described in a Paper, "On Electro-Dynamic Waves in Air, and their Reflection," in Wiedemann's *Annalen*, Vol. XXXIV., p. 609. The author had been endeavouring to find a more striking and direct proof of the finite velocity of propagation of electrodynamic waves than those which he had hitherto given; for, though these are quite sufficient to establish the fact, they can only be properly appreciated by one who has obtained a grasp of the results of the entire series of researches.

In many of the experiments which have been described, Hertz had noticed the appearance of sparks at points in the secondary conductor where it was clear from geometrical considerations that they could not be due to direct action, and it was observed that this occurred chiefly in the neighbourhood of solid obstacles. It was found, moreover, that in most positions of a secondary conductor the feeble sparks produced at a great distance from the primary became considerably stronger in the vicinity of a solid wall, but disappeared with considerable suddenness quite close to the wall. The most obvious explanation of these experiments was that the waves of inductive action were reflected from the wall and interfered with the direct waves, especially as it was found that the phenomena became more distinct when the circumstances were such as to favour reflection to the greatest possible extent. Hertz therefore determined upon a thorough investigation of the phenomena.

The experiments were made in the Physical Lecture Theatre, which is 15 metres in length, 14 metres in width, and 6 metres in height. Two rows of iron columns, running parallel to the sides of the room, would collectively act almost like a solid wall towards electro-dynamic action, so that the available width of the room was only 8·5 metres. All pendant gas-fittings were removed, and the room left empty, with the exception of wooden tables and forms, which would not exert any appreciable disturbing effect. The end wall, from which the waves were to be reflected, was of solid sandstone, with two doors in it, and the numerous gas pipes attached to it gave it, to a certain extent, the character of a conducting surface, and this was increased by fastening to it a sheet of zinc four metres high and two metres broad, connected by wires to the gas pipes and a neighbouring water pipe. Special care was taken to provide an escape for the electricity at the upper and lower extremities of the zinc plate, where a certain accumulation of electricity was to be expected.

The primary conductor was the same that was employed in the experiments described on page 456, Fig. 161, and was placed at a distance of 13 metres from the zinc plate, and, therefore, two metres from the wall at the other end of the room. The conducting wire was placed vertically, so that the E.M.F.s to be considered increased and diminished in a vertical direction. The centre of the primary conductor was 2·5 metres above the floor of the room, which left a clear space for the observations above the tables and benches. The point of intersection of the reflecting surface with the perpendicular from the centre of the primary conductor will be called "the point of incidence," and the experiments were limited to the neighbourhood of this point, as the investigation of waves striking the wall at a considerable angle would be complicated by the differences in their polarisation. The plane of vibration was therefore parallel to the reflecting surface, and the plane of the waves was perpendicular to it, and passed through the point of incidence.

The secondary conductor consisted of the circle of 35 centimetres radius, which has been already described. It was movable about an axis through its centre perpendicular to its plane, and the axis itself was movable in a horizontal plane about a vertical axis. In most of the experiments the secondary conductor was held in the hand by its insulating wooden support, as this was the most convenient way of bringing it into the various positions required. The results of these experiments, however, had to be checked by observations made with the observer at a greater distance from the secondary, as the neighbourhood of his body exerted a slight influence upon the phenomena. The sparks were distinct enough to be observed at a distance of several metres when the room was darkened, but when the room remained light they were practically invisible even when the observer was quite close to the secondary.

When the centre of the secondary was placed in the line of incidence, and with its plane in the plane of vibration, and the air space was turned first towards the reflecting wall and then away from it, a considerable difference was generally observed in the strength of the sparks in the two positions. At a distance of about 0·8 metre from the wall the sparks were much stronger when the air space was directed towards the wall, and its length could be adjusted so that, while there was a steady stream of sparks when in this position, they disappeared entirely when the air space was directed away from the wall. These phenomena were reversed at a distance of 3 metres, and recurred, as in the first case, at a distance of 5·5 metres. At a distance of 8 metres the sparks were stronger when the air space was turned away from the wall, as at the distance of 3 metres, but the difference was not so well marked. When the distance was increased beyond 8 metres no further reversal took place, owing to the increase in the direct effect of the primary oscillation and the complicated distribution of the E.M.F. in its neighbourhood.

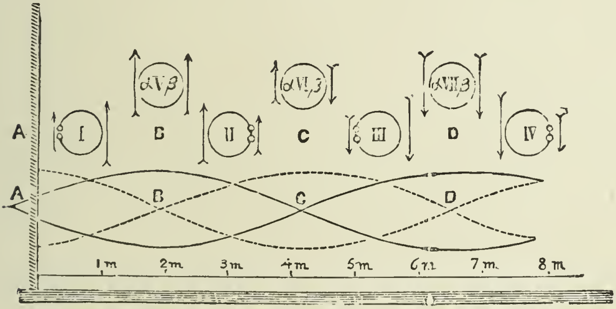

Fig. 162.

The positions I., II., III. and IV. (Fig. 162) of the secondary circle are those in which the sparks were strongest, the distance from the wall being shown by the horizontal scale at the foot. When the secondary circle was in the positions V., VI., and VII., the sparks were equally strong in both positions of the air space, and quite close to the wall the difference between the sparking in the two positions again diminished. Therefore the points A, B, C, D in the diagram may in a certain sense be regarded as nodes. The distance between two of these points must not, however, be taken as the half wave-length, for if all the electrical motions changed their directions on passing through one of these points the phenomena observed in the secondary circuit would be repeated without variation, since the direction of oscillation in the air space is indifferent.

The conclusion to be drawn from the experiments is that in passing any one of these points part of the action is reversed, while another part is not. The experimental results, however, warrant the assumption that twice the distance between two of these points is equal to the half wave-length, and when this assumption is made the phenomena can be fully explained.

For suppose a wave of E.M.F., with oscillations in a vertical direction, to impinge upon the wall, and to be reflected with only slightly diminished intensity, thus giving rise to stationary waves. If the wall were a perfect conductor, a node would necessarily be formed in its surface, for at the boundary and in the interior of a perfect conductor the E.M.F. must be infinitely small. The wall cannot, however, be considered as a perfect conductor, for it was not metallic throughout, and the portion which was metallic was not of any great extent. The E.M.F. would therefore have a finite value at its surface, and would be in the direction of the impinging waves. The node, which in the case of perfect conductivity would occur at the surface of the wall, would, therefore, actually be situated a little behind it, as shown at A in the diagram. If, then, twice the distance A B—that is to say, the distance A C—is half the wave-length, the steady waves will be as represented by the continuous lines in Fig. 162. The E.M.F.s acting on each side of the circles, in the positions I., II., III., and IV., will, therefore, at a given moment be represented in magnitude and direction by the arrows on each side of them in the diagram. If, therefore, in the neighbourhood of a node, the air space is turned towards the node, the strongest E.M.F. in the circle will act under more favourable conditions against a weaker one under less favourable conditions. If, however, the air space is turned away from the node, the stronger E.M.F. acts under less favourable conditions against a weaker one under more favourable conditions. In the latter case the resultant action must be less than in the former, whichever of the two E.M.F.s has the greater effect, which explains the change of sign of the phenomenon at each quarter wavelength.

This explanation is further confirmed by the consideration that, if it is the true one, the change of sign at the points B and D must take place in quite a different manner from that of the point C. The E.M.F.s acting on the secondary circle, in the positions V., VI., and VII., are shown by the corresponding arrows, and it is clear that in the positions B and D, if the air space is turned from one side to the other, the vibration will change its direction round the circle, and therefore the sparking must, during the rotation, vanish either once or an uneven number of times. In the position C, however, the direction of vibration remains unaltered, and therefore the sparks must disappear an even number of times, or not at all.

The experiments showed that at B and D the sparking diminished as the air space receded from $\alpha$, vanished at the highest point, and again attained its original value at the point $\beta$. At C, on the other hand, the sparking continued throughout the rotation, being a little stronger at the highest and lowest points. If, then, there is any change of sign in the position C, it must occur with very much smaller displacements than in the other positions, so that in any case there is a distinction such as is required between this and the other two cases.

Another very direct proof of the truth of Hertz's representation of the nature of the waves was obtained. If the secondary circle lies in the plane of the waves instead of in the plane of vibration, the E.M.F. must be equal at all points of the circle, and for a given position of the air space the sparking must be directly proportional to its intensity. When the experiment was made, it was found, as expected, that at all distances the sparking vanished at the highest and lowest points of the circle, and attained a maximum value at the points in the horizontal plane through the point of incidence.

The air space was then placed at such a point and close to the wall, and was then moved slowly away from the wall, when it was found that, while there was no sparking quite close to the metal plate, it began at a very small distance from it, rapidly increased, reached a maximum at the point B, and then diminished again. At C the sparking again became excessively feeble and increased as the circle was moved still further away. The sparking continued steadily to increase after this, as the motion of the circle was continued in the same direction, owing, as before, to the direct action of the primary oscillation.

The curves shown by the continuous lines in Fig. 162 were obtained from the results of these experiments, the ordinates representing the intensity of the sparks at the distances represented by the corresponding abscissæ.

The existence in the electrical waves of nodes at A and C, and of loops at B and D, is fully established by the experiments which have been described; but in another sense the points B and D may be regarded as nodes, for they are the nodal points of a stationary wave of magnetic induction which, according to theory, accompanies the electrical wave and lags a quarter wave-length behind it.

This can easily be shown to follow from the experiments, for when the secondary circle is placed in the plane of vibration with the air space at its highest point, there will be no sparking if the E.M.F. is uniform throughout the space occupied by the secondary. This can only take place if the E.M.F. varies from point to point of the circle, and if its integral round the circle differs from zero. This integral is proportional to the number of magnetic lines of force passing backwards and forwards across the circle, and the intensity of the sparks may be considered as giving a measure of the magnetic induction, which is perpendicular to the plane of the circle. Now, in this position vigorous sparking was observed close to the wall, diminishing rapidly to zero as the point B was approached, then increasing to a maximum at C, falling to a well-marked minimum at D, and finally increasing continuously as the secondary approached still nearer to the primary. If the intensities of these sparks are taken as ordinates, positive and negative, and the distances from the wall as abscissæ, the curve shown by the dotted lines in Fig. 162 is obtained, which therefore represents the magnetic waves.

The phenomena observed in the first series of experiments described above may therefore be regarded as due to the resultant electric and magnetic actions. The former changes sign at A and C, the latter at B and D, so that at each of these points one part of the action changes sign, while the other does not, and therefore the resultant action which is their product must change sign at each of these points, as was found to be the case.

When the secondary circle was in the plane of vibration the sparking in the vicinity of the wall was observed to be a maximum on the side towards the wall and a minimum at the opposite side, and as the circle was turned from one position to the other there was found to be no point at which the sparks disappeared. As the distance from the wall was increased, the sparks on the remote side gradually became weaker, and vanished at a distance of 1·08 metre from the wall. When the circle was carried further in the same direction the sparks appeared again on the side remote from the wall, but were always weaker than on the side next to it; the sparking, however, no longer passed from a maximum to a minimum merely, but vanished during the rotation once in the upper and once in the lower half of the circle. The two null points gradually receded from their original coincident positions until at the point B they occurred at the highest and lowest points of the circle. As the circle was moved further in the same direction the null points passed over to the side next to the wall, and approached each other again, until, when the centre was at a distance of 2·35 metres from the wall, the two null points were again coincident. B must be exactly half-way between this point and the similar point previously observed, which gives 1·72 metre as the distance of B from the wall—a result which agrees, within a few centimetres, with that obtained by direct observation. Moving further in the direction of C, the sparking at different points of the circle became more nearly equal, until at C it was exactly so. In this position there was no null point, and as the distance was further increased the phenomena recurred in the same order as before.

Hertz found that the position of C could be determined within a few centimetres, the determinations of its distance from the wall varying from 4·10 to 4·15 metres; he gives its most probable value as 4·12 metres. The point B could not be observed with any exactness, the direct determinations varying from 6 to 7·5 metres as its distance from the wall. It could, however, be determined indirectly, for the distance between B and C being found to be 2·4 metres, taking this as the true value, A must have been 0·68 metre behind the surface of the wall, and 6·52 metres in front of it. The half wave-length would be 4·8 metres, and by an indirect method it was found to be 4·5 metres, so that the two results agree fairly well. Taking the mean of these as the true value, and the velocity of light as the velocity of propagation, gives as the vibration period of the apparatus 1·55 hundred-millionth of a second, instead of 1·4 hundred-millionth, which was the theoretically calculated value.

A second series of experiments was made with smaller apparatus, and though the measurements could not be made with as much exactness as those already described, the results showed clearly that the position of the nodes depends only on the dimensions of the conductors, and not on the material of the wall.

Hertz states that after some practice he succeeded in obtaining indications of reflections from each of the walls. He was also able to obtain distinct evidence of reflection from one of the iron columns in the room, and of the existence of electrodynamic shadows on the side of the column remote from the primary.

In the preceding experiment the secondary conductor was always placed between the wall and the primary conductor—that is to say, in a space in which the direct and reflected rays were travelling in opposite directions, and gave rise to stationary waves by their interference.

He next placed the primary conductor between the wall and the secondary, so that the latter was in a space in which the direct and reflected waves were travelling in the same direction. This would necessarily give rise to a resultant wave, the intensity of which would depend on the difference in phase of the two interfering waves. In order to obtain distinct results it was necessary that the two waves should be of approximately equal intensities, and therefore the distance of the primary from the wall had to be small in comparison with the extent of the latter, and also in comparison with its distance from the secondary.

To fulfil these conditions the secondary was placed at a distance of 14 metres from the reflecting wall, and, therefore, about 1 metre from the opposite one, with its plane in the plane of vibration, and its air space directed towards the nearest wall, in order to make the conditions as favourable as possible for the production of sparks. The primary was placed parallel to its former position, and at a perpendicular distance of about 30 centimetres from the centre of the reflecting metallic plate. The sparks observed in the secondary were then very feeble, and the air space was increased until they disappeared. The primary conductor was then gradually moved away from the wall, when isolated sparks were soon observed in the secondary, passing into a continuous stream when the primary was between 1·5 and 2 metres from the wall—that is, at the point B. This might have been supposed to be due to the decrease in the distance between the two conductors, except that as the primary conductor was moved still further from the wall the sparking again diminished, and disappeared when the primary was at the point C. After passing this point the sparking continually increased as the primary approached nearer to the secondary. These experiments were found to be easy to repeat with smaller apparatus, and the results obtained confirmed the former conclusion—that the position of the nodes depends only on the dimensions of the conductor, and not on the material of the reflecting wall.

Hertz points out that these phenomena are exactly analogous to the acoustical experiment of approaching a vibrating tuning-fork to a wall when the sound is weakened in certain positions and strengthened in others, and also to the optical phenomena illustrated in Lloyd's form of Fresnel's mirror experiments; and as these are accepted as arguments tending to prove that sound and light are due to vibration, his investigations give a strong support to the theory that the propagation of electro-magnetic induction also takes place by means of waves excited in a medium. They therefore afford a confirmation of the Faraday-Maxwell theory of electrical action.

§ **12. Further Researches on Electro-Magnetic Radiation.**—When once the experimental proof had been given that the result of electrical oscillations in a conductor is to propagate out into surrounding space radiations which are in all respects of the same nature as *light*, except in that they cannot affect the eye, it became evident that a new and vast field of investigation had been opened, and one in which it would be possible to produce the electro-magnetic analogues of all the more familiar optical phenomena.

The reflection, refraction, dispersion, and polarisation of light waves are well-known optical phenomena. We can perform analogous experiments with rays of dark heat which differ only from light rays in having a greater wave-length, and in being thereby unable to affect the optic nerve. In performing these experiments with dark heat or non-luminous radiation we have to make use of the thermopile as a *perceiver* of the ray. The electro-magnetic radiation scattered from a conductor in which electric oscillations are set up differs again from light and dark heat in having a still longer wave-length. In performing experiments with electro-magnetic radiation we have seen that Hertz's invention of the electro-magnetic resonator put us in possession of an apparatus which is the exact equivalent of a thermopile, or the human eye, as a ray localiser; and more recent researches have shown us how to construct a large number of forms of receiver of even more sensitive character, by means of which we can detect this electro-magnetic radiation.

In these electro-magneto-optic experiments of Hertz, the source of radiation is a divided metallic cylinder about one inch in diameter and twelve inches long. This is divided in halves, and the two parts separated by a small distance. They are respectively attached to the ends of the secondary coil of a small induction coil. When the coil is put in action, electrical oscillations are set up in these cylinders which result in the outward propagation of ethereal undulations of about two feet in wave-length and having a frequency of about five hundred millions a second.

In order to *see* these waves, Hertz employed a resonator consisting of a metallic circuit having a small spark interval. With these simple appliances he has been able to show the reflection of the electro-magnetic waves from plane surfaces, and the concentration of radiation by parabolic mirrors of sheet zinc, repeating in fact the old experiment of the conjugate mirrors. The radiation from this source could, he found, be gathered up by one parabolic mirror, reflected to a second and concentrated again to a focus. Another achievement was the refraction of the rays by a great prism of pitch. Placed in the path of an electro-magnetic ray, he found that this pitch prism refracted it through an angle of 22deg., and that the material had a refractive index of 1·7 for these long waves. Again, it was found that metallic sheets were opaque to this radiation, but that it passed through such non-conductors as dry wood, and that a laboratory door, although opaque to light, is transparent to this ultra-ultra red or electro-magnetic radiation.

The reader may be referred to Dr. Lodge's book on the "Work of Hertz, and some of his Successors" for a full account of the experimental proofs that electro-magnetic radiation and that radiation we commonly call *light* are one in essential nature, although differing in wave-length. These experiments are akin to the acoustic ones in which air waves, too short to be audible, are generated; and in place of the ear, now useless, a sensitive flame is employed to find or indicate the waves, and inform us of the presence or absence of aerial wave motion. In the same way all well-known optical effects can be reproduced with ether radiation too long in wave-length to affect the eye, but capable of acting on a proper receiver.

It is a necessary corollary of Maxwell's electro-magnetic theory of light that good conducting bodies should be opaque and good insulators transparent. As a matter of fact, for disturbances of the period of light many good insulators, such as ebonite, are opaque, even in very thin sheets, and conversely gold, silver, and platinum are semi-transparent when in very thin sheets. It must be borne in mind, however, that the frequency of light oscillations falls between 400 and 700 million-million oscillations per second, or are of the order of $5 \times 10^{14}$.

We cannot by any of Hertz's methods produce electrical oscillations so rapid as this. Hence, since conductivity and insulating power of materials have generally been determined with reference either to steady currents or to moderately great oscillations, we cannot institute a comparison between these qualities as possessed by any given substance and opacity or transparency for the much greater frequency of luminous electro-magnetic waves. It has been shown that ebonite is very transparent to long waves of dark heat,* and hence there is no difficulty in understanding that it is transparent to the longer waves produced by electrical oscillations set up in moderately small conductors, whilst it is opaque to the very short ones of light. Also the transparency of thin metallic sheets to light is an indication of imperfect conductibility. We have seen that an infinitely perfect conductor is a perfect magnetic screen, and accordingly we should expect that the more perfect the conductivity of a metal the greater would be its opacity even in very thin films. It is well known that cooling copper increases its conductivity. Wroblewski showed (*Comptes Rendus*, Vol. CI., July, 1885, p. 160) that by cooling copper to $-200°$C., or to the temperature of the solidification of nitrogen, its conductivity is increased to about nine times its value at 0°C. These experiments have been greatly extended by Dewar and Fleming (*Phil. Mag.*, September, 1893), who have shown that perfectly *pure* metals have most probably no electrical resistance at the absolute zero of temperature. It would be interesting to know if the opacity of a very thin film of copper is increased by cooling to $-200°$ to any perceptible extent. With respect to electrolytes some interesting experiments have been made by Prof. J. J. Thomson.† In these experiments electrical oscillations of about $10^8$ per second in frequency were established in a primary circuit by means of an induction coil. These alternating currents were caused to induce secondary oscillations in a neighbouring parallel circuit of appropriate size. The secondary circuit oscillations were rendered visible by minute sparks at a break in that circuit. The interposition of a thin sheet of tinfoil or of the thinnest sheet of Dutch metal or gold-leaf supported on glass at once stopped completely the secondary sparks. This is a very interesting confirmation of the theory of magnetic screening laid down on p. 255 of Chap. IV. We have seen that for moderately rapid alternations the conductivity of tinfoil is not sufficient to make it opaque to electro-magnetic radiations, but for disturbances of a frequency equal to about $10^8$ the tinfoil affords a perfect screening, or, in other words, is opaque.

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* See note on "The Index of Refraction of Ebonite," by Profs. Ayrton and Perry, *Proceedings* of the Physical Society, London, Vol. IV., p. 345.

† "On the Resistance of Electrolytes to the Passage of Rapidly Alternating Currents," *Proceedings* of the Physical Society, London, Vol. XLV., No. 276, 1889.

With regard to the gold-leaf, Prof. Thomson remarks that he has not been able to get any leaf thin enough to be transparent to oscillations of this rate. On inserting a sheet of ebonite between the primary and secondary circuit, it was found to produce no effect on the sparking, indicating that ebonite, although opaque to ordinary light, is transparent to ether disturbances of the rate here employed. A thin layer of transparent electrolyte was then used as a screen, and it was found that whilst a very thin layer produced little or no effect, a depth of three to four millimetres of dilute sulphuric acid was sufficient to stop the sparking. Experiment showed that the conductivity of various electrolytic solutions was about the same for currents reversed 120 times a second as for currents reversed 100 million times a second. As, however, these electrolytes are transparent, they must, according to the electro-magnetic theory, be insulators for currents reversed about $10^{15}$ times per second, and the molecular processes on which electrolytic conduction depends must occupy a time between one-hundred millionth and one-thousand billionth of a second. Space will not permit further reference to this exceedingly promising department of future research more than to say that if the electro-magnetic theory of light is true it will be able to furnish an electrical explanation, not only of the simpler optical phenomena, but of such complex phenomena as those embraced in the sciences of spectroscopy and photography.

§ 13. **Propagation of Electro-Magnetic Energy.**—In our exposition of the various electro-magnetic phenomena we have directed attention to the facts which make it evident that even in the simple phenomenon of the propagation of an electric current in a wire we must divest ourselves of the idea that the so-called flow of current is analogous to the movement of a material fluid in a pipe. It is true that there are effects in the case of the electric current which correspond to the inertia and resistance effects in the case of water flow; but the progress of knowledge has indicated that what we are in the habit of calling the electric current is as much outside the wire as in it, and that we must release ourselves from the trammels of any ideas which cause us to concentrate attention exclusively or mainly on the actions *in* the conductor. In fact, at the absolute zero of temperature there would be no dissipation of energy in the conductor at all, if it were a pure metal, and all the processes would be confined to the medium. We are indebted to, amongst others, Prof. Poynting for an enlargement of our views on the nature of electric current propagation, and in two valuable memoirs these matters have been discussed by him.*

He says:—A space containing electric currents may be regarded as a field where energy is transformed at certain points into the electric and magnetic form by means of batteries, dynamos, thermopiles, &c., and in other parts of the field this energy is being again transformed into heat, work done by electro-magnetic forces, or any other form yielded by currents. Formerly a current was regarded as something travelling in the conductor, and the energy which appeared at any part of the circuit was supposed to be conveyed thither through the conductor by the current. But the existence of induced currents and electro-magnetic actions has led us to look on the medium surrounding the conductor as playing a very important part in the development of the phenomena. If we believe in the continuity of the motion of energy, we are forced to conclude that the surrounding medium is capable of containing energy, and that it is capable of being transferred from point to point. We are thus led to consider the problem—how does the energy connected with an electric current pass from point to point, by what paths does it travel, and according to what laws? Let us put a specific case. Suppose a dynamo at one spot generates an electric current, which is made to operate an electric motor at a distant place. We have here in the first place an absorption of energy from the prime motor into the dynamo. We find the whole space between and around the conducting wires magnetised and the seat of electro-magnetic energy. We have, further, a re-transformation of energy in the motor. The question which presents itself for solution is to decide how the energy taken up by the dynamo is transmitted to the motor.

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* "On the Transfer of Energy in the Electro-Magnetic Field," by Prof. J. H. Poynting, *Philosophical Transactions* of the Royal Society, 1884, Part II., Vol. CLXXV., p. 343. Also "On the Connection between Electric Current and the Electric and Magnetic Inductions in the Surrounding Field," by Prof. J. H. Poynting, *Philosophical Transactions* of the Royal Society, 1885, Part II., Vol. CLXXVI., p. 277.

Briefly stated, the general tendency of recent views is that this energy is conveyed through the electro-magnetic medium, or ether, and that the function of the wire is to localise the direction or concentrate the flow in a particular path, and is at the same time also a *sink* or place in which energy is dissipated. A consideration of the whole phenomenon has enabled Prof. Poynting to formulate an important law, as follows:—*At any point in the magnetic field of conductors conveying currents the energy moves perpendicularly to the plane containing the lines of electric force and the lines of magnetic force, and the amount crossing a unit of area of this plane per second is equal to the product of the intensities of the two forces multiplied by the sine of the angle between them and divided by* $4\pi$. If E denote the electric force, or force on a very small body charged with a unit of positive electricity, and H denote the magnetic force, or force on a unit magnetic pole, and if at any point in the electro-magnetic field these forces are inclined at an angle $\theta$, then there is a flow of energy $e$ at this point in a direction perpendicular to the planes of E and H, and equal per second to the value of

$$\frac{\mathrm{E\,H}\sin\theta}{4\pi}.$$

The full proof of this law is given in the first of the Papers mentioned on the preceding page.

Prof. Poynting has here introduced the important notion of a flow of energy. We may remark in passing that this notion does no violence to previous notions of energy. Energy, like matter, is *conserved*—that is, it is unalterable in total amount; and if in any circumscribed space some form of energy makes its appearance, then we know that either an equal quantity must have passed into that space from outside, or that an equivalent quantity of some other form already in the enclosure must have been transformed. If energy disappears at one point and reappears at an adjacent point in equal amount, we can with perfect propriety speak of it as having been transferred from one point to the other, although we are unable to identify the respective portions of it as we can in the case of the movement of matter. Applying this view to the simple phenomena of a battery producing heat in a conducting wire, the notion to be grasped is that the potential energy of the chemical combinations in the battery causes energy to be radiated out along certain lines, the means of conveyance being the electro-magnetic medium; this energy flows into the wire at all points, and is there re-transformed into heat or light. A simple illustration of Poynting's law is to consider the case of a section of a straight conductor traversed, as we usually say, by a current. Let the conductor be a right cylinder, or round wire, of length $l$, radius $r$, and let E be the electric force at any point in the wire, and H the magnetic force at the surface; also let V be the potential difference between the ends, C the steady current, and R the total ohmic resistance. Consider the energy flowing in on this section of the wire through its surface. It is equal per second to the area of the surface, multiplied by $\frac{\mathrm{E\,H}}{4\pi}$, or to $\frac{2\pi\, r\, l\, \mathrm{E\,H}}{4\pi}$.

Now $2\pi\, r\, \mathrm{H}$ is the line integral of the magnetic force taken round the wire following the circular surface, and this, as previously shown (p. 25) is equal to $4\pi\,\mathrm{C}$. Also we have the potential difference at the ends of the cylinder equal to the line integral of the electric force, or to $l\,\mathrm{E}$. Since, then, $2\pi\, r\, \mathrm{H} = 4\pi\, \mathrm{C}$ and $\mathrm{E}\, l = \mathrm{V}$, we get, by substitution in the value of the energy sent per second into the section of the wire, viz., $\frac{2\pi\, r\, l\, \mathrm{H\,E}}{4\pi}$, the equivalent C V. But by Ohm's law $\mathrm{C\,R} = \mathrm{V}$; hence the energy absorbed per second by the conductor is $\mathrm{C}^2\mathrm{R}$, and we know by Joule's law that this is the measure of the energy dissipated per second in the wire as heat. We see, then, that the energy dissipated in each section of the conductor is absorbed into it from the dielectric, and the rate of this supply can be calculated by Poynting's law for each element of the surface. None of the energy of a current travels along the wire, but enters into it from the surrounding non-conductor, and as soon as it enters it begins to be transformed into heat, the amount crossing successive layers of the wire decreasing till, by the time the centre, where there is no magnetic force, is reached, it has all been transformed into heat.

In the Paper another simple case treated is that of a condenser discharged by a wire. In this case, before the discharge, we know that the energy resides in the dielectric between the plates. If the plates are connected by an external wire, according to these views the energy is transferred outwards, along the electrostatic equipotential surfaces, and moves *on* to the wire, and is there converted into heat. According to this hypothesis, we must suppose the lines or tubes of electrostatic induction running from plate to plate to move outwards as the dielectric strain lessens and, whilst still keeping their ends on the plates, finally to converge in on the wire and be there broken up and their energy dissipated as heat. At the same time the wire acquires transient magnetic qualities. This means that some part of the energy of the expanding lines of electrostatic induction is converted into magnetic energy. The magnetic energy is contained in ring-shaped tubes of magnetic force, which expand out from between the plates and then contract in upon some other part of the circuit.

The whole history of the discharge may be divided into three parts. First, a time when the energy associated with the system is nearly all electrostatic and is represented by the energy of the lines or tubes of electrostatic induction running from plate to plate; second, a period when the discharge is at its maximum, when the energy exists partly as energy associated with lines of electrostatic induction expanding outwards, and partly in the form of closed rings or tubes of magnetic force expanding from and then contracting back on the wire; and lastly, a period when nearly all the energy has been absorbed or buried in the wire, and has there been dissipated in the form of heat, which is radiated out again as energy of dark or luminous radiation. The function of the discharging wire is to localise the place of dissipation, and also to localise the place where the magnetic field shall be most intense; and all that observation is able to tell us about a conductor which is conveying that which we call an electric current is that it is a place where heat is being generated, and near which there is a magnetic field. These conceptions lead us to fresh views of very familiar phenomena. Suppose we are sending a current of electricity through a submarine cable by a battery, say, with zinc to earth, and suppose the sheath is everywhere at zero potential, then the wire will be everywhere at a higher potential than the sheath, and the level surfaces will pass through the insulating material to the points where they cut the wire. The energy which maintains the current and which works the receiver at the distant end travels through the insulating material, the core serving as a means to allow the energy to get into motion or to be continually propagated. The energy absorbed by the core is, however, transformed into heat and radiated again as dark heat.

In the case of an arc or glow-lamp worked by an alternating current, we have to consider that the energy which moves in on the carbon is returned again, with no other change than that of a shortened wave-length, and the carbon filament performs the same kind of change on the electromagnetic radiation as is performed when we heat a bit of platinum foil to vivid incandescence in a focus of dark heat. If we adopt the electro-magnetic theory of light, it moves out again still as electro-magnetic energy, but in a different form, with a definite velocity and intermittent in type. We have, then, in the case of the electric light this curious result—that energy moves in upon the arc or filament from the surrounding medium, there to be converted into a form in which it is sent out again, and which, though the same in kind, is now able to affect our senses. A current passing through a seat of electromotive force is therefore a place of divergence of energy from the conducting circuit into the medium, and this energy travels away and is converged and transformed by the rest of the circuit. From this aspect the function of the copper conducting wire fades into insignificance in interest in comparison with the function of the dielectric. When we see an electric tramcar, or motor, or lamp, worked from a distant dynamo, these notions invite us to consider the whole of that energy, even if it be thousands of horse-power per hour, as conveyed through the electro-magnetic medium, and the conductor as a kind of exhaust valve, which permits energy to be continually supplied to the dielectric.

Consider, for instance, the simple case of an alternating-current dynamo connected to an incandescent lamp by conducting leads. We have in this case a closed conducting loop, consisting partly of the armature wire, partly of the leads, and lastly of the lamp filament. The action of the dynamo when at work consists in alternately inserting into and withdrawing tubes of magnetic induction from a portion of this enclosed area or loop. The insertion of this induction causes an electro-magnetic disturbance which travels away through the enclosed dielectric in the form of an ether displacement in its most generalised sense. In reaching the surface of the enclosing conductor this wave begins to soak into it, the electro-magnetic energy at the same time dissipating itself in it in the form of heat. By a suitable arrangement of the resistances and surfaces of various portions of the circuit, we are able to localise the principal place of transformation, and to control its rate so as to compel this transformation of energy to take place at a certain rate in a limited portion of the conductor. Energy is then sent out again in a radiant form—partly in the form of ether waves capable of exciting the retina of the eye, but very largely in the form of dark heat. The ether, or electro-magnetic medium, is, therefore, the vehicle by which the energy is carried to the lamp and conveyed away from it in an altered form; and, whatever be the translating device employed, the ether is the seat of the hidden operations, which are really the fundamental ones, and the visible apparatus is only the contrivance by which the nature of the energy transformation is determined and its place defined.

These views are the outcome of that half-century of scientific thought which dates from the period of Faraday's conception of an electro-magnetic medium. We can without hesitation predict that the ideas which have thus guided to so much discovery are destined to conduct to further revelations of the nature of the unseen mechanism which lies behind the apparent actions taking place in the electro-magnetic field, and of which these actions are the result.

§ **14. Propagation of Currents along Conductors.**—We have in a previous section enunciated the modern ideas on this subject, according to which the propagation of a current in a conductor depends upon actions taking place in the dielectric, and various illustrations were given in explanation of this process. We owe to Hertz, however, an absolute experimental demonstration of the correctness of the opinions on this matter, which had previously, from a mathematical standpoint, been put forward by Oliver Heaviside, Poynting, and Lodge. Hertz placed on record his experimental demonstrations in a remarkably interesting Paper which we shall quote here almost *verbatim.*[*]

Dealing with the deductions made by the above writers from Maxwell's equations Hertz described (*loc. cit.*) his experiments in confirmation of their views, and makes the following remarks :—

"Mathematical investigation points to the conclusion that the electric force which determines the current is in no wise propagated in the wire itself, but under all circumstances enters the wire from without and spreads itself in the metal comparatively slowly, and according to laws similar to those governing the changes of temperature in a conductor of heat. If the forces in the neighbourhood of the wire are continually altering in direction, the effect of these forces will only enter to a small depth into the metal; the more slowly the changes take place, so much deeper will the effect penetrate; and if, finally, the changes follow one another infinitely slowly, the force has time to fill the whole interior of the wire with uniform intensity."

The first and most important question with regard to this theory is, whether it agrees with fact. Since, in the experiments which Hertz carried out on the propagation of electric force, he made use of electric waves in wires which were of extraordinarily short period, it was convenient to prove, by means of these, the accuracy of the inferences drawn. In fact, the theory was proved by the experiments which will now be described; and it will be found that these few experiments suffice to confirm in the highest degree the view of Messrs. Heaviside and Poynting. Analogous experiments, with similar results, but with quite different apparatus, had already been made by Dr. O. J. Lodge,[*] chiefly in the interest of the theory of lightning-conductors. Up to what point the conclusions which were drawn by Dr. Lodge in this direction from his experiments are just must depend in the first place on the velocity with which the alterations of the electrical conditions really follow each other in the case of lightning. The apparatus and methods which are here mentioned are those which Hertz described in full in previous memoirs.[†] The waves used were such as had in wires a distance of nearly three metres between the nodes.

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[*] This Paper was translated from Wiedemann's *Annalen*, XXXVII., p. 395, July, 1889, by Dr. J. L. Howard for the *Phil. Mag.* of August, 1889, and by kind permission of the translator is given here.

"If a primary conductor acts through space upon a secondary conductor, it cannot be doubted that the effect penetrates the latter from without. For it can be regarded as established that the effect is propagated in space from point to point; therefore it will be forced to meet first of all the outer boundary of the body before it can act upon the interior of it. But a closed metallic envelope is shown to be quite opaque to this effect. If we place the secondary conductor having a spark gap in such a favourable position near the primary one that we obtain sparks 5mm. to 6mm. long, and surround it with a closed box made of zinc plate, the smallest trace of sparking can no longer be perceived. The sparks similarly vanish if we entirely surround the primary conductor with a metallic box. It is well known that, with relatively slow variations of current, the integral force of induction is in no way altered by a metallic screen. This is, at the first glance, contradictory to the present experiments. However, the contradiction is only an apparent one, and is explained by considering the duration of the effects. In a similar manner, a screen which conducts heat badly protects its interior completely from rapid changes of the outside temperature, less from slow changes, and not at all from a continuous raising or lowering of the temperature. The thinner the screen is the more rapid are the variations of the outside temperature which can be felt in its interior. In this case the electrical action must plainly penetrate into the interior, if we only diminish sufficiently the thickness of the metal—a box covered with tinfoil, protected completely, and even a box of gilt paper, if care were taken that the edges of the separate pieces of paper were in metallic contact. In this instance the thickness of the conducting metal was estimated to be barely $\frac{1}{20}$mm. Hertz then fitted the protecting envelope as closely as possible round the secondary conductor. For this purpose its spark-gap was widened to about 20mm., and, in order to detect electrical disturbances in it, an auxiliary spark-gap was added exactly opposite to the one ordinarily used. The sparks in this latter were not so long as in the ordinary spark-gap, since the effect of resonance was now wanting, but they were still very brilliant. After this preparation the conductor was completely enclosed in a tubular conducting envelope as thin as possible, which did not touch it, but was as near it as possible; and in the neighbourhood of the auxiliary spark-gap (in order to be able to use it) the envelope contained a wire-gauze window. Between the poles of this envelope brilliant sparks were produced, just as previously in the secondary conductor itself; but in the enclosed conductor not the slightest electrical movement could be recognised. The result of the experiment is not affected if the envelope touches the conductor at a few points; the insulation of the two from each other is not necessary in order to make the experiment succeed, but only to give it the force of a proof. Clearly we can imagine the envelope to be drawn more closely round the conductor than is possible in the experiment; indeed, we can make it coincide with the outermost layer of the conductor. Although, then, the electrical disturbances on the surface of our conductor are so powerful that they give sparks 5mm. to 6mm. long, yet at $\frac{1}{20}$mm. beneath the surface there exists such perfect freedom from disturbance that it is not possible to obtain the smallest sparks. We are brought, therefore, to the conclusion that what we call an induced current in the secondary conductor is a phenomenon which is the result of actions taking place in the surrounding dielectric.

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[*] Lodge, *Journ. Soc. of Arts,* May, 1888; *Phil. Mag.* [5], XXVI., p. 217 (1888).

[†] Hertz, Wied. *Ann.*, XXXIV., p. 551 (1888).

"One might grant that this is the state of affairs when the electric disturbance is conveyed through a dielectric, but maintain that it is another thing if the disturbance, as one usually says, has been propagated in a conductor. If we place near one of the end plates of our primary conductor a conducting-plate, and fasten to it a long straight wire, we have already seen in the previous experiments how the effect of the primary oscillation can be conveyed to great distances by the help of this wire. The usual theory is that a wave travels along the wire in this case. But we can show in the following manner that all the alterations are confined to the space outside and on the surface of the wire, and that its interior knows nothing of the wave passing over it. A piece about 4 metres long was removed from the wire conductor and replaced by two strips of zinc plate 4 metres long and 10cm. broad, which were laid flat one above the other, with their ends permanently connected together. Between the strips along their middle line, and therefore almost entirely surrounded by their metal, was laid along the whole 4 metres length a copper wire covered with gutta-percha. It was immaterial for the experiments whether the outer ends of this wire were in metallic connection with or insulated from the strips; however, the ends were mostly soldered to the zinc strips. The copper wire was cut through in the middle, and its ends were carried, twisted round each other, outside the space between the strips to a fine spark-gap, which permitted the detection of any electrical disturbance taking place in the wire. When waves of the greatest possible intensity were sent through the whole arrangement, there was nevertheless not the slightest effect observable in the spark-gap. But if the copper wire was then displaced anywhere a few decimetres from its position, so that it projected just a little beyond the space between the strips, sparks immediately began to pass. The sparks were the more intense according to the length of copper wire extending beyond the edge of the zinc strips and the distance it projected. The unfavourable relation of the resistances was, therefore, not the cause of the previous absence of sparking, for this relation had not been changed; but the wire being in the interior of the conducting mass, was at first deprived of the influence coming from without. Moreover, it is only necessary for us to surround the projecting part of the wire with a little tinfoil in metallic communication with the zinc strips in order to immediately stop the sparking again. By this means we have brought the copper wire back again into the interior of the conductor. If we bend another wire into a fairly large arc round the projecting portion of the gutta-percha wire, the sparks will be likewise weakened; the second wire takes off from the first a certain amount of the effect due to the outer medium. Indeed, it may be said that the edge of the zinc strip itself takes away in a similar manner the induction from the middle of the strip. For if we now remove one of the strips, and leave the insulated wire simply resting on the other one, we certainly obtain sparks continuously in the wire; but they are extremely weak if the wire lies along the middle of the strip, and much stronger when near its edge. Just as in the case of distribution under electrostatic influence the electricity would prefer to collect on the sharp edge of the strip, so also here the current tends to move along the edge. Here, as there, it may be said that the outermost parts screen the interior from outside influence."

The following experiments are somewhat neater and equally convincing. Hertz inserted into the conductor transmitting the waves a very thick copper wire, 1·5 metre long, whose ends carried two circular metallic discs of 15cm. diameter. "The wire passed through the centres of the discs; the planes of the discs were at right angles to the wire; each of them had on its rim 24 holes, at equal distances apart. A spark-gap was inserted in the wire. When the waves traversed the wire they gave rise to sparks as much as 6mm. long. A thin copper wire was then stretched across between two corresponding holes of the discs. When this was done the length of the sparks sank to 3·2mm. There was no further alteration if a thick copper wire was put in the place of the thin one, or if, instead of the single thin wire, twenty-four of them were taken, provided they were placed near each other through the same two holes. But it was otherwise if the wires were distributed over the rim of the discs. If a second wire was inserted opposite the first one, the spark-length fell to 1·2mm.

When two more wires were added midway between the first two, the length of the spark sank to 0·5mm.; the insertion of four more wires still in the mean positions left sparks of scarcely 0·1mm. long; and after inserting all the twenty-four wires at equal distances apart, not a trace of sparking was perceptible in the interior. The resistance of the inner wire was nevertheless much smaller than that of all the outside wires taken together. We have also a still further proof that the effect does not depend upon this resistance. If we place by the side of the partial tube of wires, and in parallel circuit with them, a conductor in all respects similar to that in the interior of the tube, we have in the former brilliant sparks, but none whatever in the latter. The former is unprotected, the latter is screened by the tube of wires. We have in this an electro-dynamic analogue of the electrostatic experiment known as the electric birdcage."

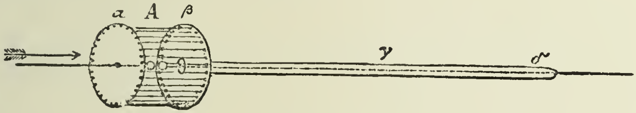

Fig. 163.

Hertz again altered the experiment, in the manner depicted in Fig. 163.

"The two discs were placed so near together that they formed, with the wires inserted between them, a cage (A) just large enough for the reception of the spark-micrometer. One of the discs, $\alpha$, remained metallically connected with the central wire; the other, $\beta$, was insulated from the wire by means of a circular hole through its centre, at which it was connected to a conducting-tube, $\gamma$, which, insulated from the central wire, surrounded it completely for a length of 1·5 metre. The free end of the tube, $\delta$, was then connected with the central wire. The wire, together with its spark-gap, is once more situated in a metallically protected space; and it was only to be expected, from the previous experiments, that not the slightest electrical disturbance would be detected in the wire in whichever direction waves were sent through the apparatus. So far, then, this arrangement showed nothing new, but it had the advantage over the previous one that we could replace the protecting metallic tube, $\gamma$, by tubes of smaller and smaller thickness of wall, in order to investigate what thickness is still sufficient to screen off the outside influence. Very thin brass tubes—tubes of tinfoil and Dutch metal—proved to be perfect screens. Glass tubes were taken which had been silvered by a chemical method, and it was then perfectly easy to insert tubes of such thinness that, in spite of their protecting power, brilliant sparks occurred in the central wire. But sparks were only observed when the silver film was no longer quite opaque to light, and was certainly thinner than $\frac{1}{100}$mm. In imagination, although not in reality, we can conceive the film drawn closer and closer round the wire, and finally coinciding with its surface; we should be quite certain that nothing would be radically altered thereby. However actively, then, the real waves play round the wire, its interior remains completely at rest; and the effect of the waves hardly penetrates any more deeply into the interior of the wire than does the light which is reflected from its surface. For the real seat of these waves, therefore, we ought not to look in the wire, but rather to assume that they take place in its neighbourhood; and, instead of asserting that our waves are propagated in the wire, we should be more accurate in saying that they glide along on the wire.

"Instead of placing the apparatus just described in the circuit in which we produced waves indirectly, we can insert it in one branch of the primary conductor itself. In such experiments results similar to the previous ones are obtained. Our primary oscillation, therefore, takes place without any participation of the conductor in which it is excited, except at its bounding surface; and we ought not to look for its existence in the interior of the conductor.

"To what has been said above about waves in wires we wish to add just one remark concerning the method of carrying out the experiments. If our waves have their seat in the neighbourhood of the wire, the wave progressing along a single isolated wire will not be propagated through the air alone; but, since its effect extends to a great distance, it will partly be transmitted by the walls, the ground, &c., and will thus give rise to a complicated phenomenon. But if we place opposite each pole of our primary conductor, in exactly the same way, two auxiliary plates, and attach a wire to each of them, carrying the wires straight and parallel to each other to equal distances, the effect of the waves makes itself felt only in the region of space between the two wires. The wave progresses solely in the space between the wires. We can thus take precautions to propagate the effect through the air alone or through another insulator, and the experiments will be more convenient and free from error by this arrangement. For the rest, the lengths of the waves are nearly the same in this case as in isolated wires, so that with the latter the effect of the disturbing causes is apparently not considerable.

"We can conclude from the above results that rapid electric oscillations are quite unable to penetrate metallic sheets of any thickness, and that it is, therefore, impossible by any means to excite sparks by the aid of such oscillations in the interior of closed metallic screens. If, then, we see sparks produced by such oscillations in the interior of metallic conductors which are nearly, but not quite, closed, we shall be obliged to conclude that the electric disturbance has forced itself in through the existing openings. This view is also correct, but it contradicts the usual theory in some cases so completely that one is only induced by special experiments to give up the old theory in favour of the new one. We shall choose a prominent case of this kind, and, by assuring ourselves of the truth of our theory in this case, we shall demonstrate its probability in all other cases. We again take the arrangement which we have described in the previous section and drawn in Fig. 163; only we now leave the protecting tube insulated from the central wire at $\delta$. Let us now send a series of waves through the apparatus in the direction from A towards $\delta$. We thus obtain brilliant sparks at A; they are of similar intensity to those obtained when the wire was inserted without any screen. The sparks do not become materially smaller, if, without making any other alteration, we lengthen the tube $\gamma$ considerably, even to 4 metres. According to the usual theory it would be said that the wave arriving at A penetrates easily the thin, good-conducting metal disc $\alpha$, then it leaps across the spark-gap at A, and travels on in the central wire. According to our view, on the contrary, we must explain the phenomenon in the following manner. The wave arriving at A is quite unable to penetrate the metallic disc; it therefore glides along the disc over the outside of the apparatus and travels as far as the point $\delta$, 4 metres away. Here it divides: one part, which does not concern us at present, travels on immediately along the straight wire, another bends into the interior of the tube and then runs back in the space between the tube and the central wire to the spark-gap at A, where it now gives rise to the sparking. That this view, although more complicated, is still the correct one, is proved by the following experiments. Firstly, every trace of sparking at A disappears as soon as we close the opening at $\delta$, even if it be only by a stopper of tinfoil. Our waves have only a wave-length of 3 metres; before their effect has reached the point $\delta$ the effect at A has passed through zero and changed sign. What influence, then, could the closing of the distant end $\delta$ have upon the spark at A, if the latter really happened immediately after the passage of the wave through the metallic wall? Secondly, the sparks disappear if we make the central wire terminate inside the tube $\gamma$, or at the opening $\delta$ itself; but they reappear when we allow the end of the wire to project even 20cm. to 30cm. only beyond the opening. What influence could this insignificant lengthening of the wire have upon the sparks in A, unless the projecting end were just the means by which a part of the wave breaks off and penetrates through the opening $\delta$ back into the interior? Thirdly, we insert in the central wire between A and $\delta$ a second spark-gap B, which we also completely cover with a gauze cage like that at A. If we make the distance of the terminals at B so great that sparks can no longer pass across, it is also no longer possible to obtain visible sparks at A. But if we hinder in like manner the passage of the spark at A, this has scarcely any influence on the sparks in B. Therefore, the passage of the spark at B determines that at A, but the passage of a spark at A does not determine that at B. The direction of propagation in the interior is therefore from B towards A, not from A to B.

"We can moreover give further proofs, which are more convincing. We may prevent the wave returning from $\delta$ to A from dissipating its energy in sparks, by making the spark-gap either vanishingly small or very great. In this case the wave will be reflected at A, and will now return again from A towards $\delta$. In doing so it must meet the direct waves from $\delta$ to A, and combine with them to form stationary waves, thus giving rise to nodes and ventral segments. If we succeed in proving their existence, there will be no longer any doubt as to the truth of our theory. For this proof we must give somewhat different dimensions to our apparatus in order to be able to introduce electric resonators into its interior. Hertz therefore led the central wire through the axis of a cylindrical tube 5 metres long and 30 centimetres diameter. It was not constructed of solid metal, but of 24 wires arranged parallel to each other along the generating surface, and resting on seven equidistant and circular rings of strong wire, as shown in Fig. 164. He made the requisite resonator in the following manner:—A closely-wound spiral of 1cm. diameter was formed from copper wire of 1mm. thickness; about 125 turns of this spiral were taken, drawn out a little, and bent into a circle of 12cm. diameter; between the free ends an adjustable spark-gap was inserted. Previous experiments had shown that this circle responded to waves 3 metres long in the wire, and yet it was small enough in size to admit of its insertion between the central wire and the surface of the tube. If now both ends of the tube were open, and the resonator was then held in the interior in such a way that its plane included the central wire, and its spark-gap was not directed exactly inwards or outwards, but was turned towards one end or the other of the tube, brilliant sparks of $\frac{1}{2}$mm. to 1mm. length were observed. On now closing both ends of the tube by four wires arranged crosswise and connected with the central conductor, not the slightest sparking remained in the interior, a proof that the network of the tube is a sufficiently good screen for our experiments. The end of the tube on the side $\beta$, that, namely, which was furthest away from the origin of the waves, was now removed. In the immediate

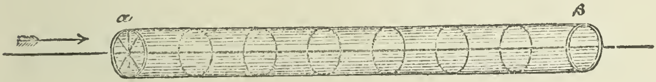

Fig. 164.

neighbourhood of the closed end—that is, at the point $\alpha$, which corresponds to the spark-gap A of our previous experiments—there were now no sparks observable in the resonator. But on moving away from this position towards $\beta$ sparks appeared, became very brilliant at a distance of 1·5 metre from $\alpha$, then decreased again in intensity, then almost entirely vanished at 3 metres distance from $\alpha$, and increased again until the end of the tube was reached. We thus find our theory borne out by fact. That we obtain a node at the closed end is clear, for at the metallic contact between the central wire and the surface of the tube the electric force between the two must necessarily vanish. It is different when we cut the central conductor at this point just near the end, and insert a gap of several centimetres length. In this case the wave will be reflected in a phase opposite to that of the previous case, and we should expect a ventral segment at $\alpha$. As a matter of fact we find brilliant sparks in the resonator in this case; they rapidly decrease in strength if we move from $\alpha$ towards $\beta$, almost entirely vanish at a distance of 1·5 metre, and become brilliant again at a distance of 3 metres; moreover they give a second well-marked node at 4·5 metres distance—that is, 0·5 metre from the open end. The nodes and loops which we have described are situated at fixed distances from the closed end, and alter only with this distance; they are, however, quite independent of the occurrences outside the tube, for example, of the nodes and loops formed there. The phenomena occur in exactly the same way if we allow the wave to travel through the apparatus in the direction from the open to the closed end; their interest is, however, smaller, since the mode of transmission of the wave deviates from that usually conceived less in this case than in the one which has just been under our consideration. If both ends of the tube are left open with the central wire undivided, and stationary waves with nodes and loops are now set up in the whole system, there is always found for every node outside the tube a corresponding node in the interior, which proves that the propagation takes place inside and outside with, at any rate approximately, the same velocity.

"On looking over the experiments which we have described, and the interpretation put upon them, as well as the explanations of the physicists referred to in the introduction, a difference will be noticed between the views here put forward and the usual theory. According to the latter, conductors are represented as those bodies which alone take part in the propagation of electric disturbances; non-conductors are the bodies which oppose this propagation. According to the modern view, on the contrary, all transmission of electrical disturbances is brought about by non-conductors: conductors oppose a great resistance to any rapid changes in this transmission. One might almost be inclined to maintain that conductors and non-conductors should, on this theory, have their names interchanged. However, such a paradox only arises because one does not specify the kind of conduction or non-conduction considered. Undoubtedly metals are non-conductors of electric force, and just for this reason they compel it, under certain circumstances, to remain concentrated instead of becoming dissipated, and thus they become conductors of the apparent source of these forces, electricity, to which the usual terminology has reference."

§ **15. Experimental Determination of Electromagnetic Wave Velocity.**—Space does not permit us to make further mention of the important and valuable work which has been carried out in recent years in confirming and extending this work of Hertz. We refer the reader specially, however, on this subject, to Dr. Lodge's monograph on this subject.*

We shall conclude this chapter by presenting an abstract of an interesting research by Messrs. Trowbridge and Duane on the "Velocity of Electric Waves"† because it furnishes a proof, having a high degree of accuracy, that the velocity of an electromagnetic wave is identical with the velocity of light.

Broadly speaking, the method employed consisted in establishing stationary waves in a conducting circuit and determining the period of oscillation by photographing the oscillating spark in a spark gap, and at the same time measuring the wave-length of the stationary waves induced in a secondary circuit turned to resonance with the primary. In the following paragraphs the description of their experiments is taken from the Paper by Messrs. Trowbridge and Duane. They say :—

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* "The Work of Hertz and some of his Successors." By Dr. O. J. Lodge, published by "The Electrician" Printing and Publishing Co., London.

† *Phil. Mag.*, August, 1895; also *The Electrician*, Vol. XXXV., p. 712.

"The first point in the course of the investigation worth detailed description is the production of electric waves along parallel wires in such a manner that they are actually visible to the eye. The arrangement of the apparatus to accomplish this was as follows :—

"A primary condenser, A B (Fig. 165), was held with its plates in vertical planes by means of suitable wooden supports (not represented in the figure), and was joined in a circuit, B C, consisting of two wires about 75cm. long, placed 4cm. apart. In reality this circuit B C should be represented as perpendicular to the plane of the paper (which is taken as the horizontal plane passing through the centre of the apparatus).

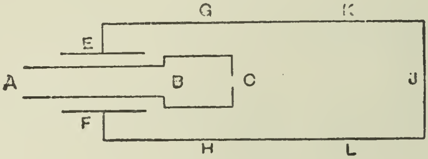

Fig. 165.

The plates of the condenser A B were sheets of tinfoil 101 × 40cm., glued to hard rubber sheets, and the dielectric between them consisted of other similar sheets of hard rubber sufficient in number and thickness to make the distance between the condenser plates 4·2cm. Outside the primary condenser plates, and separated from them by hard rubber plates (total thickness 0·6cm.), were two secondary plates, E and F, each 40cm. square. To these plates was attached the secondary circuit E G J H F, the form of which is represented in the figure. This latter circuit consisted of copper wire, diameter 0·13cm., and its total length from E to F was 4,200cm. A spark-gap with spherical terminals 2·5cm. in diameter was placed at C in the primary circuit, and another spark-gap with pointed terminals was sometimes inserted at J in the secondary circuit, although this latter spark-gap had no effect upon the phenomena to be described. The primary condenser was charged by means of a large Ruhmkorff coil excited by five storage cells with a total voltage of 10 volts. The current from these cells was made and broken by an automatic interrupter. Every time the primary condenser was charged a spark passed at C, causing an oscillatory discharge. A convenient method of forming a mental picture of the oscillation excited in the secondary circuit is the conception of Faraday tubes elaborated by J. J. Thomson in his 'Recent Researches in Electricity and Magnetism.' The oscillations of the primary acted inductively upon the secondary and sent out groups of Faraday tubes which travelled along the secondary circuit, with their ends on the wires, and lying chiefly in the space between them. At the end J they reversed their direction and travelled back along the circuit. The period of oscillation of the primary circuit was altered, until by trial it was found that groups of returning tubes met groups of advancing tubes between the points G and H. As the two sets of moving tubes were oppositely directed they annulled each other and produced a node. Thus a system of stationary waves is set up with a node at J, another node at G H, and a ventral segment at K L. The method of discovering when the circuits were in tune and of investigating the shape of the waves will be described later. The point to be noticed here is that the vibrations were sufficiently powerful to cause a luminous discharge on the surface of the wire where the accumulation of tubes was a maximum, *i.e.*, at K L, while at the nodal points J and G H the wire was entirely dark. Still further, the wave formation could be made apparent to the sense of hearing as well as that of sight; for, placing the ear within a few centimetres of the wire and walking beside it, a distinct crackling sound could be heard at the points K and L, whereas no such sound could be heard at G, J and H. By placing bits of glass tubing on the wire the sound was much intensified at the points K and L, and the phenomena made more striking. It might be supposed that by decreasing the capacity of the primary condenser, and therefore the period of its oscillation, the secondary circuit could be broken up into a new set of shorter stationary waves, with nodes at J and at points somewhere near K, L, G and H, and ventral segments between them. This was tried with perfect success, except that it was not possible to cause the light at K and L to actually disappear. There was decidedly less light at these points, however, than on either side of them. The light, of course, is simply that which always appears around wires carrying very high-potential currents, the interesting point being that it appears in some places on the circuit and not in others. The experiment showing how the circuit breaks up in several different ways would form a most beautiful lecture experiment.

"As a means of ascertaining when the circuits were in resonance, and of investigating the form of the wave in the secondary circuit, a bolometer similar to that designed by Paalzow and Rubens* was used.

"The bolometer as an instrument for measuring electric waves is so well known, that it is not necessary to state here more than its fundamental principles. It consists essentially of a well-balanced Wheatstone bridge, to one of the arms of which are metallically connected two small conductors. These conductors are brought near the circuit to be tested, and the oscillating charges induced in them and sent through the arm of the Wheatstone bridge develop enough heat to throw the bridge out of balance. By moving the conductors along the circuit different deflections are produced according to the magnitude of the charges on the wire in their neighbourhood, and thus an excellent estimate of the wave formation can be obtained. In the present case the conductors that were brought near the secondary circuit consisted of two pieces of wire insulated with rubber, bent into circles of about 2cm. radius, and fastened to a bit of pine-wood by means of a heavy coating of paraffin. The two wires of the secondary circuit passed through holes in this bit of wood in such a manner as to pass through the centres of the two circles. In the early part of the investigation the bolometer and galvanoscope were placed at a sufficient distance from the oscillating circuits to prevent any direct action of one on the other, and the leads running from the circular conductors to the bolometer consisted of long fine wires. Later, when longer circuits and longer waves were experimented with, great inconvenience

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* "Anwendung des Bolometrischen Princips auf Electrische Messungen," Wied. *Ann.*, XXXVII., p. 529.

was experienced from the long leads, since their relative position had considerable effect upon the galvanoscope deflections. In order to obviate this difficulty, short leads of heavily insulated wire were used, and the bolometer was placed on wheels and moved along from place to place. A bolometric study of the circuit just described showed the character of the oscillation to be that mentioned—namely, nodes at the points J and G H, and a ventral segment at K L. A careful run was made from one end of the circuit to the other, which furnished data from which a very regular curve was drawn.

"The insertion of a small spark-gap (1mm. to 3mm.) at the point in the secondary circuit marked J (Fig. 165, p. 500) had no appreciable effect upon the position of the nodal point G H, or of the point of maximum accumulation K L. The form of the wave was slightly altered for a metre on each side of J, and the bolometer showed a slight accumulation in the immediate neighbourhood of the spark-gap. This was probably due to the charging of the spark terminals to a sufficiently high potential to break through the dielectric. The fact that the insertion of a spark-gap into a secondary circuit in the manner described has no effect upon the length of the waves set up in that circuit was tested for a number of different cases (in none of which, however, was the length of the waves greater than in the present case), and found to be true in each one of them.

"In order to determine the time of vibration, we used a concave rotating mirror, and the images of the oscillating sparks were thrown on a sensitive plate. If the mirror rotated about a horizontal axis the photographs showed bright horizontal lines, perpendicular to which at their extremities extended two series of dots. The distance between successive dots was the distance on the plate through which the image of the spark-gap moved during the time of a complete oscillation. Hence, by determining the speed of the mirror, and measuring the distances from the mirror to the plate, the time of oscillation could be calculated. To measure the sparks we used a sharp pointer, moved at the end of a micrometer screw under a magnifying glass of low power. The instrument was originally intended for microscopic measurements, and was very accurately constructed. The rotating mirror was driven by an electric motor by means of a current from a storage battery of extremely constant voltage. To give great steadiness a heavy flywheel was attached to the axis of the mirror. The speed of the mirror was determined to within about one part in 500 by means of an electric chronograph. This apparatus, requiring great technical skill, was made for us by the mechanician of the laboratory. The mirror consisted of a thick piece of glass with a concave surface accurately ground for this research and silvered by ourselves.

"There are many advantages in photographing the secondary spark rather than the primary. In the first place, to properly photograph a spark it is necessary to use pointed terminals; but experiment has shown that the waves excited in a secondary circuit depend to a large extent upon the character of the primary spark, and that the most active sparks are those between metallic spheres with polished surfaces. It is true that waves can be produced by sparks between points, but the oscillations are not so powerful or well marked. In the second place, from the results obtained by Bjerknes, one would expect the oscillations in the secondary circuit to be much less damped than those in the primary. This expectation has been fully realised. Photographs show from ten to twelve times as many oscillations in the secondary as in the primary. The longest secondary spark we counted indicated 60 complete oscillations. In the third, and by no means the least important case, the question how close the resonance is does not affect the accuracy of the results. By photographing the sparks in the secondary the period of oscillation is determined, not of a circuit that is altered until by trial it is found to have as nearly as possible the same period of vibration as the circuit on which the length of the wave is measured, but that of the circuit along which the wave itself is actually travelling; and hence the conclusions in regard to the effect of damping reached by Bjerknes in his admirable Paper on 'Electric Resonance'* do not affect the accuracy of the results.

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* "Ueber electrische Resonanz," Wied. *Ann.*, LV., p. 121 (1895).

"The great difficulty to be overcome is the production of secondary oscillations that will produce sparks sufficiently bright to photograph. It is comparatively an easy task to photograph the primary spark, but in order to photograph the secondary the dimensions of the circuit must be chosen with great care.

"With a view to increasing the light of the spark, together with the length of the waves, it seemed desirable to lengthen the period of oscillation by enlarging the condensers rather than by increasing the self-induction of the primary circuit. A castor-oil condenser, therefore, was designed and constructed on the following plan:—Eight plates (25cm. by 20cm.) were cut out of sheet zinc, and were held in vertical planes side by side 2cm. apart by a suitable hard-rubber frame. The plates were entirely immersed in castor-oil contained in a glass jar. They were connected together in the manner shown in Fig. 166. The plates marked *a*, *c* and *e* were fastened to the conductor A B, and formed one armature of the condenser. Those marked *d*, *f* and *h* were joined to C D, and formed the other armature. The two ends of the secondary circuit E, G, J, H, F were fastened to the plates *h* and *g*. The plane of the secondary circuit was 50cm., and that of the primary 3cm. above the upper edge of the condenser plate. The total length of the secondary circuit from one condenser plate through E, G, J, H, F to the other plate was 6,338cm. The circuit consisted of copper wire (diameter 0·215cm.) supported at each end by suitable wooden frames, and also once in the middle by hard-rubber hooks, fastened by long pieces of twine to a wooden crossbar above. The

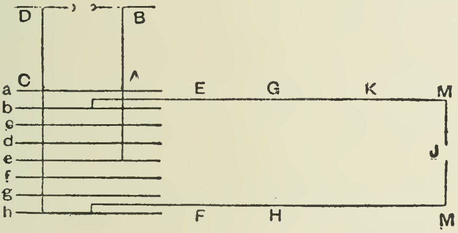

Fig. 166.

distances from F to E and from K to L were 30cm., and a spark-gap with pointed tin terminals was inserted at J. The primary circuit consisted of copper wire (diameter 0·34cm.). The distances between the two parts A B and C D were 45cm. The portion B D contained a spark-gap with platinum-faced spherical terminals, and was made so as to slide back and forth, to and from the condenser. The motion of this movable piece varied the self-induction, and therefore the period of oscillation of the primary circuit. By this means the circuits were brought into resonance. With certain arrangements of the condensers the resonance was very sharp, and the position of the movable portion could be determined to within 0·25cm. In the arrangement which was finally adopted the resonance was not so sharp. Even in this case the distance of the sliding part from plate *a* could not have been in error by more than 2cm. The length 65cm. was finally chosen for its value.

"The automatic current interrupter that worked so beautifully in connection with the Hertz vibrator would not operate well when used to excite the circuits just described. After trying many devices, we finally adopted an ordinary reed interrupter with a comparatively large hammer-and-anvil arrangement, which gave little trouble.

"At first it was found impossible to produce anything but a complex vibration in the secondary circuit when the spark-gap was open. Some slight evidence of resonance was obtained, but nothing of a decided character. When, however, the spark-gap was closed, very good resonance ensued, and a wave the length of which could be measured to within 0·4 per cent. was excited. Some photographs were taken of the spark in the secondary circuit, and they showed immediately the character of the complex wave formation. The secondary circuit could and did oscillate in three different ways, and the ratios of the periods were those of the notes in an open organ pipe, namely 1:2:3. Usually the lowest or fundamental oscillation together with one of the overtones was present; but several sparks were noticed that furnished unmistakable evidence of the simultaneous existence of all three. We have observed in a circuit 10,000cm. long the same peculiarities of oscillation, excited by a primary circuit that, judging from its dimensions, could not have been in resonance with the secondary. It was evident that the oscillation having a node between the points marked E and F (Fig. 166) is that whose period is one-third of the fundamental.

"A number of measurements of this period have been made, and from these values the velocity of the waves has been calculated. The results appear in the table below. As an average of five measurements of the wave length, none of which differed from the mean by more than 20cm., the value 5,888cm. was chosen. The distance from the mirror to the photographic plate in each case except the last was 300·1cm. Each of the first five values in the second column of the table is an average of 30 measurements of distances ranging in the neighbourhood of 1cm.

"The last line in the table contains the results of measurements on photographs of the primary spark instead of the secondary. In this case the distance from the mirror to the photographic plate was 311·5cm.

"These results were published as a preliminary record in the *American Journal of Science* for April, 1895. Since then the authors succeeded in producing much better waves and much more regular sparks, and discovered a phenomenon which renders a measurement on a photograph over a space where the dots are obliterated a questionable proceeding. The new data have given a value for the velocity more in accord with theory.

| Number of revolutions of mirror per second. | Distance between two successive points on plate. Centimetres. | Velocity of waves. Centimetres. |
|---|---|---|
| 71·2 | 0·05608 | 2·819 × $10^{10}$ |
| 70·85 | 0·05600 | 2·810 × $10^{10}$ |
| 70·7 | 0·05532 | 2·835 × $10^{10}$ |
| 71·3 | 0·05637 | 2·808 × $10^{10}$ |
| 70·8 | 0·05611 | 2·808 × $10^{10}$ |
| | Average...... | 2·816 × $10^{10}$ |
| 69·2 | 0·05340 | 2·988 × $10^{10}$ |

"Since the waves in the secondary were not well formed when the spark-gap was inserted, it seemed desirable to try to find an arrangement that would produce simultaneously a good wave and a photographable spark. A number of condensers with plates of different sizes and shapes and different substances for the dielectric were tried, and the apparatus to be described was finally adopted. The difficulties to be overcome were these. Too strong a reaction between the primary and secondary condensers could not be employed, because the increase in the damping of the primary due to the large amount of energy drawn off by the secondary made good resonance impossible. The amount of energy in the primary at full charge must be much greater than that in the secondary. On the other hand, the capacity of the primary condenser must not be too great; for the self-induction of the primary circuit would have to be proportionately small, and this, too, means an increase in the damping. The secondary condenser, too, must have a capacity of less than a certain magnitude in order that the node may fall on the circuit and not in the condenser plate. These points seem to indicate that small condensers are preferable to large ones; but a decrease in the size of the plates means a decrease in the light of the secondary sparks, and the sparks are at best barely photographable. Practically, therefore, our choice was much limited, and the particular arrangement to give the best results had to be selected by experiment after a long series of trials. The arrangement and dimensions of the apparatus finally adopted were as follows:—

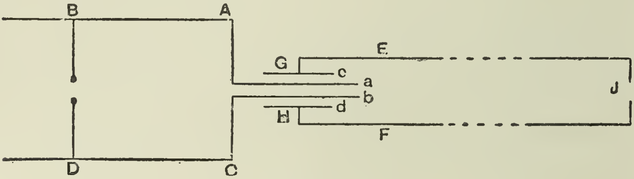

FIG. 167.

"Two metallic plates, *a* and *b* (Fig. 167), 30 × 30cm., placed in vertical planes, formed the primary condenser. The dielectric between them consisted of the best French plate glass obtainable (K = 8 + probably) and was 2cm. thick. Outside the plates *a* and *b*, and separated from them by a hard-rubber dielectric (K = 2 + about) 1·8cm. thick, were the secondary plates, 26 × 26cm. The primary and secondary circuits were joined to the condenser plates as indicated in the figure. The primary circuit lay in the horizontal plane passing through the centres of condenser plates, and consisted of copper wires 0·34cm. in diameter. In order to control the period of oscillation of the primary circuit, the portion B D containing a spark-gap with spherical terminals was made, as before, so as to slide along parallel to itself. The distance between the straight portions A B and C D was 40cm., and the lengths of A B and C D finally chosen for best resonance were 85cm. Most of the secondary circuit lay in a horizontal plane 16cm. above that of the primary. The lengths G E and H F, however, were bent down and fastened to the middle points G and H of the secondary plates. The circuit consisted of copper wire (diameter 0·215cm.), and its total length from G through J to H was 5,860cm. At J was a spark-gap with pointed terminals. With this apparatus we succeeded in producing a very regular wave formation, as indicated by the bolometer, even when there was a spark-gap at J. So many curves have been plotted and published to illustrate the characteristics of electrical waves that it does not seem worth while to add to the number here. It will be sufficient to state that the ratio of the maximum and minimum deflections in the bolometer was about 15 : 1, and that there was a node at J and another about 40cm. to the right of E and F.

"Upon photographing the secondary spark some curious phenomena were observed. In the first place, the dots usually appeared in pairs. There would be two black dots followed by a space where two or three dots either appeared faintly or were absent altogether; after that two black dots would reappear, followed again by a faint space, and so on for six or seven repetitions. All this, of course, occurred in a single spark.

"The explanation that first presents itself is that the two black dots are the result of the first two oscillations in the primary circuit, which, owing to the damping, are much more powerful than the others. If this were the true reason, the first of the pair of dots always ought to be blacker than the second, and every third dot ought to be the first of a pair. This is not the case, however. On the other hand, the phenomena cannot be explained as the result of a complex vibration, for the bolometer readings, taken only a few minutes before the photographic plates were exposed, and with exactly the same arrangement of apparatus, indicated extremely regular waves. A clue to the mystery was furnished by several sparks in which the dots made by one spark terminal had the characteristics just described, whereas those made by the other were quite regular. Following out this hint, we found that the particular substances used for the secondary spark terminals had a large effect upon the characteristics of the photographs. We tried spark terminals made of a number of different metals—tin, aluminium, magnesium, fuse-metal, &c.—and finally adopted cadmium as productive of the best sparks. In the case of cadmium the characteristics described are much less marked, and we have succeeded even in producing a few sparks in which no difference in blackness could be detected between one dot and the next. The photographs from cadmium terminals, too, are far more distinct and far more easily measured than those from terminals of any other metal that we tried.

"An interesting question arose here as to whether the distance between two successive dots would depend upon the period of oscillation of the primary circuit if the secondary were unaltered. To test this point the circuits were brought into resonance, and a photograph taken. The self-induction of the primary circuit was increased by about 20 per cent. of its value, and a second photograph taken. In the first case the distances between successive dots were all within 2 or 3 per cent. of the average obtained by measuring over several dots and dividing by the number of intervening spaces; whereas in the second case the measurements of some of the single spaces were from 8 to 12 per cent. greater than before, the average from long measurements being the same. This indicates that the vibrations of the secondary circuit are not necessarily perfectly regular, and at a distance apart fixed by the character of the circuit, but are to be looked upon as a series of pulses started travelling along the circuit and keeping at a distance from each other that is determined by the exciter. Owing to the fact that the damping of the primary is much greater than that of the secondary, the seventh and eighth pulses started are too weak to obliterate the first and second, which have travelled the length of the circuit and back. We should expect from this that the bolometer throws, which measure the average length of the wave, would not indicate a shifting of the node when the circuits are thrown slightly out of resonance, but that the minimum throws would be greater than when the circuits are exactly in resonance. This, as is well known, is what happens.

"The improved sparks which the new arrangement of apparatus and the use of cadmium as material for the spark terminals have enabled us to produce, have brought to light another interesting fact, namely, that even when the best resonance is obtained and the most regular wave formation is excited, the distances between the first three or four dots are slightly greater than the distances between three or four dots taken farther down the spark. The explanation we offer for this is the following, and it applies as a criticism to all cases in which waves are excited in a circuit by a neighbouring circuit possessing a much larger damping factor: The fact that the secondary waves last longer than the primary oscillations means that the last times that the waves travel over the circuit they do so under different end conditions from the first few times. The capacity of the secondary plates is slightly less after the primary spark has stopped than it was before, and therefore the length of the wires equivalent to the secondary plates is slightly less, and it takes a shorter time for the waves to travel along the circuit and back. Hence the observed decrease in the distance between the spark points and a certain mixing up of the dots, which occurs after the sixth or seventh oscillation (*see* Fig. 166). The sixth dot in the figure, apparently following its predecessor after about half an interval, is not a usual characteristic. In the vast majority of sparks the first few dots are far more powerful than those that follow them, and only occasionally do sparks occur that indicate more than five or six good complete oscillations. Hence these first few oscillations have the preponderating influence in fixing the length of the waves as indicated by the bolometer. In examining the sparks, therefore, we measured from the first oscillation as far down the spark as we could without passing over a space where dots were obliterated; and hence in every case we knew the number of dots between the points from which measurements were taken, and did not have to assume that good oscillations had occurred without affecting the plate.

"The following table, containing the results of our measurements with the improved apparatus, explains itself. The distance from the mirror to the photographic plate was 302cm. in each case:—

| Number of revolutions of mirror per sec. | Distance between successive points on plate. Centimetre. | Time of oscillation. Seconds. | Length of wave. Centim. | Velocity of wave. Centim. per sec. |
|---|---|---|---|---|
| 70·8 | 0·05028 | $1·871 \times 10^7$ | 5,670 | $3·030 \times 10^{10}$ |
| 73·7 | 0·05247 | $1·876 \times 10^7$ | 5,670 | $3·022 \times 10^{10}$ |
| 75·2 | 0·05536 | $1·940 \times 10^7$ | 5,670 | $2·923 \times 10^{10}$ |
| 69·5 | 0·05002 | $1·897 \times 10^7$ | 5,690 | $3·000 \times 10^{10}$ |
| 68·9 | 0·04900 | $1·874 \times 10^7$ | 5.690 | $3·036 \times 10^{10}$ |
| 69·0 | 0·04974 | $1·899 \times 10^7$ | 5,690 | $2·996 \times 10^{10}$ |
| 71·2 | 0·05075 | $1·878 \times 10^7$ | 5,660 | $3·014 \times 10^{10}$ |
| Average Value of Velocity ........... ..... | | | | $3·003 \times 10^{10}$ |

"With the exception of three preliminary trials, which gave values differing from the mean by 10 per cent. or by 12 per cent., these are the only determinations we have made. In some cases the waves in the circuit were just as good with the spark-gap as without it. In others there was a decided wave formation when sparks occurred, but the node was not quite so well marked. For this reason, and since it did not appear to make any difference in their length, the waves usually were measured without the spark-gap. As the sparks were quite regular, the difference in the bolometer readings must have been due to Faraday tubes that were reflected from the spark-gap without forming a spark and reversing themselves. The variation in the number of revolutions of the mirror per second is due to the fact that different cells were used to drive the motor on different occasions."

As an example of the data taken to ascertain the position of the node the authors give the following table. The top line contains the distances of the bolometer terminals from a pair of arbitrary fixed points on the circuit:—

| "Distances from fixed points ......... | 20cm. | | 40cm. | | 60cm. |
|---|---|---|---|---|---|
| Bolometer deflections .................. | 4·3<br>4·5<br>4·5 | ...<br>...<br>... | 4·0<br>4·1<br>4·0 | ...<br>...<br>... | 4·3<br>4·4<br>4·2 |
| Average deflections ............... | 4·43 | ... | 4·03 | ... | 4·3" |

The authors conclude with the following remarks:—

"From these deflections the position of the node was estimated. It appears from the best results that we have obtained that the velocity of short electric waves travelling along two parallel wires differs from the velocity of light by less than 0·2 per cent. of its value. It has been shown theoretically, that the velocity of such waves travelling along a single wire should be the velocity of light approximately. Our results, therefore, in a certain sense confirm the theory to an accuracy within their probable error. Theoretically, too, the velocity should be approximately equal to the ratio between the two systems of electrical units. The average of the best measurements of this ratio is 3·001, which is nearer the average velocity obtained by us than it is to the velocity of light."

# CHAPTER VI.

## THE INDUCTION COIL AND TRANSFORMER.

§ 1. **General Description of the Action of the Transformer or Induction Coil.**—In the previous chapters we have prepared the way, by a general study of the phenomena of the induction of electric currents, to enter upon a particular examination of the structure and action of the induction coil and transformer. The most logical method of procedure would be to trace first the historical development of these appliances from the initial scientific principles and facts accumulated by the early investigators. It will, however, be more advantageous to the student to defer this historical survey to a later portion of this treatise, and to direct attention at present to the actual electrical and magnetic operations which go on in the induction coil and transformer.

The induction coil and transformer, or converter as it is sometimes called, consists essentially of two conducting circuits which are both linked with a third or magnetic circuit, the three circuits being called respectively the primary circuit, the magnetic circuit, and the secondary circuit. The magnetic circuit may consist wholly of material having a magnetic permeability equal to that of air. A core of this kind may be obtained by winding the primary and secondary circuits on a ring of wood or on a paper tube, but whatever may be the exact material used, a transformer having a core made of a material, the magnetic permeability of which is equal to that of air, is generally called an *air core* transformer. Not very much interest attaches to the actions of an air-core transformer, for the reason that all practically-used transformers possess magnetic circuits consisting either partly or wholly of iron. If the magnetic circuit consists wholly of iron, the transformer is called a *closed-circuit* transformer; and if it consists partly of iron and partly of air, or other material of unit permeability, it is called an *open-circuit* transformer. The ordinary induction coil is of this last type. It has a core formed of a bundle of iron wires, and the magnetic circuit lies partly through this core and partly through the air outside it. The two conducting circuits consist generally of copper wires or bands insulated in various ways and wound on the core in sections or in overlying coils. In the chapter devoted to the practical construction of the transformer, the various methods of carrying this into effect will be described; meanwhile it will suffice to state that the two circuits, which are called respectively the primary and the secondary circuits, are well-insulated conducting circuits, the several turns of which are insulated from each other, the two circuits as a whole being also carefully insulated. The number of convolutions of each circuit may be, and generally is, very different. These are briefly spoken of as the primary turns and secondary turns. The iron core is constructed of laminated iron or iron wire, and the thickness or diameter of this is most usually about ·013 or ·014 of an inch. The object of this lamination is to prevent the production of local electric currents, called eddy currents, in the iron, which would represent an energy loss; but, as previously explained, this lamination does not, of course, prevent the hysteresis loss caused by the reversal of the magnetisation of the core.

The general action of the transformer consists in the production of a current, called a secondary current, by means of the variation in the magnetic induction in a magnetic circuit linked with it, and this induction is produced by means of another current called a primary current, the variation of the primary current producing a change of magnetic induction in a core, or magnetic circuit, which in turn creates an electromotive force in the secondary circuit linked with it.

Assuming that periodic currents are employed, it is evident, also, that the relative number of primary and secondary turns will be an important factor in determining the ratio between the mean-square value of the potential difference across the primary terminals and that across the secondary terminals of the transformer, and that it is in our power to increase or diminish this ratio. It is of course obvious, also, from first principles, that there can be no creation of energy, but only a transformation, and we can only alter the potential difference of the terminals of the two circuits at the expense of a change of corresponding current strength.

The most fundamental and valuable quality of an induction coil or transformer is, then, that it enables us to increase or reduce electrical potential difference or current strength in a definite ratio, and it is this transformation of energy which gives the apparatus its name. Transformers may therefore be classified according to the nature of the change in the character of an electric energy supply they are intended to produce.

Transformers may be constructed to act as (1) constant-potential transformers, or (2) constant-current transformers, and these may furthermore be divided into *step-up* transformers or *step-down* transformers, according as they are designed to increase or diminish in a certain ratio a potential difference or a current. Thus a transformer may be designed to work off a circuit of constant potential difference and to reduce that pressure in a certain ratio, called the transformation ratio. If it lowers the pressure it would be called a step-down constant-pressure transformer. In the same way a transformer may be employed to change a current strength in a certain ratio, or to convert from constant pressure to constant current.

The ordinary induction coil is a step-up transformer as generally used.

It is unnecessary to make any special classification depending on the character of the change of current employed in varying the induction, but it will be obvious to the reader that a closed iron-circuit transformer can only be used with alternating currents, and that for use with interrupted currents, as in the case of the ordinary induction coil, an open iron circuit or air core transformer must be employed.

Hence we have the following classification of transformers, understanding by this term any electrical arrangement consisting of two conducting circuits, both linked with a magnetic circuit, and in which the variation of current in one circuit gives rise to a production of electromotive force in the other:—

1. Transformers may be—
   - (*a*) Iron core transformers, with core wholly or partly of iron;
   - (*b*) Air core transformers, with core wholly of non-magnetic substance.

Iron core transformers may be—
   - (*c*) Closed iron circuit transformers, with core wholly of iron;
   - (*d*) Open iron circuit transformers, with core partly of iron.

Transformers may be used—
   - (*e*) To transform a potential difference in a constant ratio, called constant-pressure transformers;
   - (*f*) To transform a current strength in a constant ratio, called constant-current transformers.

Transformers may be employed—
   - (*g*) To raise pressure or current, called then step-up transformers;
   - (*h*) To lower pressure or current, called then step-down transformers.

The above is not a complete or exhaustive classification, but is sufficient to mark out broadly the various forms of the apparatus.

In whatever form it is used, the primary current must give rise to a variation of the magnetic induction in the core, and this in turn gives rise to an electromotive force in the secondary circuit.

The transformer or induction coil can evidently be operated either by intermittent, continuous, or by alternating currents. Whichever mode is adopted, the instrument is, of course, essentially and merely an energy-translating device. The current passing through the primary circuit magnetises the core. The intermittance or reversal of the primary current causes a variation or reversal of the magnetisation of the core. The variation or reversal of the magnetic induction in the core creates an electromotive force in the secondary circuit which is linked with it, and this sets up a secondary current in the secondary circuit if it is closed. The energy supplied to the primary circuit partly reappears in the secondary circuit, and the difference is represented by energy losses, called the copper losses, caused by the resistance of the conducting circuits, and partly by energy losses in the core, called the iron losses, and due to the hysteresis and eddy-currents set up in it.

The complete examination of the transformer involves, therefore, a knowledge of the manner in which the variation of the primary current, the magnetic induction in the core, the secondary current, and the secondary terminal potential difference is taking place when a certain assigned and varying primary terminal potential difference is created. It involves, also, a knowledge of the magnitude of the energy losses above described, and of the efficiency of transformation or the ratio between the power given to the external secondary circuit and the power given to the primary circuit. In addition to this, the relation of the values of the primary and secondary currents, and the primary and secondary terminal potential differences, for various states of the transformer from no load to full load, has to be ascertained. The following is, then, a summary of the operations going on in the transformer or induction coil which must be known before we can consider the action as fully understood; and the problem of transformer construction is to predetermine these variables from certain data, so as to foretell the result of construction. Given a potential difference created between the primary terminals of the transformer following any assigned law of variation, we shall have the following effects taking place as a consequence, and the practical problem is to determine or predetermine their mode and magnitude.

(1.) We have a primary current in the primary circuit following a certain mode of variation in strength, and causing a definite copper loss in the primary circuit;

(2.) A magnetic induction in the core following a definite mode of variation, and having a certain magnitude at every instant, this magnetic induction causing, by its variation, iron core losses due to hysteresis and eddy currents in the iron core;

(3.) A secondary current and secondary terminal potential difference following some definite law of variation and accompanied by a copper loss in the internal secondary circuit due to its resistance and a contribution of power to the external secondary circuit.

The transformer problem in all its completeness would be solved if we could in all case predetermine the above effects from the known primary potential difference. This, however, is not capable of being effected in a perfect manner, for reasons presently to be stated.

In early discussions of the transformer problem it was customary to make arbitrary assumptions as to the mode of the variation of the currents, the induction and potential differences. These artificial assumptions did not, however, assist real knowledge. The only useful method is to endeavour, in the first place, to ascertain what does go on inside the transformer, and then, on the basis of this analysis, to construct as far as possible a true working theory of the transformer. We have accordingly abandoned all discussion of imaginary transformers with air cores and currents and inductions, which are simple sine functions of the time and base, for such theory as we are able to build up on an actual knowledge of what does take place in the transformer. The only scientific method of treating the problems involved is, we repeat, first to endeavour to ascertain what are the actions really taking place, and to make them the basis for further reasoning.

**§ 2. The Delineation of Periodic Curves of Current and Electromotive Force.**—The method which has proved most fertile in enabling us to understand the operations taking place in the transformer is that which consists in graphically representing the form and relative position of the curves of periodic current and electromotive force in the two circuits and deducing that of the magnetisation of the core. When the primary electromotive force is an alternating one, derived from a single alternating-current dynamo which is accessible, the method of obtaining a graph of the various periodic quantities required is some modification of the arrangement first suggested by Joubert,* in which a circuit is closed for a very short but assigned period during the phase, and puts some electrical instrument intermittently into connection with the circuit, so that it reads, not the mean-square value of the current or potential difference, but the *instantaneous value* at the assigned instant. The modern method of delineating transformer curves is as follows:

Let the ordinates of the periodic curve in the Fig. 168 represent the varying potential difference between two conductors connected to an alternator, and let a condenser be connected across the circuit in series with a switch. If the switch is permanently closed, the condenser has a flow of current into and out of it, and the potential difference of its terminals varies periodically. If, however, the switch is closed intermittently at intervals which are equal to the periodic time of the alternator, then the condenser has a series of short contacts made with it, and its terminal potential difference is equal to the instantaneous value of the periodic potential difference of the circuit corresponding to the instant when the contact is broken. The difference of the potentials of terminals of the condenser has then to be determined. Several methods may be adopted. The condenser plates may be connected to the terminals of an electrostatic voltmeter, and the potential difference of the condenser plates thus determined. The condenser may be discharged through a galvanometer, and the

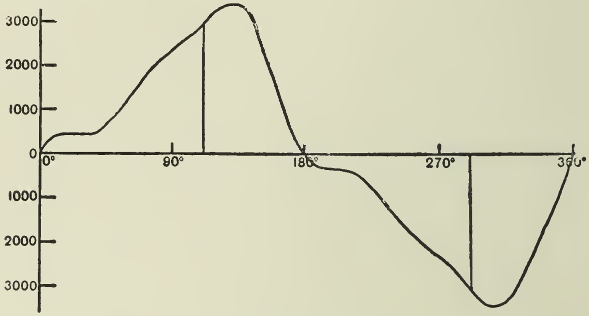

FIG. 168.

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* *Comptes Rendus* of the Academy of Science, France, Vol. XCI., July, 1880, p. 161.

condenser plate discharge determined by the quantity of the charge found in the condenser. We can thus charge the condenser by a series of contacts made with the circuit, for a very short time, always at the same position during the complete cycle or period of the varying potential difference. The condenser acquires, after a short time, a potential difference between its terminals which is exactly equal to that of the instantaneous value of the potential difference of the circuit at the instant, when the contact is made. If, instead of connecting the condenser across the circuit, it is connected to the extremities of a suitable non-inductive resistance inserted in any alternating-current circuit, we can obtain from its terminal potential difference the instantaneous values of the periodic current.

We have next to consider the various practical details of the process. If the alternator is accessible, or if a single alternator is providing the current, then the intermittent contact may be made by an apparatus fixed on the shaft of the alternator. If the alternator is not accessible, and if the primary potential difference is derived from a battery of alternators running in parallel, then it is necessary to operate an intermittent contact by means of a synchronous alternating-current motor, driven from that part of the circuit which is accessible. As this last method is capable of so much more general application than that of the contact-maker driven off the shaft of the alternator, we shall describe in some detail the arrangement of a suitable motor and associated apparatus. In addition to the early experiments by Joubert above mentioned, the delineation of periodic curves of current and electromotive force by means of an intermittent contact on the shaft of the alternator was suggested and carried out by Dr. Louis Duncan, and experiments by this method were effectively conducted by Messrs. Duncan, Hutchinson and Wilkes, and also by Prof. Ryan in the United States.* It has also been largely employed by Dr. J. Hopkinson, M. Blondel, by the author, and by many others for alternating current researches. The general arrangement of the apparatus required for the complete study of the periodic quantities in a transformer, under any circumstances, is as follows:—The principal instrument required is a small alternating-current synchronous motor. A suitable form has been devised and used by the author in investigations of this character,* and a view of it is shown in Fig. 169 on next page.

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* See a Paper by Messrs. Duncan, Hutchinson and Wilkes in the *Electrical World* of New York for March, 1888, referred to in *The Electrician*, Vol. XXI., p. 111, June 1, 1888. Also see Prof. Harris J. Ryan on Transformers in the *Transactions* of the American Institute of Electrical Engineers, Vol. VII., January, 1890.

The general details of the machine are as follows:—The motor, as constructed by the author, consists of two sets of field magnets, M M, which are secured to two cast-iron discs. Between these field magnets revolves a small armature, A, the iron core of which is formed of a strip of very thin transformer iron, wound up into a ring, the armature coils being wound upon this ring. The armature coils are joined up in series with one another, so as to give a series of contrary polarities round the iron ring. The diameter of this armature is about 6in. The field magnets have eight poles, and the armature eight coils. The field-magnet cores are bobbins about 2in. long and 1½in. in diameter, and when joined up in series in the proper manner the field magnets take a current of about 4 amperes to give them the proper amount of saturation. The armature is carried upon a hard wood boss fixed to a steel shaft. This steel shaft is carried through small ball bearings like bicycle bearings, the shaft being borne upon seven or eight balls carried in gun-metal cells. In order to prevent any side shake of the armature, there are at the opposite ends of the base cast iron pillars with a gun-metal screw at each end, against which the rounded end of the shaft bears. The shaft can thus be adjusted with great nicety, and runs with great freedom from friction. The ends of the armature coils are brought to two small insulated collars, fixed on the shaft, against which press two light brass brushes, marked B B, kept gently against the collars by means of an expanding steel wire, W. On the armature shaft is an ebonite disc, which carries a transverse steel slip let into it. Two insulated springs, S S, are carried upon a rocking arm, H; the rocking arm can be traversed over through half a circumference, and is centred upon the gun-metal end screw, which prevents side shake in the shaft. A pointer and

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* See *The Electrician*, Vol. XXXIV., p. 460, February 15, 1895, "On the Delineation of Alternating Current Curves."

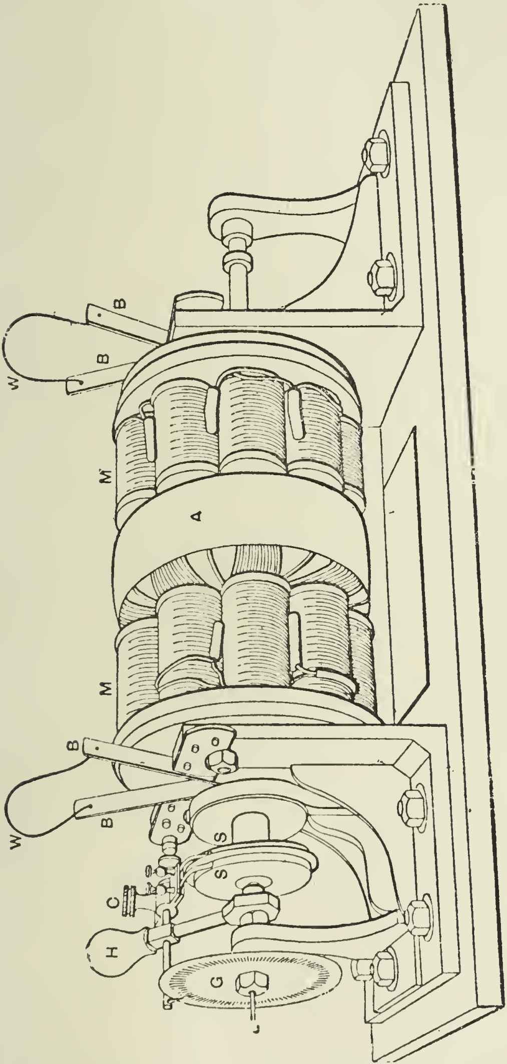

FIG. 169.—Alternating-Current Curve Tracer.

graduated scale, G, enables the exact angular position of the contact springs, S S, to be determined.

One of the springs, S, is carried on a small adjusting screw, so that one spring can be given a little lead over the other, and in this manner the duration of the contact made when the steel transverse piece passes underneath and electrically connects the springs S S is determined. By means of a set screw the springs can be lifted off from the ebonite disc, and their pressure also adjusted. This little synchronising motor with its attached contact-breaker forms the apparatus for determining the form of the current and electromotive-force curves. The motor is started in step with the alternating current flowing through the armature coils by passing round the end of the steel shaft which projects at the end opposite to the contact-breaker a tape or thin leather strap sprinkled with rosin. To start the motor the following arrangements are made:—The field magnets are excited by current obtained from a small secondary battery, or from any other constant source of continuous current. The armature circuit requires about 2 amperes to make it run properly. Let us assume that the potential difference curve is to be taken from two 100 volt alternating-current mains which come into a building. The armature of the motor is joined across these mains in series with two or three incandescent lamps placed in parallel. The field magnets being excited in the proper direction by a continuous current from a few secondary cells, the operator passes the strap or tape half round the shaft, and by pulling on one side of the tape the motor can gradually be set in rotation with an increasing speed. If the frequency of the alternating current is, say, 100 ∿, then the 8-pole motor has to be brought up to run at something approaching to 1,500 revolutions per minute before it will drop into step; but at a certain speed the incandescent lamps in series with the armature begin to blink, and by a little skill in adjusting the speed by suitable pulls on the tape the motor will drop into step and continue to run in synchronism with the circuits. If the springs are then put down gently upon the revolving contact piece, a contact is made from one spring to the other at an assigned position during the phase of electromotive force, depending on the position of the rocking arm. If the maximum electromotive force to be read does not exceed 160 volts, then by far the most convenient instrument to employ for reading the electromotive force, and the one which has been constantly employed in these tests, is Lord Kelvin's vertical or horizontal pattern multicellular voltmeter. As these voltmeters only begin to read at about 60 or 80 volts, it is necessary to add a constant electromotive force in series with them, and this is done by employing a set of small secondary cells. About fifty cells in one tray form a convenient arrangement, provided they have contacts at every cell, so as to take off any required electromotive force. The battery is joined up in series with the electrostatic voltmeter, and the terminals of the voltmeter are short-circuited by a condenser having a capacity of about half a microfarad. This arrangement of voltmeter and battery is then connected across the two points between which the potential is to be determined through the two springs S S. The motor being started, the needle of the voltmeter takes a certain deflection, which is due to the electromotive force of the cells, plus the value of the difference of potential between the mains at an instant depending upon the position of the rocking handle. By blocking up the voltmeter in this way, and using more or less cells as required, so as to add a known amount to the electromotive force to be measured, the electrostatic voltmeter can be employed to measure potential differences over the whole range varying from zero to 160 volts in either direction. These observations are taken at equal short intervals as the rocking arm H is swept over through a quarter of a circle. It is possible to thus measure the instantaneous values of the alternating potential difference between the two points at equi-distant instants throughout the phase. It has been found by experiment that this small alternating-current motor, when working on the circuits of any alternator of a size such as would be used in a generating station, does not sensibly affect the form of the curve of electromotive force. The motor is only used as a means of making the contact with a voltmeter at an assigned instant during the phase. The current which passes through its armature is not in any way measured or taken account of; the motor simply acts as a synchronising arrangement, which connects the contact-breaker *electrically* to the distant alternator.

The synchronising motor can, therefore, be set to run in step with any alternating-current circuit, and to make a contact or close a circuit for a short instant during every period at an assigned instant in the phase, which depends on the position of the rocking arm carrying the contact-making springs.

This apparatus may be employed to determine any of the curves of current or potential of a transformer, as follows:— Let us suppose the transformer is one intended to be operated with a primary terminal potential difference of 2,000 volts, and that the secondary terminal potential difference is 100 volts. Across the primary terminals of the transformer a non-inductive resistance is connected, which is divided into two sections in the ratio of 1 to 19, and in series with the primary circuit of the transformer is placed another non-inductive resistance having such a magnitude that, when traversed by the primary current of the transformer, it will create a fall of potential of about 100 volts. The synchronising motor is then suitably arranged to be operated from the same circuit which supplies the primary current for the transformer, and the armature circuit of the motor may be fed through a *step-down* transformer, which reduces this circuit pressure to a convenient magnitude.

The motor contacts are then arranged to close the circuit of a voltmeter, which is placed across one or other of the resistances. It is found necessary to connect a condenser across the terminals of the voltmeter to increase its capacity, or else the leakage of the voltmeter in the intervals between the moments when the contact is made causes irregularity and uncertain deflections of the instrument. The process of getting the complete set of curves of current and electromotive force of a transformer is then as follows: The curve of primary potential difference is obtained by connecting the voltmeter through the motor contacts across the smaller section of the divided resistance which bridges over the primary terminals. The motor being started, the voltmeter will read a potential difference, which is one-twentieth of the whole primary potential difference, and if the rocking arm of the motor is moved over step-by-step the indications of the voltmeter will successively give the values of this fraction of the primary potential difference corresponding to the different intervals of the whole period.

In the same way the curve of primary current can be obtained by connecting the voltmeter circuit across the terminals of the resistance inserted in series with the primary circuit of the transformer. The curves of secondary potential difference and secondary current, if necessary, can be obtained by connecting the contact-maker and voltmeter across the secondary terminals of the transformer when closed by a known non-inductive resistance.

The curves of current and potential thus obtained can be set down in a chart, the horizontal abscissæ in which represent fractions of the complete periodic time, and the vertical ordinates represent the instantaneous values of the potential differences or currents.

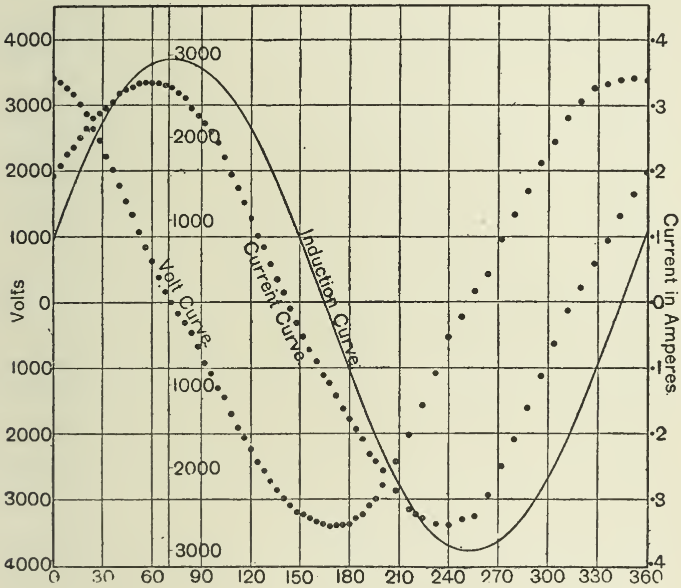

Fig. 170.—Primary Current, Primary Terminal, Potential Difference, and Induction Curves of Ganz 10 H.P. Transformer taken off a Kapp Alternator by the Alternating-current Curve Tracer.

In Fig. 170 is shown a set of curves taken from a Ganz transformer connected to a Kapp alternator. The dotted curve marked *volt curve* is the curve of primary electromotive force, or difference of potential at the primary terminals of the transformer. The dotted curve marked *current curve* is the curve of primary current, and the dots show the actual position of the observations. The figures on the horizontal line indicate degrees of phase. The scale on the left-hand side is the scale of volts, and that on the right-hand side the scale of current. From the curves of current and potential difference we can obtain the curve of magnetic induction in the core as follows:—Let $b$ be the induction density in the iron core—that is, the number of lines or unit tubes of induction per square centimetre of cross-section of the core. This induction density will not in general be the same in all parts of the core or the same at full load as at no load. If, in the first place, we consider the case of the transformer when the secondary circuit is open, we have a definite relation between the current and the primary circuit, the magnetic induction in the core, and the primary terminal potential difference or electromotive force at any instant. Let $i$ be the instantaneous value of the current in the primary circuit, $e$ the instantaneous value of the primary potential difference, and $b$ the induction density in the core. If $N_1$ is the number of turns of the primary circuit, R the resistance of the primary circuit, and S the area of cross-section of the core, then from the ordinary current equation for inductive circuits we have the relation

$$N_1 S \frac{db}{dt} + R i = e,$$

or

$$b = \int \frac{e - R i}{N_1 S} dt$$

$$= \frac{1}{N S} \int e\, dt - \frac{R}{N_1 S} \int i\, dt$$

$$= \frac{R}{N_1 S} \left\{ \int \frac{e}{R} dt - \int i\, dt \right\}.$$

In most cases of closed iron circuit transformers, the second term or integral on the right-hand side of the last equation is a very small quantity compared with the first term, and may be neglected. Hence when $\int i\, dt$ is small we can obtain the value of $b$ by integrating the primary E.M.F. curve, or by finding the value of $\int e\, dt$ between proper limits. Take, for instance, the case of the Ganz transformer, the curves of which are given in Fig. 170. The resistance of the primary circuit R is 2·5 ohms and the maximum value of the primary current $i$ is nearly 0·34 amperes. Hence the value of $\mathrm{R}i$ never exceeds 0·85 of a volt. The value of $e$, the primary electromotive force, varies from 0 to nearly 3,000 volts, and hence at any instant, except very near the moment when $e$ is zero, the value of $\mathrm{R}i$ is quite negligible compared with that of $e$.

In order to obtain the induction curve we have to integrate the curve of primary electromotive force, and to do this properly the following procedure must be followed. The whole area included by the primary E.M.F. curve must be obtained, and the integration of the curve must be started from that point on the horizontal axis which corresponds with the bisection of the area of the E.M.F. curve. Starting from this point the area of the E.M.F. curve is obtained by successive increments, and corresponding to the limit of each increment an ordinate is set up whose length on some scale is proportional to the whole area of the E.M.F. curve measured from the abcissa corresponding to the semi-area of the curve to the limit considered. This ordinate will then be an ordinate of the curve of induction. In making this integration the area of the E.M.F. curve below the time axis must be reckoned as negative. To obtain the absolute value of the induction at any point, the area of the E.M.F. curve must be reckoned out in volt-seconds and then divided by the value of $\mathrm{N}_1\,\mathrm{S}$, S being measured in square centimetres. The result must be multiplied by $10^8$ to reduce to C.G.S. measure and give the induction density in C.G.S. units.

In this manner the firm line curve which is marked *induction curve* in Fig. 170 was obtained. If the curve of secondary terminal potential difference has been obtained, we can, by a similar integration of this curve, obtain another induction curve which is generally practically identical with that obtained from the primary E.M.F. curve if the transformer secondary circuit is unloaded, but which does not agree with it if the secondary circuit is closed and a secondary current is being produced therein. Into the causes of this we shall enter later. The full set of transformer curves for the currents, potential differences and induction constitutes what may be called the *indicator diagram* of the transformer, and shows us all that is going on inside. The quick description of these curves becomes, therefore, an important matter. Many investigators have devised methods for expediting this process. One effective method was described by M. A. Blondel* in 1891. M. Blondel employs a rotating contact-maker with two brushes, and the contacts are so arranged that a condenser is periodically charged at a certain moment during the complete phase of the potential and then immediately afterwards is discharged through a galvanometer. The two brushes are fixed to an arm movable about an axis co-axial with that of the revolving motor or alternator, and this brush holder is revolved by clockwork at a regular rate. Hence the galvanometer indicates a current which is varied as the brush holder rotates. If the brush holder is held at rest, the galvanometer has a series of rapid charges from the condenser sent through it, and takes a steady deflection. If the brush holder rotates, this deflection varies from moment to moment, but at any instant is proportional to the instantaneous potential at which the condenser is being charged. If a mirror d'Arsonval galvanometer is employed, and the image of an illuminated opening thrown on a photographic scale which is moved transversely to the motion of the spot of light, a photographic trace of the alternating-current curve can be obtained. By using a pair of contact-makers and two galvanometers the current and E.M.F. curves can be delineated at the same time. Such a photographic record of the current and E.M.F. curve for an alternating-current arc lamp worked off a Meritens alternator is shown in Fig. 171.

A very similar arrangement has been described by Messrs. Barr, Burnie and Rodgers.† These investigators employ a revolving contact-maker of a particular kind. It is thus described by them: The shaft of the alternator or motor is fitted with a contact-making disc, and the contact brush is moved slowly and continuously through its successive angular positions. Contact is thus made each time at a slightly different position of the armature. Thus the potential difference at each contact differs slightly from that at the preceding contact. This potential difference is used to charge a condenser across the terminals of which is connected either a reflecting electrometer or a high-resistance galvanometer.

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* See *La Lumière Electrique*, September 12, 1891, and September 16, 1893; also see *The Electrician*, Vol. XXVII., p. 603.

† See *The Electrician*, September 27, 1895.

The deflection of the instrument so used follows the value of the potential difference of the wave form to be determined, and accurately follows it, for the mean rate of variation of the potential differences between the terminals of the condenser is exceedingly small in comparison with the rate of change of the electromotive force to be investigated.

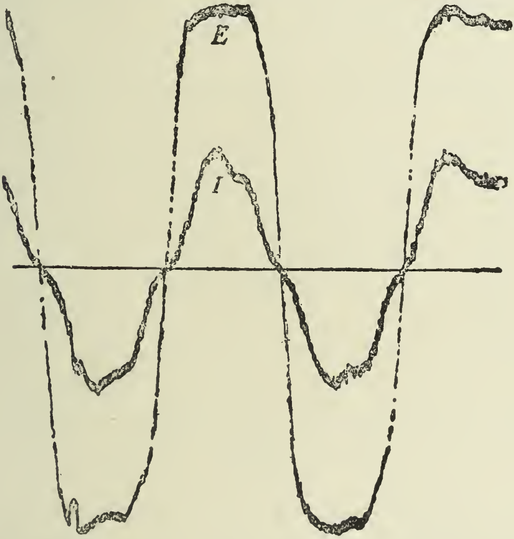

Fig. 171.—Photographic Trace of Current Curve I and Electromagnetic Force Curve E of an Alternating Current Arc Lamp.

Fig. 172 shows the arrangement of the contact disc and accessories, a galvanometer being used, but for which an electrometer might be substituted. In this diagram, for the sake of clearness, the vulcanite foundation work is omitted and the brass only shown. $D_1$ is the contact disc, with knife edge and contact brush, which is rigidly fixed to the shaft of the dynamo or motor. The rings $D_2$ and $D_3$, and the rods and brushes $B_1$ and $B_4$, are mounted on a vulcanite sleeve loose upon the shaft, and are revolved slowly. The brush $B_1$ is joined to the ring $D_2$, with which the brush $B_2$ is in permanent connection, so that when the brush $B_1$ makes contact with the knife edges the condenser C is charged to the potential difference between the terminals $T_1$, $T_2$. The condenser is throughout the whole revolution of the disc $D_1$ discharging through the galvanometer G by way of $B_1$, $B_4$, $B_5$, and the resistance R.

As the brushes $B_1$ and $B_4$ are moved slowly round, a succession of charges passes through the galvanometer, the value of each of which is proportional to the potential of the condenser—that is, to the potential difference of the points $T_1$, $T_2$. This contact apparatus, in fact, performs the operation of charging a condenser at a definite instant during the period at the terminals $T_1$ and $T_2$, which are the terminals of the alternating-current circuit under investigation, and then immediately afterwards discharges this condenser through a galvanometer. The galvanometer, therefore, gives a steady deflection which is proportional to the instantaneous potential difference between the points $T_1$ and $T_2$ at the instant corresponding to the moment when the contact with the condenser is broken.

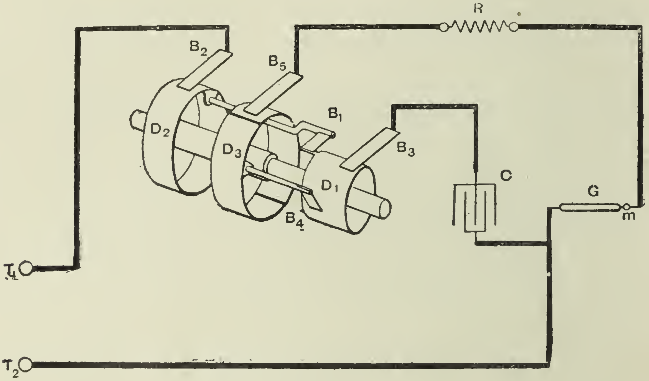

Fig. 172.

The rapidity with which the curves of instantaneous potential can be determined depends to a large extent upon the perfection of the insulation of the condenser and voltmeter.

If these leak to any sensible degree they lose charge in the intervals between the contacts, and the resulting permanent deflection is too small, and the time required for the voltmeter to take its full steady deflection when the place of contact is changed is greatly increased. Hence it is necessary to examine this question of leakage carefully before placing implicit reliance on the voltmeter and condenser actually used.

The value of the instantaneous potential may also be determined by balancing it against some point on a slide wire down which a known fall of potential is created by a battery. The arrangement known as a potentiometer consists of a uniform fine wire stretched over a scale down which a uniform fall of potential is created by a cell or two of a secondary battery attached to its extremities. If a sliding contact moves over this wire, we can insert between one end of the potentiometer wire and this slider any source of electromotive force, and, by moving the slider, balance the fall of potential down any length of the slide wire against this other potential difference. If the revolving contact-maker, connected in series with a condenser, is placed across a proper section of a divided resistance, which resistance is across the terminals of the transformer, the contact-maker will close the circuit of the condenser at equal periodic intervals and give it a potential which depends upon the position of the contact of the contact-maker. The potential of this condenser can then be measured on the slide wire, and, knowing the value of the two sections of the divided resistance, we are able to determine the value of the instantaneous potential difference between the terminals of the transformer.

A revolving contact-maker for determining alternating-current and potential curves has also been devised by Prof. Hicks.* In this instrument the same principle is adopted as in the one just described. A revolving contact-maker connects a condenser intermittently, but at definite instants in the period, to a source of alternating potential, and then in between these contacts discharges the condenser through a galvanometer. The shifting of the brush contacts varies the galvanometer deflection, but so that it is always proportional to the instantaneous value of the potential charging the condenser. Many other forms of apparatus have been described, but the principles are practically the same as those above referred to. In all cases a condenser is charged through a contact-maker, and the potential of the condenser determined by either a galvanometer or voltmeter. A few practical suggestions in connection with the construction of such contact-makers may be useful. In the first place, if a condenser and electrostatic voltmeter are used, care must be taken to see that both are very highly insulated. The use of the condenser is to act as a reservoir and supply the electrical leakage of the voltmeter. If a condenser of large capacity is employed, then the contact must be suitably prolonged, or else the condenser will not be charged completely during the contact. The contact springs sometimes give trouble by making imperfect contact with the disc, and too much pressure must not be applied to the brushes, or else they create a trail of metallic deposit on the insulating disc. The author has found a material called *stabilit* a very suitable insulating material for the construction of the insulating disc of the contact-maker, and the contact piece may be a transverse slip of steel or hard brass let into it. The contact springs are best made of steel, tempered and well cleaned at the contact surfaces. The contact-maker is best constructed by attaching a circular disc of stabilit or ebonite to the shaft of the motor or alternator and turning it up very accurately on the shaft. The metal contact slip is then let into the disc and a pair of insulated springs are carried on a rocking arm which moves round an axis co-axial with that of the motor or alternator. As the disc revolves the contact slip passes under the springs, and connects them together for an instant. These springs are connected to the circuit of the voltmeter and condenser, so that when the contact is made between the springs the condenser and voltmeter in parallel with it are connected to the alternating circuit under test for a short instant. The instant during the period when the contact is made can be varied by rocking over the arm carrying the springs.

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* See *The Electrician*, Vol. XXXIV., 1895, p. 698.

Prof. Ryan has suggested and employed a jet of salt water as a means of making an electric contact. Through this jet a steel needle passes at an assigned instant during the revolution.

With care and proper construction the steel spring contact-maker works very well, and is much more convenient to use than a contact in which a liquid jet is employed.

M. Blondel has described several forms of instrument, which he calls *oscillographs*, for the direct representation by optical means of the form of alternating-current curves, enabling us to project on to a screen a luminous line having the form of the alternating-current curve. For a description of these we must refer the reader to his Paper in the *Comptes Rendus*, Vol. CXVI., No. 10, March 6, 1893, p. 502, and to *The Electrician*, Vol. XXX., March 17, 1893, p. 571.

§ 3. **Discussion of Transformer Diagrams.**—The methods, some of which have been described in the previous section enable us, as it were, to look inside the transformer and observe the nature and order of the electrical operations taking place in it. We shall proceed to discuss some of the experimental results which have been thus obtained. In the first place, it must be noted that the curve of primary potential difference, or as it is generally called, the curve of primary E.M.F.,

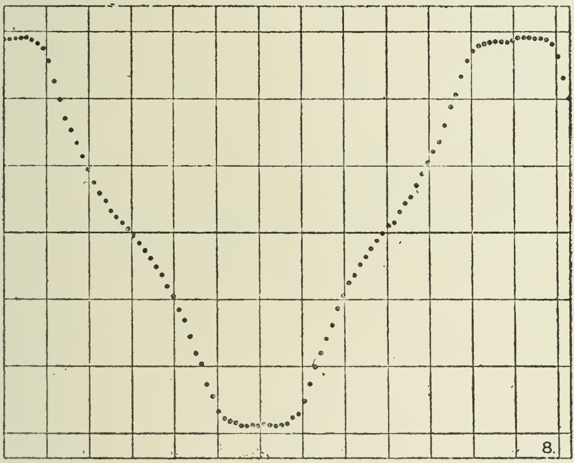

FIG. 173.—Curve of Electromotive Force of Thomson-Houston Alternator on Open Circuit.*

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* In Figs. 173, 174, 175, the dots represent the actual position of observations.

depends upon the construction of the alternator producing the electromotive force, and also upon the nature of the circuit, whether inductive or non-inductive, which that alternator is supplying. Any assumption that the curve of primary potential difference is always a simple sine curve is very far from true. The form of the curve of primary electromotive force is not even a fixed and independent attribute of the alternator. The form of the E.M.F. curve of the alternator may be quite different when taken on open circuit to that which it is when taken at the terminals of the alternator when this last is loaded with an inductive or non-inductive load of transformers. In Fig. 173 is shown the curve of electromotive force of a Thomson-Houston alternator at no load or on open circuit, and in Fig. 174 the E.M.F. curve of the same machine when actuating a load of transformers, the secondary circuits of which are lightly loaded. It will be seen that the second curve is quite different to the first, and that neither of them is even approximately a simple periodic curve. In Fig. 175 is shown the E.M.F. curve of a Mordey alternator at full load on a water resistance. It is, then, clear that no assumption must be made as to the con-

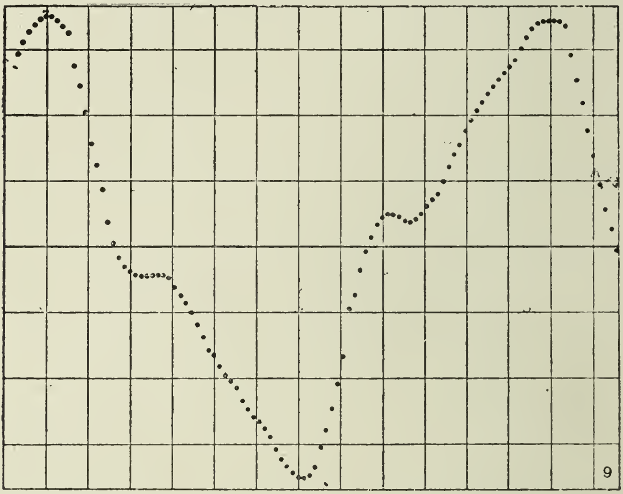

FIG. 174.—Curve of Electromotive Force of Thomson-Houston Alternator working on an Inductive Circuit.

stancy of the form of the curve of electromotive force of any alternator, but that the form of the curve of primary terminal potential difference of the transformer under test must always be determined.

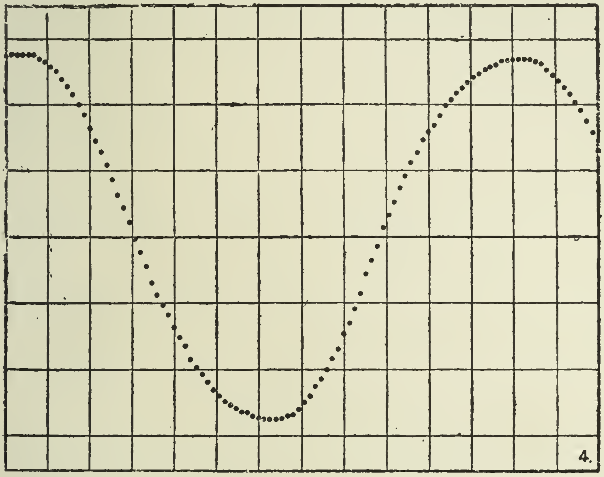

Fig. 175.—Curve of Electromotive Force of Mordey Alternator on Water-Resistance Load.

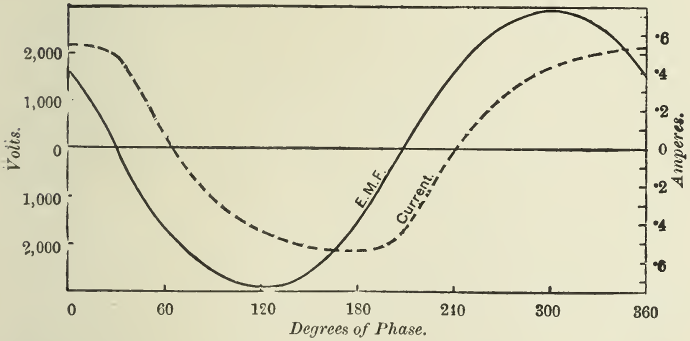

Fig. 176.—Primary E.M.F. and Primary Current Curves of Mordey Transformer, on Open Secondary Circuit, supplied off Mordey Alternator, with no other load.

We have, in the next place, to consider the case of the transformer when the secondary circuit is open or unloaded, and to inquire what under those circumstances is the form and relative position of the primary terminal potential difference curve and the primary-current curve. A number of examples of such curves are given in the diagrams on pages 537 to 541. In Figs. 176 to 183 are shown the primary-current curves and primary E.M.F. curves for transformers on open secondary circuit made by the Brush Electrical Engineering Company

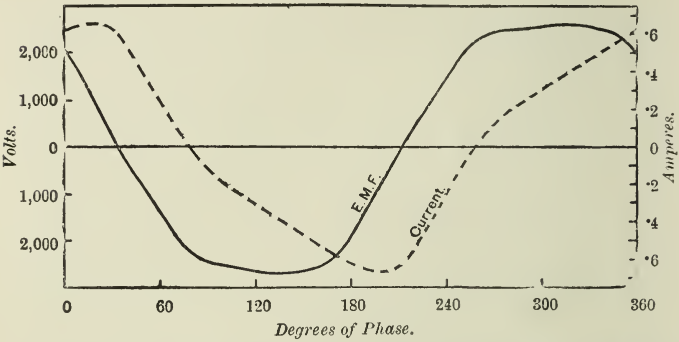

FIG. 177.—Primary E.M.F. and Primary Current Curves of Mordey Transformer, on Open Secondary Circuit, supplied off Mordey Alternator, furnishing Current also to other transformers lightly loaded.

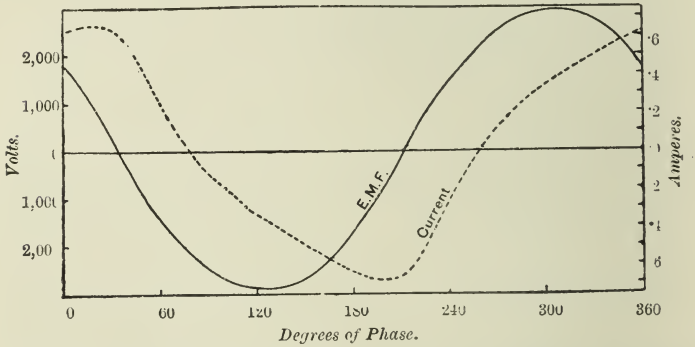

FIG. 178.—Primary E.M.F. and Primary Current Curves of Thomson-Houston Transformer, on Open Secondary Circuit, supplied off Mordey Alternator, with no other load.

and the Thomson-Houston Company, the electromotive force being supplied by Mordey or Thomson-Houston alternators in various states of load. The Mordey-Brush transformers are 50 kilowatt size and the Thomson-Houston are 30 kilowatt size.

It will be seen that the primary current under these conditions always lags behind the curve of primary E.M.F. or primary terminal potential difference. The primary current, when the secondary circuit of the transformer is open, is called the magnetising current of the transformer. Even if the curve of primary potential difference is nearly a true sine curve, the curve of primary current is not of a similar character, but is always more irregular. The form of the primary current curve depends not merely upon the form of

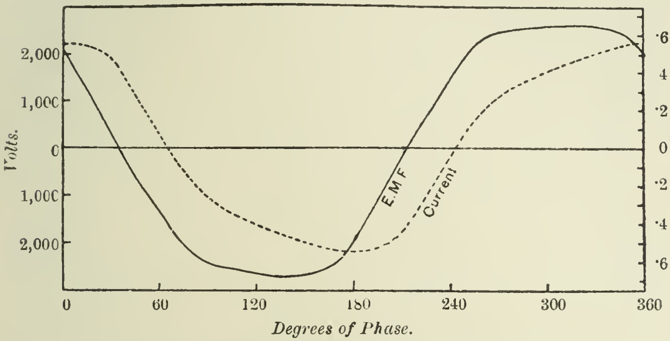

Fig. 179.—Primary E.M.F. and Primary Current Curves of Thomson-Houston Transformer, on Open Secondary Circuit, supplied off Mordey Alternator, furnishing Current also to other Transformers lightly loaded.

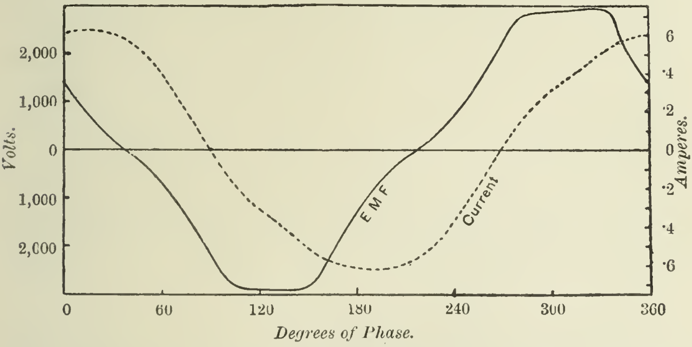

Fig. 180.—Primary E.M.F. and Primary Current Curves of Mordey Transformer, on Open Secondary Circuit, supplied off Thomson-Houston Alternator, with no other load.

the primary E.M.F. curve, but upon the nature of the iron used in the iron core and upon the structure of the transformer generally, so that the primary-current curves of two transformers by different makers will have different forms of

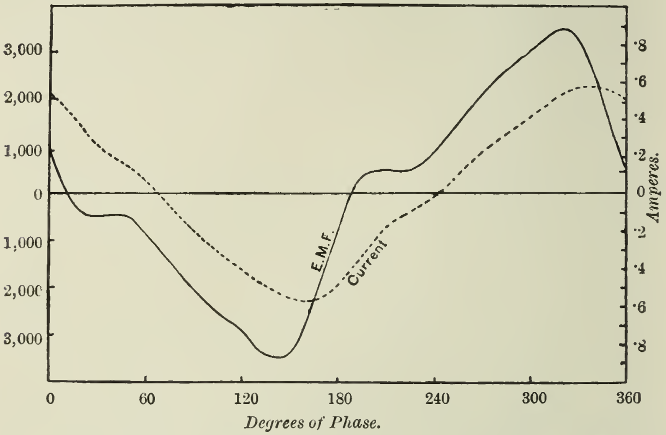

FIG. 181.—Primary E.M.F. and Primary Current Curves of Mordey Transformer, on Open Secondary Circuit, supplied off Thomson-Houston Alternator, furnishing Current also to other Transformers lightly loaded.

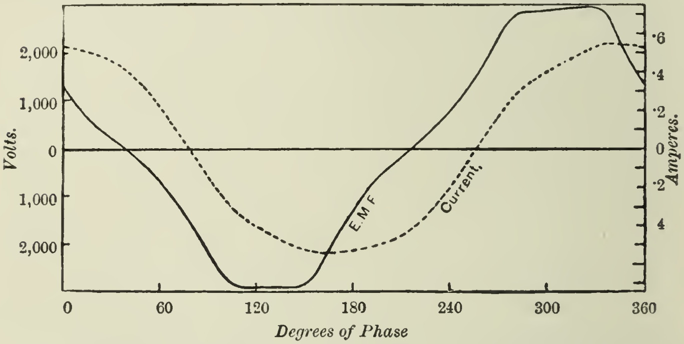

FIG. 182.—Primary E.M.F. and Primary Current Curves of Thomson-Houston Transformer on Open Secondary Circuit supplied off Thomson-Houston Alternator with no other load.

magnetising current curve, even if worked off the same alternator. This is well shown in the curves in Figs. 176 and 178, in which a Brush and Thomson-Houston transformer are worked off the same Mordey alternator. The curve of primary E.M.F. is the same in each case, but the curve of primary current is of a quite different form. This is brought about by differences in the reluctance of the iron circuit producing small differences in the form of the curve of magnetic induction in the core.

By comparing Figs. 176 and 181 it will be seen that the form of the curve of primary current is also dependent upon the form of the curve of primary E.M.F., and for the same transformer the curves of current may be considerably altered by supplying it off a different alternator, or off the same alternator in different states of load. In some alternators, such as the Mordey alternator, the armature reaction is very small and the form of the curve of electromotive force given by the machine is not very different whether the machine is worked on open circuit or on full load, on water resistance or on an inductive load. In the case of a machine with large armature reaction, the form of the curve of electromotive force will, under these various conditions, be greatly altered, and hence the form of the primary-current wave of trans-

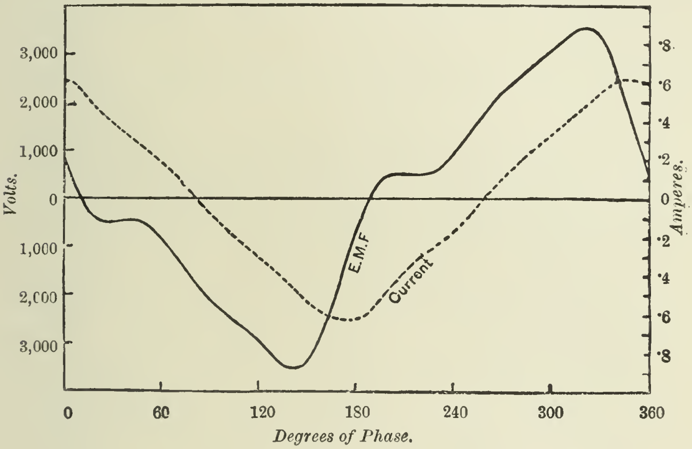

FIG. 183.—Primary E.M.F. and Primary Current Curves of Thomson-Houston Transformer, on Open Secondary Circuit, supplied off Thomson-Houston Alternator, furnishing Current also to other Transformers lightly loaded.

formers on open secondary circuit connected to it will be quite different also.

We pass on next to consider the form and position of the curve of secondary terminal potential difference or secondary E.M.F. when the transformer secondary circuit is open.

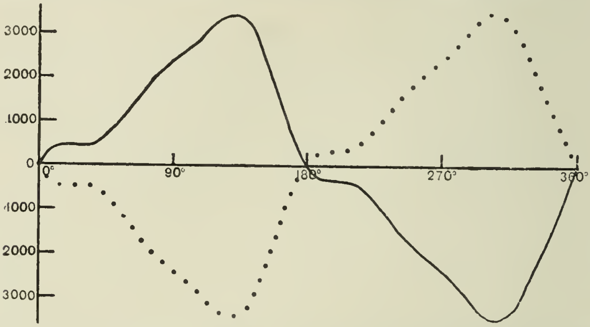

FIG. 184.—The Primary E.M.F. Curve (firm line) and Secondary E.M.F. Curve (dotted line) of a Thomson-Houston Transformer taken off a Thomson-Houston Alternator. The Secondary Curve is drawn to a scale which makes its Maximum Ordinate equal to that of the Primary Curve, and the Curves are seen to be identical in form.

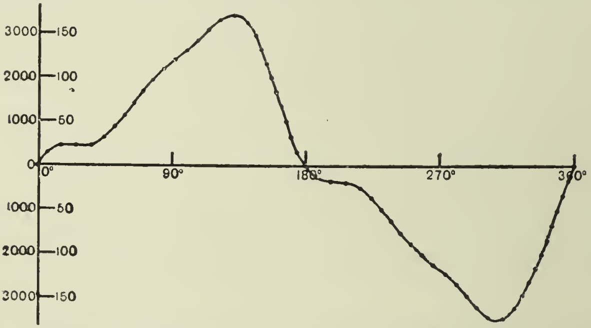

FIG. 185.—The same Primary and Secondary E.M.F. Curves, delineated in Fig. 184, are here drawn with the Secondary Curve (dotted) reversed and superposed on the Primary Curve to show its exact coincidence with the Primary Curve.

It is found that this curve of secondary electromotive force is, under these conditions, an exact copy on a reduced scale of the curve of primary E.M.F., and that it is in exact opposition to it in phase. In Fig. 184 are shown the curves of primary and secondary terminal potential difference of a Thomson-

Houston transformer at no load. The primary terminal potential difference curve or primary E.M.F. curve is represented in Fig. 184 by a firm line, and the secondary E.M.F. curve by a dotted line. The secondary curve has been drawn to such a scale that the ordinates of the secondary curve are equal to those of the primary curve. In Fig. 185 the curve of secondary E.M.F., represented by a dotted line, has been reversed and drawn over the primary to show the exact coincidence of the two curves.

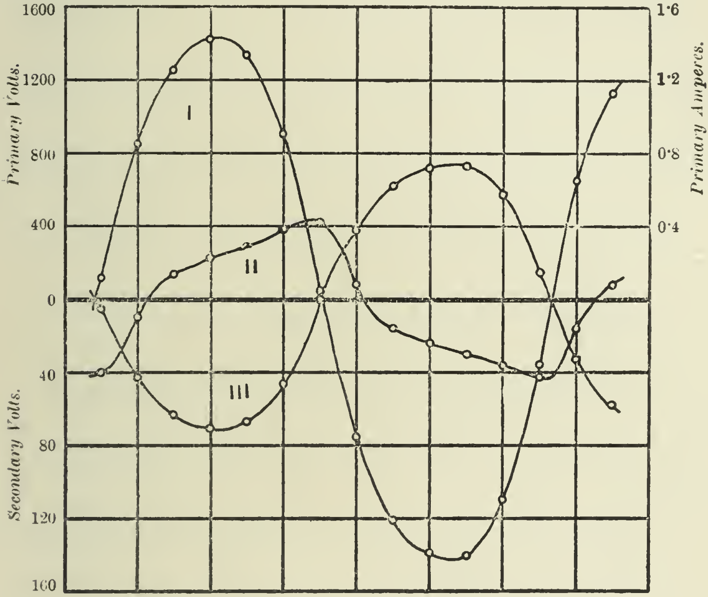

Fig. 186.—Primary E.M.F. Curve (I), Primary Current Curve (II) and Secondary E.M.F. Curve (III) of 10-light Westinghouse Transformer on Open Secondary Circuit.

This constitutes one of the most valuable properties of the transformer, viz., that it copies varying or periodic potential difference exactly to a reduced or increased scale. Hence, if we have a pair of terminals between which there is a periodically-varying potential difference having a $\sqrt{\text{mean-square}}$ value of, say, 2,000 volts, and we attach the primary circuit of a suitably-wound transformer to these terminals, we can produce a periodically-varying potential difference of lower or higher value, and the curve of which is an exact copy to a reduced or increased scale of the original. We shall see later on that useful applications can be made of this fact.

If the secondary circuit of the transformer is closed by a non-inductive resistance, such as incandescence lamps, then the curve of secondary terminal potential difference or secondary electromotive force undergoes a displacement and is brought forward or lags behind the curve of primary electromotive force. The reason for this is to be found in the magnetic leakage across the magnetic circuit which then takes place, and which will be discussed in a later section. The act of closing the secondary circuit of the transformer and producing a secondary current also effects a displacement in the position of the primary-current curve. As the transformer is loaded up the primary-current curve is displaced backwards, so that the lag in phase between the primary current and primary electromotive force is decreased. At full load the

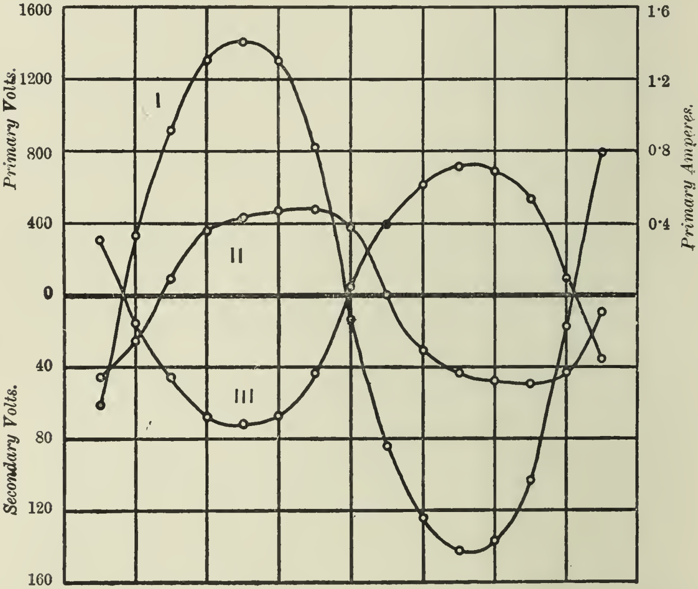

Fig. 187.—Primary E.M.F. Curve (I), Primary Current Curve (II) and Secondary E.M.F. Curve (III) of 10-light Westinghouse Transformer, loaded to one-tenth of full load.

primary current and secondary electromotive force are nearly in opposition of phase. This is seen to be the case by examining the series of curves in Figs. 186 to 189, which were taken by Prof. Ryan from a small Westinghouse transformer, in which magnetic leakage is not by any means absent.

We have next to consider the position of the curve of magnetic induction. The curve of induction is obtained, as already described, by integrating one or other of the curves of electromotive force. The process of obtaining a second curve, by taking as ordinates the area up to successive abscissæ of a first curve and plotting these areas as new ordinates to the limiting abscissæ, is a process which always has the effect of smoothing out irregularities in the original curve, so that if the first curve is one not far removed in form from a simple sine curve the second or integration curve will be more nearly still a simple sine curve.

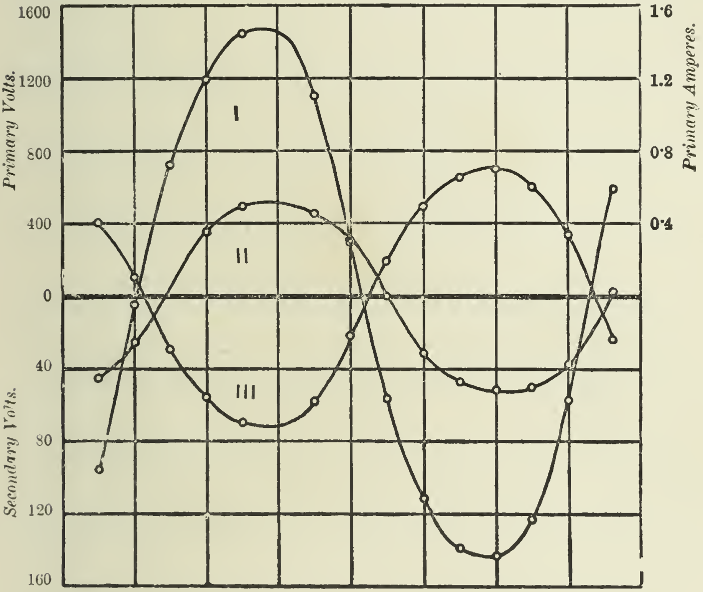

Fig. 188.—Primary E.M.F. Curve (I), Primary Current Curve (II) and Secondary E.M.F. Curve (III) of 10-light Westinghouse Transformer loaded to half load.

An analytical proof of this is as follows: If the ordinate $y$ of a periodic curve is represented by a Fourier series, as it can always be if periodic and single valued, then $y$ may be expressed by the series

$$y = \mathrm{A} \sin pt + \mathrm{B} \cos pt + \mathrm{C} \sin 2pt + \mathrm{D} \cos 2pt + \&c.,$$

where A, B, C, &c., are constants. Hence,

$$\int y\,dt = \frac{\mathrm{B}}{p} \sin pt - \frac{\mathrm{A}}{p} \cos pt + \frac{\mathrm{D}}{2p} \sin 2pt - \frac{\mathrm{C}}{2p} \cos 2pt + \&c.,$$

or $$p\int y\,dt = \mathrm{B} \sin pt - \mathrm{A} \cos pt + \frac{\mathrm{D}}{2p} \sin 2pt - \frac{\mathrm{C}}{2p} \cos 2pt + \&c.$$

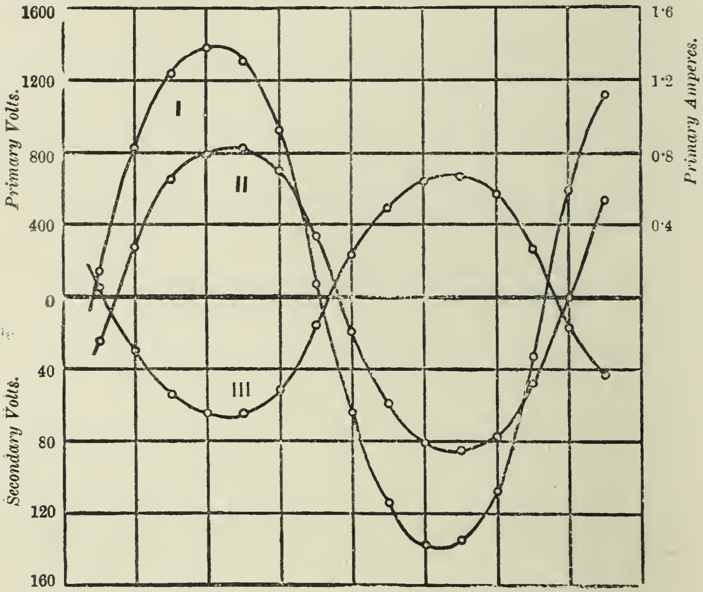

Fig. 189.—Primary E.M.F. Curve (I), Primary Current Curve (II) and Secondary E.M.F. Curve (III) of 10-light Westinghouse Transformer at full load.

It will be seen that the result of the integration has been to effect a change of phase of all the components and to weaken the higher harmonics by diminishing the coefficients which denote their amplitudes. Hence the process of forming a new periodic curve by taking as ordinates the area of a first periodic curve up to successive abscissæ always has the effect of wiping out irregularities of form of the primary curve, and yielding a curve more nearly a simple sine curve. It follows, therefore, that the curve of magnetic induction is always less irregular than the curve of primary or secondary electromotive force from which it is derived. From what has been already said, it will be seen that the curve of magnetic induction in the core has its maximum value at the moment when the electromotive force curve from which it is derived has its zero value. We may, then, sum up the general facts about transformer indicator diagrams by saying that when a transformer is at work we have—

1st. A varying potential difference between the primary terminals which follows a certain wave form depending—

(*a*) On the nature of the alternator;

(*b*) On the state of the load of that alternator, whether full or light, inductive or non-inductive;

(*c*) On the nature and construction of the transformer connected to the alternator.

No assumptions must be made as to the form of this curve, but in every case its true form at the terminals of the transformer under test must be determined. The curve of primary electromotive force has widely different forms in the cases met with in practice.

2nd. If the transformer has its secondary circuit open, we have a primary current flowing into its primary circuit which is called the magnetising current, and which lags in phase behind the curve of primary electromotive force. As the transformer secondary circuit is loaded up this curve of primary current is brought more into step with the primary electromotive force under the conditions that the load on the secondary circuit is a non-inductive load.

The curve of primary current is an irregular periodic curve the form of which is affected by the form of the curve of primary electromotive force and by the nature of the transformer, and may have very different forms as these two operating causes are changed.

3rd. We have a curve of secondary terminal potential difference which is in exact opposition to the curve of primary terminal potential difference when the transformer is on open secondary circuit, and which is an exact copy of the curve of primary potential difference to a reduced scale. This curve of secondary potential difference may be shifted forward in phase as the transformer secondary is loaded up, so as to come more nearly into opposition with the curve of primary current.

4th. We have a curve of magnetic induction, and this induction is not the same in different parts of the core, or the same on open secondary circuit as at full load. The form of the curve is always more nearly a simple periodic curve than is the form of the curves of primary and secondary terminal potential difference.

Each of these curves being a periodic single-valued curve, can be expressed by a Fourier series and analysed into constituent harmonics. Thus the ordinate $e_1$ of the curve of primary potential difference corresponding to any instant $t$ reckoned from the beginning of the phase can be expressed by the series

$$e_1 = \mathrm{E}_1 \sin p\,t + \mathrm{F}_1 \cos p\,t + \mathrm{E}_3 \sin 3\,p\,t + \mathrm{F}_3 \cos 3\,p\,t \\ + \mathrm{E}_5 \sin 5\,p\,t + \mathrm{F}_5 \cos 5\,p\,t + \&c.$$

The constant or first term of the Fourier series is zero, because the curve is always symmetrical above and below the axis of time. Moreover, only the odd harmonic constituents are present, viz., the harmonics whose wave lengths are one-third, one-fifth, &c., of the fundamental wave length, and if by any form of harmonograph we mechanically resolve any of these transformer curves, we find that they can be quite adequately represented by the first three odd terms of the Fourier series—that is to say, we can build up any transformer curve by adding together the ordinates of three simple periodic curves the wave lengths of which are in the ratio of 1:3:5, the amplitudes and relative positions being suitably chosen. The reason for the absence of the even harmonic constituents—viz., those whose wave lengths are $\frac{1}{2}$, $\frac{1}{4}$ that of the fundamental—is to be found in the peculiar symmetry of these transformer curves. On looking at any transformer curve it will be seen that it is of such a character that, if the portion below the time axis be considered to be reversed, we should get a repetition of the same form. Thus, a curve of electromotive force in Fig. 190, when so treated, becomes *rectified* into that in Fig. 191.

On considering, then, the form of any curve, it will be seen that the harmonic constituents must be such that if we move forward 180deg. along the time axis the value of the ordinate becomes negative but remains the same in magnitude.

If $e$ represents the ordinate of any curve at any point corresponding to an instant $t$, and if $p$ as usual is $2\pi n$, where $n$ is the frequency, then we can represent the value of $e$ by the series

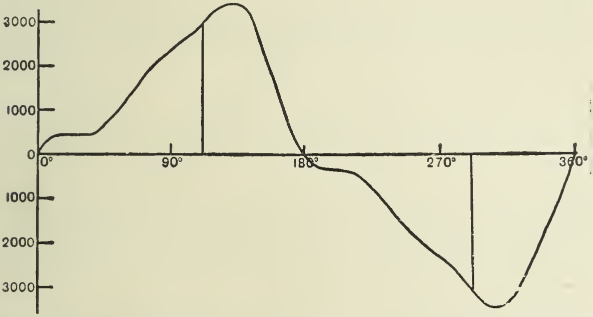

FIG. 190.—A Transformer Curve showing the typical symmetry of all transformer curves.

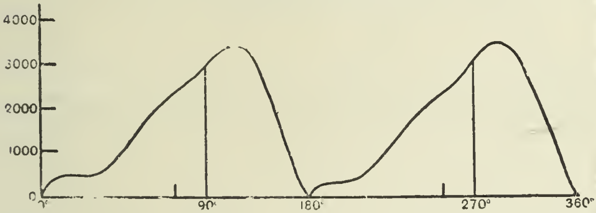

FIG. 191.—The same Transformer Curve shown in Fig. 190, but with the second half of the wave rectified to show the typical symmetry of the curve.

$$e = E_1 \sin pt + F_1 \cos pt + E_2 \sin 2\,pt + F_2 \cos 2\,pt$$
$$+ E_3 \sin 3\,pt + F_3 \cos 3\,pt + \&c.$$

The harmonic constituents must be such that if we put $(pt+\pi)$ for $pt$ the value of $e$ becomes $-e$.

It is easily seen that, since $\sin\,(pt+\pi) = -\sin pt$ and $\sin\,\{3\,(pt+\pi)\} = -\sin 3\,p\,t$, &c., whereas $\sin\,\{2\,(pt+\pi)\} = \sin 2\,pt$, the essential condition is that only the odd harmonics must be present. Hence the expansion of the ordinate of the real transformer curve can only contain the 1st, 3rd, 5th, &c., terms. As a matter of experience it is found that any transformer curve met with in practice can very nearly be represented by the first three odd terms of the series, and hence any observed transformer curve can be very quickly analysed into its constituents by the arithmetical process explained on page 92.

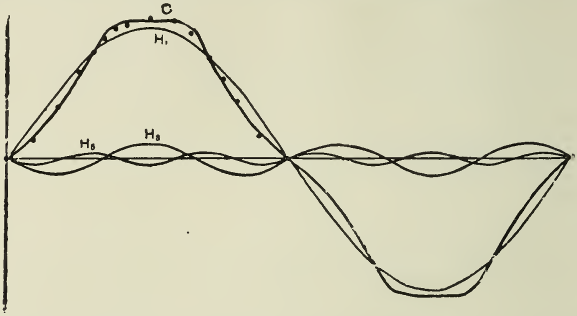

FIG. 192.—The Harmonic Analysis of the Curve of E.M.F. of a Thomson-Houston Alternator at no Load. The thick Curve C is the Curve of E.M.F., and the Curves marked $H_1$, $H_3$, $H_5$, are the Harmonic Constituents with Wave Lengths in the ratio of 1, 3, 5.

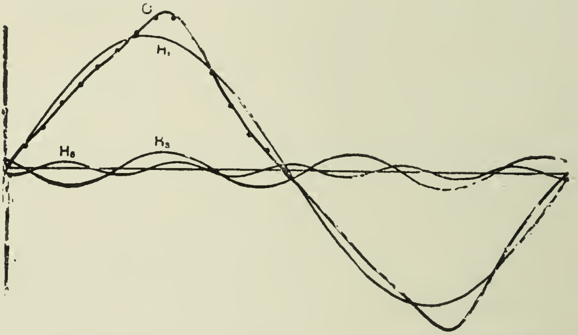

FIG. 193.—The Harmonic Analysis of the Curve C of E.M.F. of a Thomson-Houston Alternator partly loaded up on water resistance. The Harmonic Constituents of the Curve are represented by the Curves marked $H_1$, $H_3$, $H_5$.

By the use of mechanical harmonographs or analysers this can, of course, be very easily done, and an illustration is given in Figs. 192, 193 and 194 of the E.M.F. curve of a Thomson-Houston alternator so analysed. The curves given in Figs. 192, 193 and 194 were analysed for the author by Mr. G. U. Yule, with his mechanical harmonograph. It will be noticed that when the curve C is symmetrical, the harmonic constituents start from the same point, and have no lag relatively to one another.

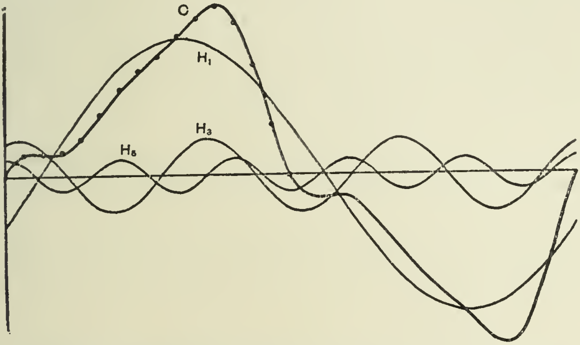

Fig. 194.—The Harmonic Analysis of the Curve C of E.M.F. of a Thomson-Houston Alternator, partly loaded up on Inductive Resistance. The Harmonic Constituents of the Curve are represented by the Curves marked $H_1$, $H_3$, $H_5$, with wave lengths in the ratio of 1, 3, 5.

§ 4. **Derivation of Curves of Power and Hysteresis.**—From the curves of current, electromotive force, and induction obtained as above described we can construct two other curves which give us the variation of the total power supplied to the transformer, and the total loss in the iron core per cycle. These curves are obtained as follows:—Let us assume that a set of transformer curves has been taken when the transformer is on open secondary circuit. Taking the curves of primary current and primary terminal potential difference, we multiply together (as explained on page 153, § 23, of Chapter III.), the corresponding ordinates of the two curves for abscissæ taken at equidistant points on the time axis, and set up a new ordinate representing the value of the product $e\,i$, where $e$ is the primary potential difference and $i$ the primary current at the same instant. This product set off as an ordinate defines another curve called the *power curve*, and the true mean ordinate of this power curve gives us the mean power taken up in the transformer at no load. To obtain the true mean ordinate of the power curve we have to integrate the whole area included between the power curve and the time axis, and to reckon those areas which lie above the time axis as positive and those which lie below as negative. The total area of the positive and negative parts algebraically added, and divided by the length of the axis representing one complete period, gives us the true mean ordinate of the power curve. Hence, we can, from the transformer diagram taken on open secondary circuit, determine the mean power taken up in the transformer. The amount dissipated in heat in the copper of the primary circuit is generally an exceedingly small fraction of the total loss, and hence the mean power obtained as above is practically the value of the power taken up in the iron core.

The analytical expression of this fact is as follows: Taking the fundamental equation for the transformer on open secondary circuit, viz.,

$$e_1 = \mathrm{R}_1 i_1 + \mathrm{S}\,\mathrm{N}_1 \frac{d\,t}{d\,t},$$

we multiply the equation all through by $i_1$ and obtain

$$e_1 i_1 = \mathrm{R}\, i_1^2 + \mathrm{S}\,\mathrm{N}_1\, i_1 \frac{d\,b}{d\,t},$$

or

$$e_1 i_1\, d\,t = \mathrm{R}\, i_1^2\, d\,t + \mathrm{S}\,\mathrm{N}_1\, i_1\, d\,b.$$

If this last equation is integrated between the limits 0 to $\frac{\mathrm{T}}{2}$, where T is the complete periodic time, and each integral multiplied by $\frac{2}{\mathrm{T}}$, we obtain an expression for the mean power given to the transformer during one half-period. Thus,

$$\frac{2}{\mathrm{T}}\int_0^{\frac{\mathrm{T}}{2}} e_1 i_1\, d\,t = \frac{2\,\mathrm{R}}{\mathrm{T}}\int_0^{\frac{\mathrm{T}}{2}} i_1^2\, d\,t + \frac{2\,\mathrm{S}\,\mathrm{N}_1}{\mathrm{T}}\int_0^{\frac{\mathrm{T}}{2}} i_1\, d\,b.$$

The first term on the left hand side represents the true mean power given to the transformer in one half-period. The second term represents the power dissipated as heat in the primary circuit in one half-period, and the third term represents the power dissipated on eddy currents and hysteresis in the core in the same time. If a horizontal line is taken, and from an origin distances are set off right and left to represent the varying values of the primary current $i$ during the period, and vertical ordinates corresponding to these abscissæ taken to represent the values of the induction density $b$ in the core at the same instant, then a curve will be defined which will be a cyclic curve, and will give us the total core loss per cycle when the numerical value of its area is multiplied by the factor $N_1 S$. If the iron core is well laminated, eddy current loss will be practically absent; and the value of this area, therefore, will give us the true hysteresis loss in the iron.

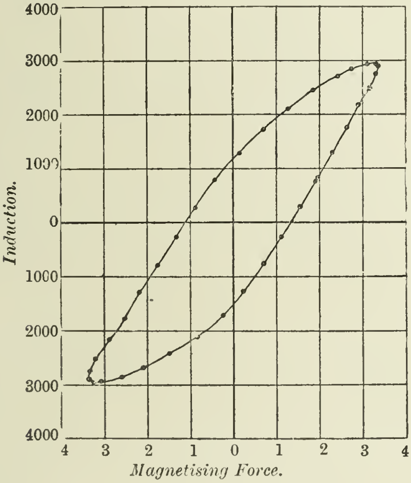

FIG. 195.—Hysteresis Curve of a Ganz Transformer.

Hence, such a curve is called the hysteresis curve of the core. In Fig. 195 is shown the hysteresis curve of the Ganz transformer, so obtained from the current and induction curves of the same transformer as given in Fig. 170.

In order to obtain the correct numerical value of the hysteresis loss per cycle it must be noted that if all the quantities S, $i$ and $b$ are measured in C.G.S. measure, the value of the integral $S N_1 \int i \, db$ will give us, when taken round one complete cycle, the value of the core loss in *ergs* during one complete period. And this value has to be divided by $10^7$ to reduce it to *joules*. Since the integral $S N_1 \int i\,db$ can be written $\int (N i)\, d(S b)$, we see that the core loss per cycle in joules can at once be obtained by taking the area of a loop curve, the horizontal ordinates of which represent the periodic values of the primary ampere-turns, and the vertical ordinates the corresponding total core induction during one complete period, taken in a unit equal to $10^8$ C.G.S. units of magnetic induction.

If the frequency is $n$, then $n$ times the above integral gives the loss in the core per second; and this should have the same numerical value as the mean ordinate of the power curve which measures the same quantity. The practical rule, therefore, for obtaining the core loss in the transformer due to the hysteresis and eddy current loss which may be present is as follows: Draw two axes at right angles; on the horizontal axis set off right and left from the origin distances which represent the primary ampere-turns for the different instants during one complete period. At these points set up ordinates which represent the total induction in the core measured in units each equal to $10^8$ C.G.S. units of magnetic induction, and complete the looped curve defined by these ordinates. The area of this curve, measured in terms of the area of a rectangle one side of which is the length taken to represent one ampere-turn and the other side is the length taken to represent $10^8$ C.G.S. units of induction, will give the value of the core loss per cycle in joules, and multiplication of this value by the frequency $n$ will give the mean loss of power in the core in *watts*. The number so obtained will agree closely with the value of the mean ordinate of the power curve in those cases in which the copper loss in the primary circuit when the transformer is not loaded can be neglected.

The form of this hysteresis loop will depend upon the manner in which the magnetic induction in the core varies with the magnetising force, and, as we shall see presently, the area of this hysteresis loop depends, amongst other things, upon the form of the curve of primary impressed electromotive force.

§ 5. **The Efficiency of Transformers.**—If the secondary circuit of the transformer is closed through a resistance, and the transformer is therefore loaded up, the power given to the primary circuit in part reappears in a transformed form in the external secondary circuit. As by far the most frequently presented case in practice is that in which the resistance which closes the secondary circuit consists of incandescence lamps or other practically non-inductive resistances, we shall, therefore, in the first instance assume that the external secondary circuit is an inductionless resistance. Under these conditions the secondary current is, to a close approximation, in step or synchronism with the secondary electromotive force, and the mean power given to the external secondary circuit is measured by the product of the mean-square value of the secondary current strength and the mean-square value of the potential difference of the secondary terminals. If we denote by $P_2$ the power thus given up to the external secondary circuit, and similarly by $P_1$ the power given up to the primary circuit, the ratio of $P_2$ to $P_1$ is called the *efficiency* of the transformer. This efficiency is generally expressed as a percentage, and will be denoted by the symbol $\epsilon$. Hence

$$\epsilon = 100 \frac{P_2}{P_1}.$$

The difference between $P_1$ and $P_2$ is represented by the power lost in the core and dissipated in the copper circuits of the transformer. If the symbol $C_1$ stands for the mean-square value ($\sqrt{\text{mean}^2}$) of the primary current, and $C_2$ for that of the secondary currents at any time, and if $R_1$ and $R_2$ are the resistances of these circuits when warm and at that time, then the power wasted in the primary and secondary circuits respectively is $C_1^2 R_2$ and $C_2^2 R_2$, and if H is the core loss, viz., the hysteresis and eddy-current loss, then

$$P_1 - P_2 = C_1^2 R + C_2^2 R_2 + H,$$

on the assumption that there are no eddy-current losses or energy dissipations in the copper circuits, or in the iron case or framework of the transformer.

One of the most important measurements, therefore, which it is necessary to make in connection with transformers is the measurement of the power given to the primary circuit. We shall defer to a later chapter on transformer testing a full discussion of the various methods which can be employed for determining either the magnitude of the quantity $P_1$ or of the difference $P_1 - P_2$ in the case of a transformer at any load. It may suffice to state at present that one way in which this measurement can be made is by means of a properly constructed wattmeter, which measures directly the power $P_1$ given to the primary circuit. The objection to this method is that any error made in evaluating $P_1$ appears to the same extent and percentage in the ratio of $P_2$ to $P_1$, and therefore in the efficiency. Hence other methods have been devised for measuring directly the difference $P_1 - P_2$. However the value of the efficiency may be determined, the results are best set down in the form of an efficiency curve as follows: Each transformer is constructed to give safely a certain output of power to the secondary external circuit, which is called its full load, and is stated generally in watts or kilowatts. The load on the secondary circuit in any other cases can be expressed as a fraction of the full load. To draw an efficiency curve for any transformer, a horizontal line is taken, on which are marked off the decimal fractions of the full load, and at these points are set up ordinates which represent the percentage efficiencies at these loads, viz., the value of $100\frac{P_2}{P_1}$, where $P_1$ is the power given to the primary circuit and $P_2$ is the power given to the external secondary circuit. The extremities of these ordinates delineate the efficiency curve.

In Figs. 196 and 197 are shown the efficiency curves of various transformers. It will be seen that the chief difference is that the more modern transformer has a higher efficiency at the low loads. A good transformer of any moderate size should have at least 80 per cent. efficiency at one-tenth load. And larger transformers of 15 and 20-kilowatt size and upwards will reach to 90 per cent. efficiency or more at one-tenth of full load.

Another method of delineating the efficiency is to plot the difference $P_1 - P_2$ in terms of $P_2$; in other words, to plot a curve the abscissæ of which represent the secondary output $P_2$, and the ordinates of which represent the total loss of

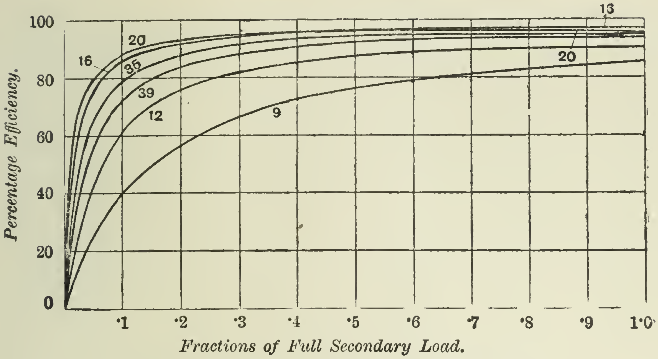

FIG. 196.—Efficiency Curves of Transformers.

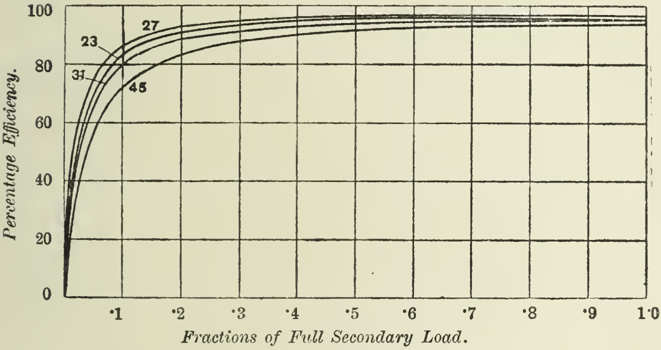

FIG. 197.—Efficiency Curves of Transformers.

In the above diagrams, horizontal distances represent the decimal fractions of full secondary load, and vertical ordinates the percentage efficiency corresponding thereto. The numbers against the curves refer to the following transformers :—

| | | | |
|---|---|---|---|
| No. 9. | 5 horse-power | Ferranti Transformer ... ... | 1885 type. |
| „ 12. | 5 „ | „ „ ... ... | 1885 type rewound. |
| „ 20. | 15 , | „ „ ... ... | 1892 type. |
| „ 23. | 15 „ | „ „ ... ... | 1892 type rewound. |
| „ 27. | 20 „ | „ „ ... ... | 1892 type. |
| „ 31. | 6 kilowatt | Mordey Transformer ... ... ... | 1892 type. |
| „ 35. | 4·5 „ | Thomson-Houston Transformer ... | 1892 type. |
| „ 39. | 4 „ | Kapp Transformer ... ... | 1892 type. |
| , 45. | 3 „ | "Hedgehog" (Swinburne) Transformer | 1892 type. |
| „ 16. | 6·5 „ | Westinghouse Transformer ... | 1892 type. |

power in the transformer, viz., $P_1 - P_2$. In Fig. 198 is shown such a curve drawn for a 6,500 - watt Westinghouse transformer. The ordinates of the upper curve give the value of $P_1 - P_2$ corresponding to the secondary output $P_2$.

One important question which arises in this connection is whether the true iron core loss by hysteresis remains constant at all loads of the transformer. This was at one time denied. It has, however, been shown by careful experiments that the hysteresis loss in the iron core is sensibly constant at all loads.*

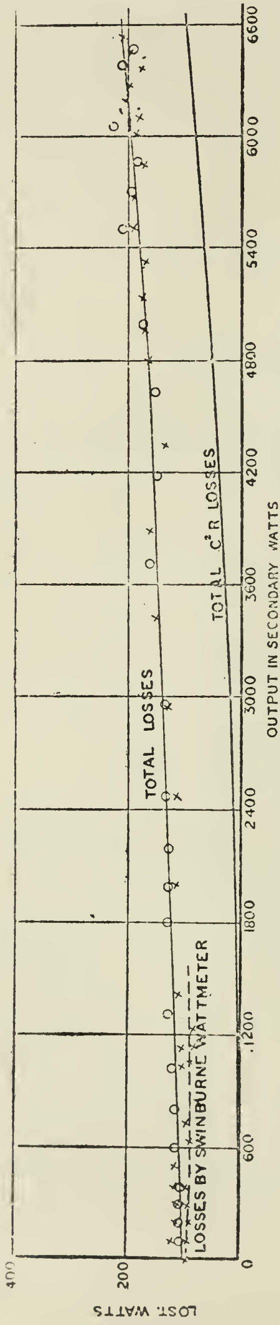

Fig. 198.—Westinghouse Transformer. Losses by Dynamometer Wattmeter.

The proof of this was obtained by careful measurements made of the total energy loss $P - P_2$ for various transformers. This value was plotted down, as in Fig. 198, in terms of the secondary output $P_2$. On the same diagram was drawn a curve representing the total copper loss or $C^2R$ loss for the primary and secondary circuits taken together. These two curves are found to be sensibly parallel to each other throughout their whole range, and hence the true iron core loss or hysteresis loss is a constant quantity at all loads. This is a necessary consequence of the fact that in constant potential transformers as designed for ordinary electric lighting work the induction in the core is constant at all loads, and this in turn is a consequence of the fact that the resultant magnetising force in the core is constant for all loads. Generally speaking, we may state that for all fairly well designed closed iron circuit constant-potential transformers the iron core loss is constant for all loads. This enables us to determine the efficiency curve for any transformer of this description by three measurements. If we measure the total power loss in the transformer at no load we have the quantity which is constant at all loads. Call this loss in watts $w$. If, then, we measure the resistances of the primary and secondary circuits and correct these values so as to obtain the true resistances $R_1$ and $R_2$ of the copper circuits at the final temperature reached by the transformer when working, we can calculate the copper losses $C_1^2 R_1$ and $C_2^2 R_2$ for various values of the output of the transformer. We can determine the value of the primary current $C_1$ corresponding to any value of the secondary current $C_2$ to a sufficient approximation for this purpose by taking it as equal to $C_2 \frac{N_2}{N_1}$, where $N_1$ and $N_2$ are the number of turns of the primary and secondary circuits respectively. Hence, the total copper loss in the transformer is very approximately equal to

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* The reader may be referred to a Paper by the author in the *Proceedings* of the Institution of Electrical Engineers, Vol. XXI., 1892, entitled "Experimental Researches on Alternate - Current Transformers," for full information on the experimental methods by which this question has been settled. See also *The Electrician*, Vol. XXX., pp. 97, 120, 162, 446.

$$C_2^2 \left\{ \left(\frac{N_2}{N_1}\right)^2 R_1 + R_2 \right\},$$

and the total power $P_1$ given to the primary circuit consequently corresponding to any secondary output $C_2 V_2$, where $V_2$ is the secondary terminal potential difference, is given by the equation

$$P_1 = w + C_2^2 \left\{ \left(\frac{N_2}{N_1}\right)^2 R_1 + R_2 \right\}$$

and $$P_2 = C_2 V_2.$$

Hence the ratio of $P_2$ to $P_1$, or the efficiency at various

EFFICIENCIES OF VARIOUS TRANSFORMERS CALCULATED FROM CURVES OF TOTAL LOSSES. FROM OBSERVATIONS WITH THE WATTMETER. THESE ARE THE RESULTS FROM WHICH THE CURVES ARE PLOTTED IN FIGS. 196 AND 197.

| SIZE AND DESCRIPTION OF TRANSFORMER. | FRACTIONS OF FULL SECONDARY LOAD. | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0 | 0·025 | 0·05 | 0·1 | 0·2 | 0·3 | 0·4 | 0·5 | 0·6 | 0·7 | 0·8 | 0·9 | 1·0 |
| | PERCENTAGE EFFICIENCIES. | | | | | | | | | | | | |
| 6,500-watt Westinghouse ... ... ... ... | 0 | 61·8 | 75·9 | 85·7 | 91·9 | 94·0 | 95·1 | 96·0 | 96·3 | 96·6 | 96·8 | 96·9 | 96·9 |
| 6,000-watt Swinburne ... ... ... ... ... | 0 | 48·4 | 65·2 | 79·0 | 88·2 | 91·8 | 93·7 | 94·8 | 95·5 | 95·9 | 96·1 | 96·1 | 96·1 |
| 3,000-watt „ ... ... ... ... ... ... | 0 | 39·3 | 56·2 | 71·7 | 82·9 | 87·6 | 89·8 | 91·2 | 92·2 | 92·8 | 93·2 | 93·4 | 93·5 |
| 4,500-watt Thomson-Houston ... ... ... | 0 | 49·1 | 65·4 | 78·8 | 87·8 | 91·1 | 92·9 | 93·8 | 94·2 | 94·6 | 95·0 | 95·0 | 94·7 |
| 20-H.P. 1892 pattern Ferranti ... ... ... | 0 | 62·0 | 76·6 | 86·5 | 92·6 | 95·0 | 95·6 | 96·1 | 96·5 | 96·8 | 96·8 | 96·6 | 96·6 |
| 15-H.P. „ „ ... ... ... | 0 | 65·5 | 79·0 | 88·1 | 93·4 | 95·0 | 95·7 | 96·0 | 96·2 | 96·3 | 96·1 | 95·8 | 95·5 |
| 15-H.P. „ (rewound) ... | 0 | 55·0 | 70·8 | 83·0 | 90·5 | 93·1 | 94·5 | 95·1 | 95·4 | 95·5 | 95·4 | 95·2 | 94·7 |
| 6,000-watt Mordey ... ... ... ... ... ... | 0 | 52·1 | 67·6 | 80·1 | 88·5 | 91·1 | 92·7 | 93·7 | 94·2 | 94·7 | 94·9 | 95·1 | 95·4 |
| 4,000-watt Kapp ... .. ... ... ... ... | 0 | 39·5 | 56·5 | 72·3 | 83·8 | 88·0 | 90·4 | 91·9 | 92·6 | 93·3 | 93·8 | 94·0 | 94·2 |
| 5-H.P. 1885 pattern Ferranti (rewound) ... | 0 | 28·8 | 44·5 | 61·4 | 75·6 | 81·9 | 85·5 | 87·7 | 89·0 | 89·8 | 90 2 | 90·5 | 90·8 |
| 5-H.P. „ „ ... ... ... | 0 | 14·6 | 25·3 | 40·3 | 57·2 | 66·5 | 72·5 | 76·4 | 79·1 | 81·5 | 83·1 | 84·5 | 85·5 |

loads, can be calculated. This is a convenient and fairly accurate method to adopt in the case where we are testing large transformers. It is sometimes very difficult or impossible then to obtain the necessary non-inductive load in the form of incandescence lamps for very large loads such as 40 or 50 kilowatts, and in that case the above procedure may be followed. The table on page 560 gives the results of a large number of transformer efficiency measurements made by the author in 1892, employing many forms of transformers then in use.

The table shows particularly what a great advance was made in transformer manufacture in the course of the seven years between 1885 and 1892.

In the case of larger transformers the efficiency curves can be made still more square-shouldered, and efficiencies of over 90 per cent. obtained at one-tenth load.

Since the core loss at no load is an important factor in determining the efficiency of the transformer, it is obvious that no transformer can have a high efficiency at light loads unless the core loss is small. The iron core loss or no-load loss in the case of transformers of 30 kilowatt size and upwards can now be made to be less than 1 per cent. of the full secondary output. That is to say, it is possible to make the iron core loss of a 50-kilowatt transformer not more than 400 watts. In the case of smaller transformers, from 1 to 15 kilowatts, the core loss will in general be from 3 to 1·3 per cent. of the full secondary output. Thus, a 1-kilowatt transformer, or one capable of giving out 1,000 watts in its external secondary circuit, is a fairly good one if it has a core loss of not more than 30 watts, or 3 per cent. of its full load; a 6-kilowatt transformer if it has a core loss of not more than 120 watts, or 2 per cent.; and a 15-kilowatt transformer is good if its core loss does not exceed 225 watts, or 1·5 per cent. These figures will be a guide to the reader to know what the core loss may be expected to be found in various cases.

We shall consider presently the causes which affect the magnitude of the core losses.

**§ 6. Current Diagram of a Transformer.**—Let a horizontal line be taken on which are set off distances proportional to

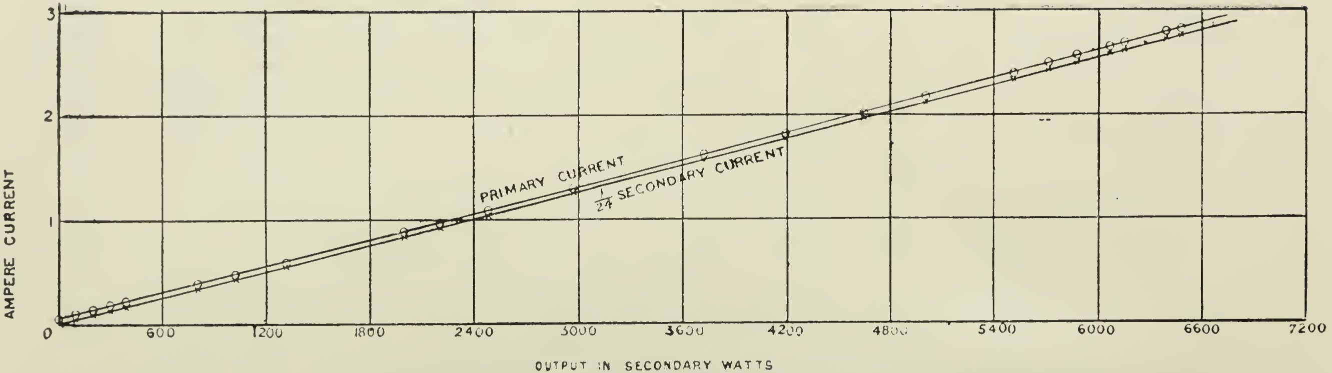


FIG. 199.—Current Diagram of 6,500-Watt Westinghouse Transformer.

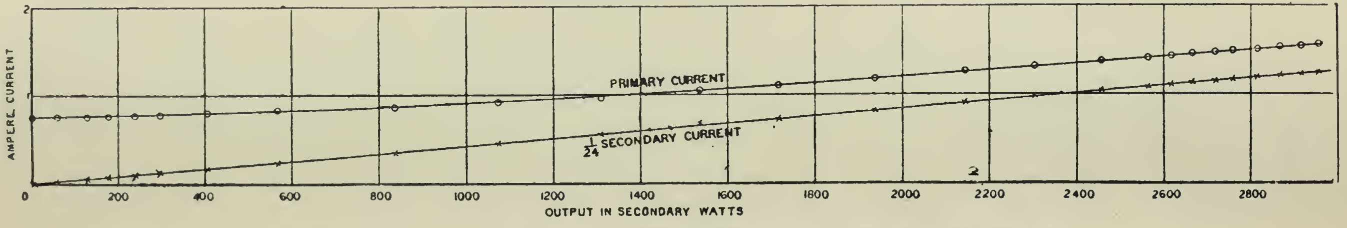


FIG. 200.—Current Diagram of 3,000-Watt Swinburne (Hedgehog) Transformer.

the power output on the external secondary circuit, that is, to the secondary load of a transformer, and let ordinates at these points be drawn to any scale representing the magnitudes of the primary and secondary currents, the scale of the primary current ordinates being taken so that if one unit of length represents one ampere of primary current, and the scale of the secondary current so that one unit of length represents $\frac{N_2}{N_1}$ times the corresponding secondary current, then we shall delineate the lines called the current curves. For any closed circuit transformer of constant potential type these current are nearly two lines running nearly parallel to each other as shown in Fig. 199. If $C_1$ stands for the mean-square value of the primary current, and $C_2$ for that of the secondary current, and if $N_1$ and $N_2$ are the numbers of the primary and secondary turns respectively, then experiment shows that a good type of closed magnetic circuit constant potential transformer $C_1 - \frac{N_2}{N_1} C_2$ is a nearly constant quantity, and that this difference is practically the same as the mean-square value of the primary current when the transformer is not loaded. Let this last be called $c_1$. Then

$$c_1 = C_1 - \frac{N_2}{N_1} C_2,$$

or

$$c_1 n_1 = C_1 N_1 - C_2 N_2.$$

In other words, the difference of the primary and secondary ampere-turns at all loads is a constant quantity, and is equal to the ampere-turns at no load. This is merely the expression of the fact that the magnetomotive force acting on this magnetic circuit is a constant quantity, and that therefore the induction is constant as well. This is, however, not the case for open-circuit transformers. In Fig. 200 is shown the current curves for a Swinburne "Hedgehog" transformer, and it will be seen that the difference of the primary and secondary ampere-turns is not constant, but increases as the load diminishes. This is a consequence of the fact that in the open circuit transformer the difference of phase between the primary and secondary currents is considerable at light loads, but becomes less as the transformer is loaded up, and that there-

**Table A.**—*Test of a Westinghouse Transformer.*

Power, 6,500 watts. Secondary volts, 100.
Frequency used, 82·5 periods per second.
Average final temperature of transformer, 96°F.
Volts on primary circuit ($V_1$) = 2,400 (kept constant).
Primary circuit resistance = 5·95 ohms at 96°F.
Secondary circuit resistance = 0·0108 ohm at 96°F.

| Secondary Circuit. | | | Primary Circuit. | | | Total power taken up in transformer $= W_1 - W_2$. | Efficiency, $= \frac{W_2}{W_1}$, in per cent. |
|---|---|---|---|---|---|---|---|
| Volts. | Amperes. | Power taken out in watts, $W_2$. | Volts. | Amperes. | Power given in watts $= W_1$. | | |
| 101·0 | 0 | 0 | 2,400 | 0·050 | 95 | 95 | 0 |
| 100·9 | 1·00 | 101 | „ | 0·100 | 205 | 104 | 49·3 |
| 100·8 | 1·98 | 200 | „ | 0·140 | 306 | 106 | 65·4 |
| 100·8 | 2·94 | 296 | „ | 0·180 | 401 | 105 | 73·7 |
| 100·7 | 3·87 | 390 | „ | 0·218 | 493 | 103 | 79·1 |
| 100·7 | 4·79 | 482 | „ | 0·250 | 597 | 115 | 80·7 |
| 100·7 | 8·00 | 806 | „ | 0·382 | 920 | 114 | 87·6 |
| 100·4 | 10·15 | 1,019 | „ | 0·472 | 1,139 | 120 | 89·5 |
| 100·3 | 13·07 | 1,311 | „ | 0·580 | 1,440 | 129 | 91·1 |
| 100·1 | 18·00 | 1,802 | „ | 0·800 | 1,930 | 128 | 93·4 |
| 100·1 | 19·90 | 1,992 | „ | 0·880 | 2,118 | 126 | 94·1 |
| 100·0 | 21·93 | 2,193 | „ | 0·960 | 2,330 | 127 | 94·1 |
| 100·0 | 24·74 | 2,474 | „ | 1·080 | 2,609 | 135 | 94·8 |
| 100·0 | 29·66 | 2,966 | „ | 1·285 | 3,096 | 130 | 95·8 |
| 99·8 | 37·20 | 3,713 | „ | 1·610 | 3,870 | 157 | 96·0 |
| 99·5 | 42·00 | 4,179 | „ | 1·810 | 4,324 | 145 | 96·7 |
| 99·3 | 46·65 | 4,633 | „ | 2·002 | 4,792 | 150 | 96·9 |
| 99·2 | 50·40 | 5,000 | „ | 2·160 | 5,174 | 174 | 96·7 |
| 99·0 | 52·16 | 5,164 | „ | 2·240 | 5,422 | 258 | 95·3 |
| 98·9 | 55·60 | 5,499 | „ | 2·383 | 5,702 | 203 | 96·1 |
| 98·8 | 57·68 | 5,700 | „ | 2·478 | 5,885 | 185 | 96·9 |
| 98·9 | 59·32 | 5,867 | „ | 2·550 | 6,041 | 174 | 97·3 |
| 98·7 | 61·32 | 6,053 | „ | 2·633 | 6,271 | 218 | 96·7 |
| 98·8 | 62·16 | 6,142 | „ | 2·672 | 6,344 | 202 | 96·8 |
| 98·7 | 63·00 | 6,218 | „ | 2·700 | 6,426 | 208 | 96·8 |
| 98·7 | 64·00 | 6,317 | „ | 2·750 | 6,522 | 205 | 96·9 |
| 98·6 | 64·74 | 6,384 | „ | 2·775 | 6,598 | 214 | 96·9 |

fore there must be an increase in mean-square or maximum value of the primary current, so that its greater value at light loads is a compensation for the greater difference of phase between the primary and the secondary current. This is on the assumption that the secondary circuit is a practically non-inductive circuit. If the carefully-drawn current diagram of

**Table B.**—*Test of a Swinburne "Hedgehog" Transformer.*

Power, 3,000 watts. Secondary volts, 100.
Frequency used, 81·1 periods per second.
Average final temperature of transformer, 145°F.
Volts on primary circuit ($V_1$) = 2,400 (kept constant).
Primary circuit resistance = 24·00 ohms at 145°F.
Secondary circuit resistance = 0·051 ohm at 145°F.

| Secondary Circuit. | | | Primary Circuit. | | | Total power taken up in transformer $= W_1 - W_2$. | Efficiency $= \frac{W_2}{W_1}$, in per cents. |
|---|---|---|---|---|---|---|---|
| Volts. | Amperes. | Power taken out in watts, $W_2$. | Volts. | Amperes. | Power given in watts $W_1$. | | |
| 101·8 | 0 | 0 | 2,400 | 0·756 | 121 | 121 | 0 |
| 101·7 | 1·00 | 102 | ,, | 0·761 | 213 | 111 | 47·9 |
| 101·5 | 2·97 | 301 | ,, | 0·786 | 414 | 113 | 72·7 |
| 101·3 | 4·84 | 490 | ,, | 0·811 | 608 | 118 | 80·6 |
| 101·3 | 6·00 | 607 | ,, | 0·829 | 730 | 123 | 83·1 |
| 101·2 | 8·00 | 810 | ,, | 0·862 | 943 | 133 | 85·9 |
| 101·0 | 10·20 | 1,030 | ,, | 0·915 | 1,161 | 131 | 88·6 |
| 100·9 | 12·00 | 1,211 | ,, | 0·960 | 1,361 | 150 | 89·0 |
| 100·6 | 14·00 | 1,408 | ,, | 1·013 | 1,551 | 143 | 90·8 |
| 100·3 | 15·87 | 1,592 | ,, | 1·066 | 1,750 | 158 | 91·1 |
| 100·0 | 17·89 | 1,789 | ,, | 1·133 | 1,951 | 162 | 91·7 |
| 100·0 | 19·80 | 1,980 | ,, | 1·197 | 2,129 | 149 | 93·0 |
| 99·9 | 21·82 | 2,180 | ,, | 1,260 | 2,344 | 164 | 93·0 |
| 99·7 | 23·66 | 2,359 | ,, | 1·321 | 2,525 | 166 | 93·3 |
| 99·5 | 25·46 | 2,534 | ,, | 1·397 | 2,732 | 198 | 92·8 |
| 99·3 | 26·46 | 2,628 | ,, | 1·430 | 2,823 | 195 | 93·2 |
| 99·1 | 27·42 | 2,718 | ,, | 1·465 | 2,914 | 196 | 93·4 |
| 99·0 | 28·28 | 2,800 | ,, | 1·500 | 2,988 | 188 | 93·7 |
| 98·9 | 29·26 | 2,896 | ,, | 1·532 | 3,103 | 207 | 93·3 |
| 99·0 | 30·20 | 2,990 | ,, | 1·566 | 3,185 | 195 | 94·0 |

any closed-circuit transformer of constant-potential type are examined, it will be seen that the difference between the ordinates of the primary and secondary current curves or lines increases slightly very near the origin, and as we shall see presently this is an indication of the fact that the primary current is for all loads, except very small ones, practically in exact opposition, as regards phase, to the secondary current. As an example of the measurements of a complete test of a closed circuit and open circuit transformer, we give on page 564 and above, in Tables A and B, the figures obtained for a Westinghouse and Swinburne transformer respectively.

§ 7. **The Power Factor of Transformers.**—If under any conditions of load on the secondary circuit we measure the true power $P_1$ being taken up by the primary circuit, and also the mean-square ($\sqrt{\text{mean}^2}$) value A of the primary current and the mean-square value V of the primary terminal potential difference, the ratio of $P_1$ to the product A V is called the *power factor* of the transformer at that load. The product A V is often called the *apparent power* or apparent watts given to the transformer, and the value of $P_1$ is the *true power* or true watts given to the transformer. Hence the power factor (F) is defined thus:—

$$\left.\begin{matrix}\text{Power fac-}\\ \text{tor (F)}\end{matrix}\right\} = \frac{\text{True power in watts given to the transformer}}{\text{Apparent power in watts given to the transformer}}$$

The above relation may be symbolically expressed by writing

$$F = \frac{P_1}{A\,V} \qquad \text{or} \qquad P_1 = (F\,A)\,V.$$

This last mode of writing it exposes the appropriateness of the term "power factor"; since we see that it is a factor by which the value of the total mean-square value of the current must be multiplied to obtain that current which, when multiplied by the mean-square value of the potential difference V, will give the true mean power being taken up in the circuit. Thus, if the power factor is denoted by F, this signifies that the portion F A of the current A is effective in conveying power, and the remainder $(1 - F)\,A$ is ineffective, or, as it is sometimes called, is the *wattless* component of the current. Hence, we may, in imagination, divide the apparent power A V into two portions: a part F A V, which is a measure of the true power given to the circuit; and a part $(1 - F)\,AV$, which is the wattless or powerless portion.

The true power $P_1$ is obtained from the correct wattmeter reading, and the apparent power A V is the value of the product of the readings of an electrostatic voltmeter used to measure the mean-square value of the potential difference and that of an alternating-current ammeter used to measure the mean-square value of the current. A very important constant with respect to any transformer is its power factor at no load or on open secondary circuit, and the Table on page 567 gives the

## POWER FACTORS OF VARIOUS TYPES OF TRANSFORMERS AT NO LOAD, OR ON OPEN SECONDARY CIRCUIT.

| TRANSFORMER. | Maximum output in watts from secondary. | Magnetising current in amperes $=A$. | Primary volts $=V$. | True power in watts taken up in transformer at no load $=P_1$. | Apparent power absorbed $=AV$. | Power factor $=\frac{P_1}{AV}$. | Alternator employed for test. |
|---|---|---|---|---|---|---|---|
| Ferranti, 1885 pattern ... ... ... | 1,875 | 0·18 | 2,416 | 288 | 432 | 0·66 | Ferranti. |
| Ditto ditto ... ... ... | 3,750 | 0·337 | 2,400 | 540 | 808 | 0·68 | ditto. |
| Ditto ditto ... .. ... | 7,500 | 0·25 | 2,435 | 444 | 600 | 0·74 | ditto. |
| Ditto ditto ... ... ... | 11,250 | 0·34 | 2,447 | 578 | 816 | 0·70 | ditto. |
| Ditto ditto ... ... ... | 15,000 | 0·57 | 2,389 | 1,019 | 1,368 | 0·75 | ditto. |
| Ferranti, 1892 pattern ... ... ... | 7,500 | 0·075 | 2,400 | 138 | 180 | 0·77 | ditto. |
| Ditto ditto ... ... .. | 11,250 | 0·076 | 2,400 | 148 | 182 | 0·81 | ditto. |
| Ditto ditto ... ... ... | 15,000 | 0·112 | 2,400 | 228 | 269 | 0·85 | ditto. |
| Swinburne (Hedgehog) ... ... .. | 3,000 | 0·74 | 2,400 | 112 | 1,775 | 0·063 | ditto. |
| Ditto ditto ... ... .. | 6,000 | 1·194 | 2,400 | 156 | 2,918 | 0·053 | ditto. |
| Westinghouse ... ... ... .. | 6,500 | 0·05 | 2,400 | 95 | 120 | 0·79 | ditto. |
| Mordey, 1892 pattern ... ... .. | 6,000 | 0·076 | 2,400 | 140 | 182 | 0·77 | ditto. |
| Ditto ditto ... ... .. | 750 | 0·032 | 2,392 | 61 | 76 | 0·81 | ditto. |
| Thomson-Houston ... ... ... ... | 4,500 | 0·083 | 2,400 | 108 | 199 | 0·54 | ditto. |
| Kapp ... ... ... ... ... ... | 4,000 | 0·145 | 2,400 | 152 | 348 | 0·61 | ditto. |
| Mordey, 1894 ... ... ... ... ... | 50,000 | 0·494 | 2,000 | 790 | 988 | 0·79 | Mordey. |
| Thomson-Houston ... ... ... ... | 30,000 | 0·562 | 2,000 | 557 | 1,124 | 0·49 | Thomson-Houston. |

values of the no-load power factor for various types of transformers taken on certain alternators.

Generally speaking, it is found that the power factor of most closed iron circuit transformers has a value lying between 0·5 and 0·8, but that for induction coils on open-circuit transformers, such as the "Hedgehog" transformer, the power factor is about one-tenth of the value for closed iron circuit transformers.

It must not be supposed, however, that the power factor of an induction coil or transformer has a constant and fixed value for any particular transformer. The value of the power factor is affected to a very considerable degree by the form of the curve of primary terminal potential difference, and may vary within wide limits according as the curve is varied in form.

*Power Factors of Transformers taken off different Alternators at the same Primary Voltage.*

| Transformer. | Size in Kilow'tts | Power Factor. | | Magnetising Currents in Amperes. | |
|---|---|---|---|---|---|
| | | On Thomson-Houston Alternator | On Mordey Alternator | On Thomson-Houston Alternator | On Mordey Alternator |
| Mordey-Brush ... | 50 | 0·609 | 0·704 | 0·668 | 0·623 |
| Thomson-Houston | 30 | 0·490 | 0·536 | 0·562 | 0·569 |
| Mordey-Brush ... | 18 | 0·685 | 0·751 | 0·326 | 0·332 |

In the Table above are given the power factors at no load of three transformers taken off a Mordey alternator having an E.M.F. curve similar to that shown in Fig. 175, and a Thomson-Houston alternator having an E.M.F. curve of the kind shown in Fig. 174, and it will be seen that the power factors and magnetising currents of the transformers are quite different in the two cases.

We must not, therefore, regard the power factor as an absolute constant for the transformer, but as a function to some degree of the form of the primary E.M.F. curve, although at the same time dependent essentially upon the nature of the magnetic circuit of the transformer.

If the primary E.M.F. curve and current curve were both simple periodic or sine curves, then the power factor would be simply the cosine of the angle of lag of the current behind the electromotive force. If the transformer has its secondary circuit loaded up, the power factor approximates to unity as this loading takes place.

In the case of most closed-circuit transformers a very little loading-up of the secondary circuit causes the power factor to become unity, but in the case of an open-circuit transformer, whilst the loading-up of the transformer increases the power factor, it never actually reaches unity.

### **Table C.**—*Test of a Westinghouse Transformer.*

Primary volts, 2,400 (kept constant).
Secondary volts at no load = 101·0.

| Primary Current, $I_1$. | Secondary Current, $I_2$. | $\frac{I_2}{24}$ | Copper Losses in Watts. | | | Total Secondary Drop in volts. | Primary Watts. | | Power-Factor, $F=\frac{W}{W_1}$ |
|---|---|---|---|---|---|---|---|---|---|
| | | | Primary, $I_1^2\times5·95$. | Secondary, $I_2^2\times0·0108$. | Total. | | True Power $=W$. | Apparent Power $=W_1$. | |
| 0·050 | 0 | 0 | 0 | 0 | 0 | 0 | 95 | 120 | 0·79 |
| 0·100 | 1·00 | 0·042 | 0 | 0 | 0 | 0·1 | 205 | 240 | 0·85 |
| 0·140 | 1·98 | 0·083 | 0·1 | 0 | 0 | 0·2 | 306 | 336 | 0·91 |
| 0·180 | 2·94 | 0·122 | 0·2 | 0·1 | 0 | 0·2 | 401 | 432 | 0·93 |
| 0·218 | 3·87 | 0·161 | 0·3 | 0·2 | 0 | 0·3 | 493 | 523 | 0·94 |
| 0·250 | 4·79 | 0·199 | 0·4 | 0·2 | 1 | 0·3 | 597 | 600 | 0·99 |
| 0·382 | 8·00 | 0·333 | 0·9 | 0·7 | 2 | 0·3 | 920 | 917 | 1·00 |
| 0·472 | 10·15 | 0·423 | 1·3 | 1·1 | 2 | 0·6 | 1,139 | 1,133 | 1·00 |
| 0·580 | 13·07 | 0·545 | 2·0 | 1·8 | 4 | 0·7 | 1,440 | 1,392 | 1·03 |
| 0·800 | 18·00 | 0·750 | 3·8 | 3·5 | 7 | 0·9 | 1,930 | 1,920 | 1·00 |
| 0·880 | 19·90 | 0·830 | 4·6 | 4·3 | 9 | 0·9 | 2,118 | 2,112 | 1·00 |
| 0·960 | 21·93 | 0·914 | 5·5 | 5·2 | 11 | 1·0 | 2,330 | 2,304 | 1·01 |
| 1·080 | 24·74 | 1·031 | 6·9 | 6·6 | 14 | 1·0 | 2,609 | 2,592 | 1·01 |
| 1·285 | 29·66 | 1·238 | 9·8 | 9·5 | 19 | 1·0 | 3,096 | 3,085 | 1·00 |
| 1·610 | 37·20 | 1·550 | 15·4 | 14·9 | 30 | 1·2 | 3,870 | 3,864 | 1·00 |
| 1·810 | 42·00 | 1·750 | 19·5 | 19·0 | 39 | 1·5 | 4,324 | 4,344 | 0·99 |
| 2·002 | 46·65 | 1·945 | 23·5 | 23·5 | 47 | 1·7 | 4,792 | 4,805 | 1·00 |
| 2·160 | 50·40 | 2·100 | 23·7 | 27·5 | 55 | 1·8 | 5,174 | 5,184 | 1·00 |
| 2·240 | 52·16 | 2·171 | 29·8 | 29·5 | 59 | 2·0 | 5,422 | 5,376 | 1·01 |
| 2·383 | 55·60 | 2·320 | 33·9 | 33·5 | 67 | 2·1 | 5,702 | 5,719 | 1·00 |
| 2·478 | 57·68 | 2·404 | 36·5 | 36·0 | 73 | 2·2 | 5,885 | 5,947 | 0·99 |
| 2·550 | 59·32 | 2·474 | 38·7 | 38·1 | 77 | 2·1 | 6,041 | 6,120 | 0·99 |
| 2·633 | 61·32 | 2·560 | 41·2 | 40·6 | 82 | 2·3 | 6,271 | 6,319 | 0·99 |
| 2·672 | 62·16 | 2·594 | 42·5 | 41·8 | 84 | 2·2 | 6,344 | 6,413 | 0·99 |
| 2·700 | 63·00 | 2·623 | 43·3 | 42·9 | 86 | 2·3 | 6,426 | 6,480 | 0·99 |
| 2·750 | 64·00 | 2·665 | 45·0 | 44·2 | 89 | 2·3 | 6,522 | 6,600 | 0·99 |
| 2·775 | 64·74 | 2·700 | 45·8 | 45·3 | 91 | 2·4 | 6,598 | 6,660 | 1·00 |

This may best be illustrated by giving the figures of test of two transformers, one of the closed-circuit type (Westinghouse) and one of the open-circuit type (Swinburne) (*see* Tables C and D). It will be seen that a very small loading of the closed-circuit type suffices to bring the power factor up to unity.

Hence it follows that for closed-circuit transformers the apparent power given to the transformer is equal to the real power at and beyond about one-tenth of full load. In Fig. 201 are shown three curves illustrating the gradual rise of the power factor towards unity in the case of three types of transformer. In the case of an open-circuit transformer at no stage of the load is the true power taken up by the transformer identical in value with the apparent power given to the transformer.

**Table D.**—*Test of a Swinburne "Hedgehog" Transformer.*

Primary volts, 2,400 (kept constant).

Secondary volts at no load = 102·00.

| Primary Current, $I_1$. | Secondary Current, $I_2$. | $\frac{I_2}{24}$. | Copper Losses in Watts. | | | Total Secondary Drop in Volts. | Primary Watts. | | Power Factor, $F = \frac{W}{W_1}$. |
|---|---|---|---|---|---|---|---|---|---|
| | | | Primary, $I_1^2 \times 24·00$. | Secondary, $I_2^2 \times 0·051$. | Total. | | True Power = W. | Apparent, Power = $W_1$. | |
| 0·756 | 0 | 0 | 13·7 | 0 | 14 | 0 | 121 | 1,816 | 0·07 |
| 0·761 | 1·02 | 0·042 | 14·0 | 0·1 | 14 | 0·1 | 228 | 1,829 | 0·13 |
| 0·786 | 2·98 | 0·124 | 14·9 | 0·5 | 15 | 0·2 | 420 | 1,886 | 0·22 |
| 0·811 | 4·84 | 0·202 | 15·8 | 1·2 | 17 | 0·4 | 621 | 1,948 | 0·32 |
| 0·829 | 6·00 | 0·250 | 16·5 | 1·8 | 18 | 0·6 | 730 | 1,988 | 0·37 |
| 0·862 | 8·00 | 0·333 | 17·8 | 3·3 | 21 | 0·8 | 943 | 2,067 | 0·46 |
| 0·911 | 10·00 | 0·417 | 19·9 | 5·1 | 25 | 1·0 | 1,152 | 2,188 | 0·53 |
| 0·960 | 12·00 | 0·500 | 22·6 | 7·3 | 30 | 1·2 | 1,353 | 2,301 | 0·59 |
| 1·013 | 14·00 | 0·584 | 24·7 | 10·0 | 35 | 1·3 | 1,538 | 2,432 | 0·63 |
| 1·066 | 15·88 | 0·663 | 27·2 | 12·9 | 40 | 1.5 | 1,746 | 2,559 | 0·68 |
| 1·133 | 17·89 | 0·746 | 30·9 | 16·3 | 47 | 1·7 | 1,932 | 2,720 | 0·71 |
| 1·197 | 19·80 | 0·825 | 34·5 | 20·0 | 55 | 1·9 | 2,129 | 2,873 | 0·74 |
| 1·260 | 21·63 | 0·902 | 38·2 | 24·0 | 62 | 2·1 | 2,320 | 3,022 | 0·77 |
| 1·321 | 23·58 | 0·983 | 42·0 | 28·3 | 70 | 2·4 | 2,510 | 3,172 | 0·79 |
| 1·395 | 25·46 | 1·060 | 46·6 | 33·1 | 80 | 2·7 | 2,706 | 3,350 | 0·81 |
| 1·426 | 26·38 | 1·098 | 48·9 | 35·4 | 84 | 2·9 | 2,829 | 3,422 | 0·83 |
| 1·460 | 27·28 | 1·136 | 51·2 | 38·0 | 89 | 3·0 | 2,890 | 3,501 | 0·83 |
| 1·498 | 28·21 | 1·175 | 54·0 | 40·8 | 95 | 3·0 | 2,993 | 3,596 | 0·83 |
| 1·529 | 29·19 | 1·215 | 56·1 | 43·5 | 100 | 3·2 | 3,042 | 3,666 | 0·83 |
| 1·567 | 30·33 | 1·262 | 59·0 | 46·9 | 106 | 3·3 | 3,163 | 3,761 | 0·84 |

In Fig. 201 the three curves show the progress of increase of the power-factor as the load on the secondary circuit is progressively increased. The upper curve represents the growth of power factor (F) for a 6,500-watt Westinghouse transformer. Beginning at 0·8, it rises up to unity at about one-tenth of full load. Hence at and after this load the apparent watts are the same as the true watts, and the real power taken up in the transformer is quite accurately given by the product of the primary terminal pressure and the primary current, mean-square ( $\sqrt{\text{mean}^2}$ ) values being understood. For an open magnetic circuit transformer like the "Hedgehog" the case is quite different. The power factor begins at a value of 0·08 or 0·06, and it never rises up above 0·8. Hence at no stage of the load is the real power taken up by the transformer equal to the "apparent watts." A transformer like the 4,000-watt Kapp appears to occupy an intermediate position, and although it has a medium power factor to start with, its power factor rises up to unity at about half-load. The importance of this fact in alternate-current station working is very great. It shows us, if we have a station wholly supplied with transformers of the type of Mordey, Westinghouse, Thomson-Houston, Ferranti, &c., that the apparent power supplied to the transformers is equal to the real power at any hour when all the transformers are more than one-tenth loaded.

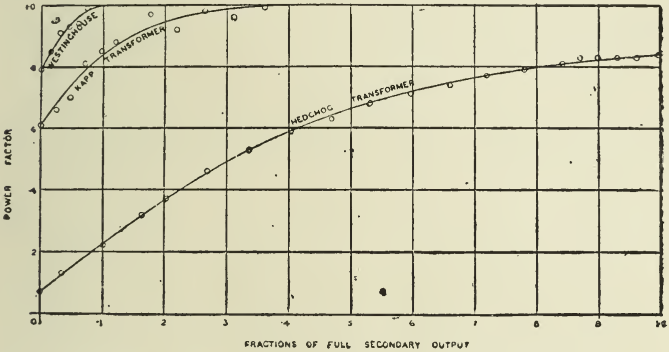


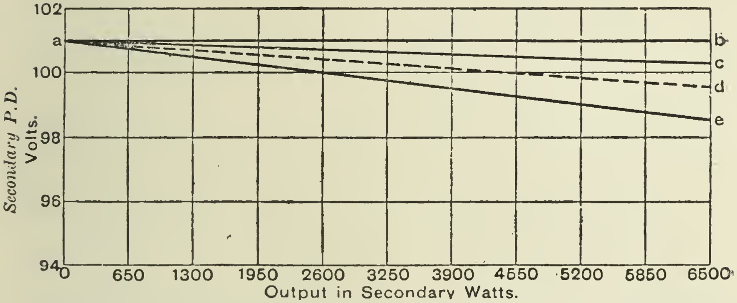
Fig. 201.—Relation of Power Factor to Secondary Output.

The reciprocal of the power factor of a transformer on open secondary circuit is a measure of the reluctance of the magnetic circuit of the transformer. In the case of a transformer with an air-iron magnetic circuit (open-circuit type) the reluctance of the iron circuit is large and the reciprocal of the power factor large also, and may be a number approximating to 16 or 17. In the case of a closed iron circuit transformer like the Mordey transformer, with very short magnetic circuit and very small reluctance, the reciprocal of the power factor is very small, and will be a number approximating to 1·2 to 1·4. The introduction of any bad magnetic joint into the iron circuit, or the employment of iron of small permeability, immediately decreases the magnitude of the power factor of that transformer. Any joint or break in the magnetic circuit accordingly increases the value of the reciprocal of the power factor, and although this alone will not affect the total core loss in the transformer, it is an indication of the increased reluctance of the magnetic circuit. The advantage of a large power factor is that it involves a small value of the magnetising current of the transformer. In the case of an alternating current station large magnetising current involves additional waste of power in the passage of this current through the distributing mains. This point will be discussed at greater length in connection with the subject of alternating current distribution.

§ 8. **Magnetic Leakage and Secondary Drop.**—If a transformer has the mean-square value of the potential difference of its primary terminals kept perfectly constant, whilst at the same time secondary currents of various magnitudes are taken from its secondary coil by altering the resistance of the external secondary circuit, we find that the mean-square value of the potential difference between the secondary terminals of the transformer changes with every change in the secondary load.

The secondary terminal potential difference (S.P.D.) becomes less as the secondary current and load increases. The difference between the secondary terminal potential difference at no load and at any load is called the *secondary drop* of the transformer due to that load.

We may represent the variation of secondary drop with secondary load by a diagram as follows : Let a horizontal line be taken on which are set off distances representing the fractions of the full secondary load, and let vertical ordinates set up at these points represent the value of the secondary potential differences at these loads. For convenience sake we may make these ordinates represent the magnitude of the secondary terminal potential difference diminished by a certain constant amount which is less than the least difference found with full load. For instance, suppose the secondary terminal potential difference at no load is 100 volts and at full load is 97 volts, we may make the vertical ordinates represent the terminal potential difference *minus* 90 volts. The curve defined, as in Fig. 202, by the extremities of these ordinates is called the secondary terminal volt curve, and it shows in a graphical manner the gradually diminishing secondary terminal potential difference as the transformer is loaded up. This "secondary drop" arises from two causes. The first is the loss of potential due to resistance, and the second is the loss of secondary potential due to magnetic leakage. Let the resistance of the secondary coil of the transformer be represented by $R_2$, and the secondary current (mean-square value) be represented by $(I_2)$. Then $R_2(I_2)$ is the loss of voltage due to secondary resistance. If the primary terminal potential difference is kept constant, then,



*a b* is the horizontal line through *a* ; Curve *a c* is the Curve of Drop due to secondary resistance ; Curve *a d*, that due to primary resistance ; and Curve *a e* is the Curve of total Drop.

FIG. 202.—Secondary Drop Curves of 6,500-watt Westinghouse Transformer.

over and above the loss of secondary voltage due to the resistance of the internal secondary circuit, there is a portion of the secondary drop which is due to loss of voltage by the resistance of the primary circuit, and, in addition to this, the loss above mentioned, which is due to *magnetic leakage*. Furthermore, the secondary drop is to a considerable extent dependent, as will be explained presently, upon the form of the curve of primary terminal potential difference. Hence the difference between the potential difference of the secondary terminals of the transformer at no load and full load, primary potential difference being constant, is dependent on four things, viz., upon—

(1) The resistance of the primary circuit;
(2) The resistance of the secondary circuit;
(3) The magnetic leakage of the transformer as affected by,
 (*a*) Its construction.
 (*b*) The form of the curve of primary terminal potential difference.

The effect called the magnetic leakage in a transformer may be generally described as follows: The primary current creates in the iron core a certain total induction, or in usual language creates a certain number of lines of induction in the core which are linked with the primary circuit. The magnetising effect of the secondary current is at any instant opposed to that of the primary, and hence creates an induction in the core in an opposite direction. The resultant, or actual induction in the core at any place is due to the difference of the opposed magnetising forces acting on the core. When the transformer has its secondary circuit open the magnetic induction in the core is that due to the primary current only, which is then generally called the magnetising current. When the secondary circuit is closed and a secondary current produced, the rise of induction in that part of the core enveloped by the secondary circuit is delayed, and its maximum value is reduced. The simplest way in which the effect of increasing the secondary current of the transformer can be regarded is as follows: Let us denote by the letter $Z_1$ the maximum value of the total magnetic induction in the core which would be produced by the primary current if it acted alone, and by $Z_2$ the same due to the secondary current, these values being the inductions just within that part of the core enveloped by the primary and secondary coils respectively. The whole of the induction $Z_1$ which is linked with the primary coil turns is not, however, linked with the secondary. Let a fraction, say $\beta Z_1$, of this primary induction escape linkage with the secondary coil, and a similar fraction, say $\beta Z_2$, of the secondary induction will escape linkage with the primary coil. Then the total induction linked with the primary coil is $Z_1 - Z_2(1-\beta)$, because the induction caused by the primary current is opposed in direction to the induction caused by the secondary current, and the inductions, like the two currents, are opposite in phase and reach their maxima nearly coincidently. Also, for the same reasons, the total induction linked with the secondary circuit is

$$Z_1(1-\beta) - Z_2.$$

$\beta$ is called the coefficient of leakage.

The value of the total induction linked with the primary circuit is therefore the product of the number of primary turns $N_1$ and the resultant induction $Z_1 - Z_2(1-\beta)$, and, similarly, the value of the total induction linked with the secondary circuit is given by the product of the number of secondary turns $N_2$ and the resultant induction $Z_1(1-\beta) - Z_2$. Hence we have the relation,

$$\left.\frac{\text{The total linkage of primary circuit and induction traversing it}}{\text{The total linkage of secondary circuit and induction traversing it}}\right\} = \frac{N_1\{Z_1 - Z_2(1-\beta)\}}{N_2\{Z_1(1-\beta) - Z_2\}} = T.$$

It will be shown presently that this fraction T represents the ratio of the mean-square value of the primary terminal potential difference to that of the secondary terminal potential difference.

This ratio, which is denoted by T, is called the transformation ratio of the transformer. Since the difference between $Z_1$ and $Z_2$ remains nearly constant as $Z_1$ and $Z_2$ increase, it is easily seen that the transformation ratio increases as $Z_1$ and $Z_2$ increase, subject to the condition that $Z_1 - Z_2$ is nearly constant at all loads.

Hence, if the mean-square value of the primary electromotive force is kept constant, that of the secondary potential difference decreases as the currents, and therefore the inductions, in the core increase, and this effect is called the "secondary drop."

The predetermination of the magnetic leakage of a transformer is a matter of some difficulty, and can only be anticipated in certain limited cases. We can obtain the relation between the leakage drop, the resistance drop, and the total drop if we assume an approximately simple periodic variation of the electromotive forces, currents and inductions, as follows:—

Let $R_1$ be the true resistance of the primary circuit and $R_2$ that of the secondary circuit of the transformer, and let S be the cross-section of the magnetic circuit or core. Let $N_1$ be the number of primary turns, $N_2$ the number of secondary turns, and $a$ stand for the ratio of $N_1$ to $N_2$. Let $b_1$ be at any instant the induction density in that part of the core enveloped by the primary coil, and $b_2$ that part enveloped by the secondary coil; the difference between these inductions may be called the density of the leakage of induction, and be denoted by $b$. Hence

$$b = b_1 - b_2.$$

In other words, if S is the cross-section of the core, then $S\,b = S\,b_1 - S\,b_2$ and $S\,b$ represents that part of the induction linked with the primary coil which is not linked with the secondary coil. If $e_1$ is the primary terminal potential difference at any instant, and $e_2$ that of the secondary terminal at the same instant, and $i_1$ and $i_2$ the currents at the same moment, then, by fundamental equations, we have

$$e_1 = R_1 i_1 + S N_1 \frac{d b_1}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (146)$$

and

$$0 = R_2 i_2 + e_2 + S N_2 \frac{d b_2}{d t}. \quad . \quad . \quad . \quad (147)$$

Let us write $\frac{N_1}{N_2} = a$ and $b = b_1 - b_2$;

we have by elimination from the fundamental equations the result

$$\frac{e_1}{a} + e_2 + R_2 i_2 - R_1 \frac{i_1}{a} = S N_2 \frac{d b}{dt}. \quad . \quad . \quad (148)$$

If $e_1$ varies in a simple periodic manner so that $e_1 = E_1 \sin p t$, then, since $e_2$ is always opposite in phase and similar in form to $v_1$, we must have

$$e_2 = - E_2 \sin p t.$$

Moreover, when the transformer is fully loaded, the currents $i_1$ and $i_2$ are in step with the electromotive forces $e_1$ and $e_2$, but $i_2$ differs 180deg. in phase from $i_1$.

Hence
$$i_1 = I_1 \sin p t$$
$$i_2 = - I_2 \sin p t.$$

We can also write $b = - B \sin p t$, because the leakage $b$ is determined by, and is in step very nearly with, the secondary current. Hence, by substitution of the above values in the equation (148) we arrive at the equation

$$\left(\frac{E_1}{a} - E_2 - R_2 I_2 - R_1 \frac{I_1}{a}\right) \sin p t = - S N_2 p B \cos p t.$$

The quantity $S N_2 p B \cos p t$ is the instantaneous value of the potential difference of the secondary circuit lost by leakage—that is to say, it is the measure of the amount by which the secondary terminal potential difference would be increased if there were no leakage. Hence the left-hand side of the above equation represents the same thing. The factors which multiply the $\sin p t$ and $\cos p t$ respectively in the above equation give, therefore, the maximum value of the "leakage," and therefore, when divided by $\sqrt{2}$, represent the mean-square value.

Hence the quantity $\left(\frac{E_1}{a} - E_2 - R_2 I_2 - R_1 \frac{I_1}{a}\right) \frac{1}{\sqrt{2}}$, or, which comes to the same thing, the quantity

$$\left(\frac{1}{a} \frac{E_1}{\sqrt{2}} - \frac{E_2}{\sqrt{2}} - R_2 \frac{I_2}{\sqrt{2}} - \frac{1}{a} R_1 \frac{I_1}{\sqrt{2}}\right)$$

represents the mean-square ($\sqrt{\text{mean}^2}$) value of the loss of potential difference of the secondary circuit due to magnetic leakage when the potential difference is measured in volts.

The quantities $\frac{E_1}{\sqrt{2}}$, $\frac{E_2}{\sqrt{2}}$, $\frac{I_1}{\sqrt{2}}$, $\frac{I_2}{\sqrt{2}}$ represent the magnitude of the currents and potentials as read in alternating-current ammeters and voltmeters. We may denote these mean-square values by the symbols $(E_1)$, $(E_2)$, $(I_1)$, $(I_2)$, and the values which these mean-square potential differences and currents have at full and at no secondary load by the symbols $(E_1)_f$, $(E_2)_f$, $(E_1)_0$, $(E_2)_0$, &c.

The total loss of secondary terminal voltage between full and no secondary load will be given by the difference between the values of the expressions

$$\left\{\frac{1}{a}(E_1)_f - (E_2)_f - R_2(I_2)_f - \frac{1}{a}R_1(I_1)_f\right\}$$

and

$$\left\{\frac{1}{a}(E_1)_0 - (E_2)_0 - R_2(I_2)_0 - \frac{1}{a}R_1(I_1)_0\right\}.$$

The value of $(I_2)_0$ is, of course, zero.

If the primary terminal potential difference is the same at no load as at full load, we have for the secondary drop due to magnetic leakage the expression

$$(E_2)_0 - (E_2)_f - \left\{R_2(I_2)_f + \frac{1}{a}R_1(I_1)_f - \frac{1}{a}R_1(I_1)_0\right\}. \quad (149)$$

The secondary terminal volts at full load being denoted by $(E_2)_f$, and that at no load by $(E_2)_0$, we see that the quantity $(E_2)_0 - (E_2)_f$ represents the total secondary drop due to all causes. The quantity

$$\left\{R_1(I_2)_f + \frac{1}{a}R_1(I_1)_f - \frac{1}{a}R_1(I_1)_0\right\} \quad . \quad . \quad (150)$$

therefore represents that part of the drop due to the resistance of the primary and secondary circuits.

Hence we have the following rule for determining the drop due to magnetic leakage:—Add together the product of the secondary resistance and secondary current and $\frac{1}{a}$ multiplied into the product of primary resistance and primary current, after deducting from the last value the primary current at no load. Subtract this sum from the total observed drop, and the remainder is the secondary potential difference due to magnetic leakage.

Testing in this way a number of transformers, the author found that where the primary and secondary circuits were intermixed, the magnetic-leakage drop was small, but that where the primary and secondary circuits were separated, and in each consisting of one coil only, the magnetic leakage drop was large.

In the diagram in Fig. 202 are shown three curves, by which the three sources of secondary drop have been distinguished, the lines $ac$, $ad$ showing the curves of drop due respectively to the primary and secondary resistance, and the curve $ae$ showing the total drop.

In designing a transformer, it is not permissible to purchase small core loss at the expense of large secondary drop. In a proper specification for a transformer a limitation should be put upon the amount of secondary drop allowed, and it is usual to express it as a percentage of the normal potential difference or voltage of the secondary circuit when the transformer is unloaded.

It is advantageous to so arrange the winding of the secondary circuit that if the drop is, say, 2 per cent., and the secondary-circuit voltage is 100, that the transformer shall give 101 volts terminal pressure at no load, and 99 volts at full load. In this way the full drop is divided and is not felt so much in working on 100-volt lamps as if the transformer were wound to give 100 volts at no load and 98 at full load.

§ 9. **Effect of the Form of the Curve of Primary Electromotive Force in the Transformer Efficiency and Currents.**—It has generally been assumed by many of those who have written on the subject of the alternate-current transformer that the efficiency, power factor and secondary drop were characteristics of the transformer only. It has already, in previous sections, been suggested that the form of the curve of primary potential difference or primary electromotive force had a considerable effect in modifying the value of these quantities, and it will now be necessary to examine the matter a little more in detail. We will consider in the first place the effect of the form of primary terminal potential difference upon the form of the current curves and magnitude of the mean-square value of the currents.

The widely-different forms which the primary current of a transformer on open secondary circuit may have is shown in the diagrams in Figs. 203 and 204. Fig. 203 shows the primary-current curve of a small transformer taken off a Ganz alternator having a peaked curve of electromotive force. The curves in Fig. 204 show the primary electromotive force and primary-current curves of the same transformer taken from a

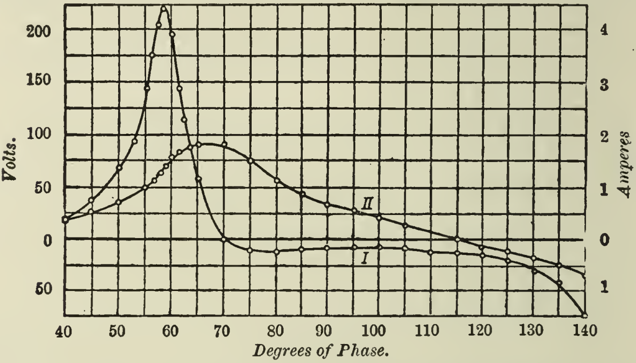


**Fig.** 203.—Primary Current Curve II and Primary Electromotive Force Curve I of a Transformer taken off Ganz Alternator.

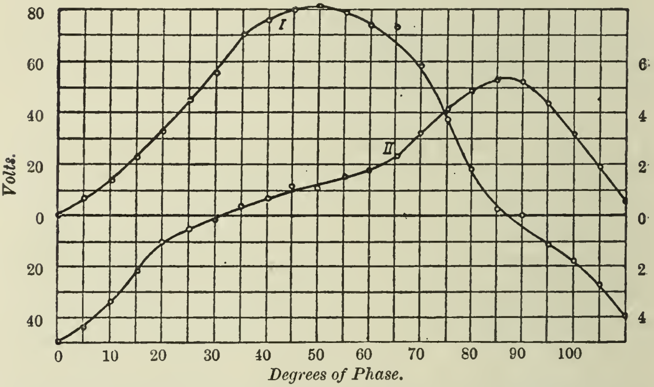


Fig. 204.—Primary Current Curve II and Primary Electromotive Force Curve I of the same Transformer taken off Wechsler Alternator.

Wechsler alternator. These and the following curves are from an interesting Paper by Dr. G. Roessler.*

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* "Das Verhalten von Transformatoren unter den Einflusse von Wechselströmen Verschiedenen Periodischen Verlaufs." A Paper read at the third annual meeting of the Verband Deutscher Electrotechniker, Munich, July 6, 1895. *See* also *The Electrician*, Vol. XXXVI., 1895, p. 150.

Not only do the forms of the current curves differ when taken with different-shaped electromotive force curves, but if the primary electromotive force is kept at the same mean-square value, and if the transformer is gradually loaded up, the mean-square values of the primary current corresponding to given secondary currents will differ if the curves of primary electromotive force have different forms. This is shown in Fig. 205, where the ordinates represent the mean-square values of the secondary and primary currents of one and the same transformer, taken off a Ganz and Wechsler machine respectively.

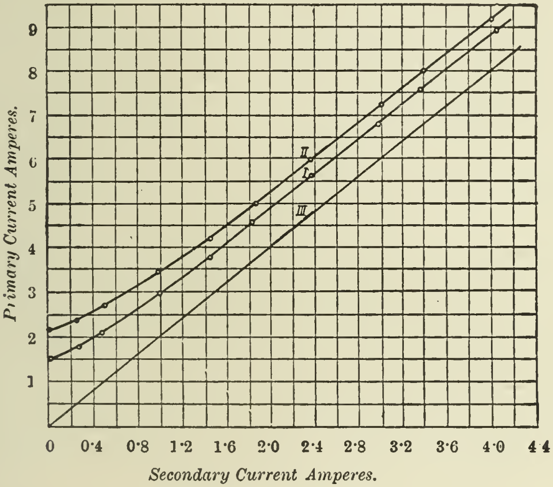

FIG. 205.—Current Diagram of a certain Transformer. Curve I, Primary Current taken with Ganz Alternator. Curve II, Primary Current taken with Wechsler Alternator. Curve III, twice value of Secondary Current. Transformation ratio of Transformer = 2 : 1.

It is thus seen that the peaked electromotive force curve gives a primary current with smaller mean-square value than a rounded curve.

The most important fact, however, is that the iron core loss in the transformer, and therefore its efficiency, is sensibly affected by the form of the curve and primary electromotive force. In Fig. 206 are shown two curves, the ordinates of which represent the total power given to a transformer for certain values of the secondary output represented by the abscissæ, the transformer being tested in the two cases on the Ganz and Wechsler alternators.

There is between the two power lines a nearly constant difference of ordinate, showing that the cause is to be sought in the difference between the iron core losses in the two cases.

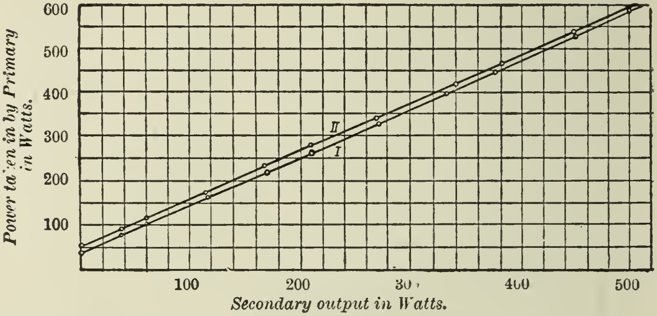

FIG. 206.—Energy Diagram of Transformer. Curve I, Ganz Alternator. Curve II, Wechsler Alte nator.

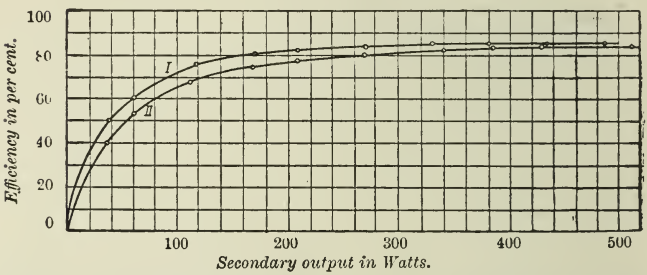

FIG. 207.—Efficiency Curve of Transformer. Curve I taken on Ganz Alternator. Curve II taken on Wechsler Alternator.

It follows that both the efficiency curves and power-factor curves of the transformer plotted in terms of the secondary output will differ if the form of the primary electromotive force curve is varied.

In Figs. 207 and 208 are shown the forms of the efficiency and power-factor curves of the same transformer when taken off the Ganz machine with peaked electromotive force curve and the Wechsler machine with a rounded curve.

We find also that the secondary drop is considerably affected by the form of the primary electromotive force curve. In

Fig. 209 are shown the secondary drop curves of the same transformer taken on the above-mentioned alternators, the primary electromotive force having in each case the same constant mean-square value.

It is clear, therefore, that a peaked electromotive force curve of the type given by the Ganz alternator causes a less iron core loss but a greater magnetic leakage than does a curve of a more rounded form similar to that of the Wechsler machine. Numerous tests and experiments made by the author with the Mordey and Thomson-Houston alternators had established this fact prior to the appearance of the Paper by Dr. G. Roessler, from which the above transformer diagrams, taken off the Ganz and Wechsler alternators, are copied. It is generally true that a sharp-peaked electromotive force curve gives a less hysteresis loss in the iron than does a rounded or sine electromotive force curve having the same mean-square value.

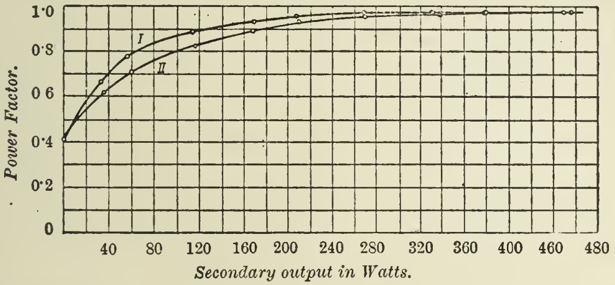

Fig. 208.—Power Factor Curves of a Transformer. Curve I taken on Ganz Alternator. Curve II taken on Wechsler Alternator.

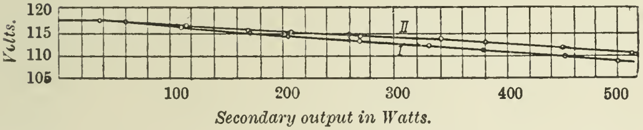

Fig. 209.—Curve of Secondary Drop of Transformer. Curve I taken on Ganz Alternator. Curve II taken on Wechsler Alternator.

**§ 10. The Form Factor and Amplitude Factor of a Periodic Curve.**—The above differences are closely connected with the magnitude of the *form factor* of the curve of primary electromotive force. This quantity is defined as the ratio of the square root of the mean of the squares of the equispaced ordinates of a curve to the true mean value of the equispaced ordinates. If we denote the first function, viz., the mean-square value, by the letters R.M.S. (root mean square), and the second function by the letters T.M. (true mean), then the form factor of any single-valued periodic curve is defined as follows:—

$$\text{The form factor} = \frac{\text{The R.M.S. value of equispaced ordinates}}{\text{The T.M. value of equispaced ordinates}} = f.$$

Take, for instance, in the case of a simple sine curve, the R.M.S. value of the equispaced ordinates is equal to the value of the maximum ordinate divided by $\sqrt{2}$. The T.M. value of the equispaced ordinates is equal to the value of the maximum ordinate multiplied by $\frac{2}{\pi}$.

Since $$\frac{1}{\sqrt{2}} = 0{\cdot}707 \quad \text{and} \quad \frac{2}{\pi} = 0{\cdot}637,$$

the ratio of the R.M.S. value to the T.M. value for a simple sine curve is $\frac{0{\cdot}707}{0{\cdot}637} = 1{\cdot}1$.

For several other simple forms of curve the form factor, R.M.S. value, and T.M. value are as below, the maximum ordinate in each case being taken as unity :—

| Curve. | T.M. value of ordinate as fraction of max. ordinate. | R.M.S. value of ordinate as fraction of max. ordinate. | Form factor $f$. |
|---|---|---|---|
| Sine .................... | 0·637 | 0·707 | 1·1 |
| Semicircle..... ......... | 0·7854 | 0·835 | 1·063 |
| Triangle ............... | 0·5 | 0·58 | 1·16 |
| Rectangle............... | 1·0 | 1·0 | 1·0 |
| Semi-ellipse........... | 0·785 | 0·816 | 1·039 |
| Parabola with axis vertical.............. | 0·666 | 0·730 | 1·096 |
| Two semi-parabolas meeting at a cusp.. | 0·33 | 0·447 | 1·35 |

The form factor of any curve can easily be obtained geometrically as follows : On one side of a straight line (*see* Fig. 210) plot a wave diagram of the curve, and on the other side of the line plot a polar diagram of the same curve, with its pole on the line of reference. Then, by the proposition on page 193, the radius of the semicircle, so drawn that its area is equal to the area of the polar curve, is the R.M.S. value of the ordinates, and the height of the rectangle described on the base line, so that its area is equal to that of the wave curve, gives the T.M. value of the ordinates. Hence the ratio of the radius of the semi-circle to the height of the rectangle is the form factor of the curve, which is represented by the wave or polar diagram.

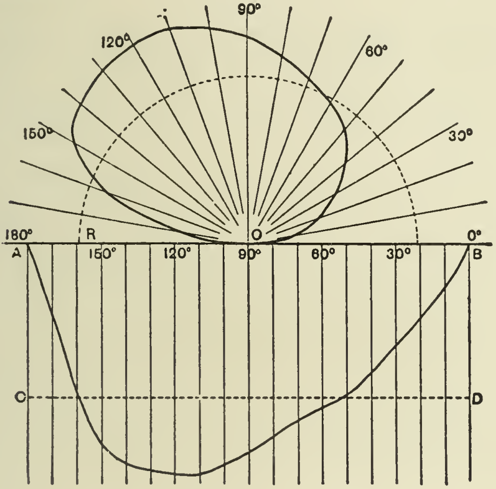

Fig. 210.

This form factor is an important quantity in the design of alternators, and by suitably proportioning the width of armature coils and field poles the form factor can be varied within wide limits.

It is evident that for the same R.M.S. value the form factor will be greater if the curve is a sharp-peaked curve than if it is a rounded curve like a semicircle or sine curve.

If some of the ordinates of any curve are increased so as to form a peak, this operation, geometrically considered, increases the R.M.S. value faster than it increases the T.M. value, and so increases the form factor of the curve.

The *amplitude factor* of a periodic curve is defined as the ratio between the root mean-square (R.M.S.) value of the ordinates and the value of the maximum ordinate, or

$$\left.\begin{array}{c}\text{The amplitude}\\ \text{factor}\end{array}\right\}=\frac{\text{R.M.S. value of equispaced ordinates}}{\text{Value of maximum ordinate}}=g.$$

For the same R.M.S. value the amplitude factor is less for a sharp-peaked curve than for a rounded or flat curve.

These two factors—the form factor $f$ and the amplitude factor $g$—are important quantities in the case of periodic curves.

§ **11. General Analytical Theory of the Transformer and Induction Coil.**—It remains, then, to indicate the manner in which the various periodic and fixed quantities concerned in the action of the transformer are connected and how they can be determined.

For convenience we may collect together the symbols employed to represent the various quantities with which we are concerned.

$e_1$ = The value of the primary terminal potential difference or primary E.M.F. at any instant.

$E_1$ = The maximum value of the same.

$m\,e_1$ = The true mean (T.M.) value of $e_1$ during the period.

$\sqrt{m\,e_2^2}$ = The root-mean-square (R.M.S.) value of $e_1$ during the period.

$\left.\begin{array}{r} e_2 \\ E_2 \\ m\,e_2 \\ \sqrt{m\,e_2^2} \end{array}\right\}$ are the same quantities for the secondary terminal potential difference.

$R_1$ = The resistance of the primary circuit.

$R_2$ = The resistance of the secondary circuit.

$N_1$ = The number of turns on the primary coil.

$N_2$ = The number of turns on the secondary coil.

$b_1$ = The density of magnetic induction in the core inside primary coil.

$B_1$ = The maximum value of induction density $b_1$.

$Z_1$ = The total induction produced in the core due to primary coil.

$b_2$, $B_2$, $Z_2$ are the same quantities for the secondary circuit.

$i_1$ = Primary current at same instant that the primary terminal potential difference is $e_1$.

$I_1$ = Maximum value of $i_1$.

$m . i_1$ = True mean value of $i_1$ during the period.

$\sqrt{m\, i_1^2}$ = Root-mean-square value of $i_1$ during the period.

$i_2$, $I_2$, $m\, i_2$, $\sqrt{m\, i_2^2}$ are the same quantities for the secondary circuit.

$h_1$ = Magnetic force due to the primary current $i_1$.

$H_1$ = Maximum value of $h_1$.

$h_2$ = Magnetic force due to the secondary current.

$H_2$ = Maximum value of $h_2$.

X = Total power loss in watts in the iron core.

Y = Hysteresis loss in watts in the iron core per cubic centimetre.

U = Eddy-current loss in watts in the core per cubic centimetre.

V = Total volume of the iron core.

S = Cross-sectional area of iron core.

$l$ = Mean length of magnetic circuit.

$f$ = The form factor = R.M.S ÷ T.M. value.

$g$ = The amplitude factor = R.M.S. ÷ maximum value.

$n$ = The frequency.

$p = 2\pi n$ = the angular velocity.

T = periodic time = $n^{-1}$.

Then the fundamental equations are as follows:—

When the secondary circuit is open and the transformer, therefore, at no load, we have

$$e_1 = R_1 i_1 + S N_1 \frac{d b_1}{d t} \quad . \quad . \quad . \quad . \quad . \quad (151)$$

The above equation holds good also when the transformer is loaded up, provided we then interpret $b_1$ to mean the resultant induction density in the iron core as affected by the current in the secondary coil.

In all good modern closed iron circuit transformers the value of $R_1 i_1$ is so small at all times during the period, when compared with $e_1$, that we may without sensible error write

$$e_1 = S N_1 \frac{d b_1}{d t}.$$

Hence $$e_1\, d t = S N_1\, d b_1 \quad . \quad . \quad . \quad . \quad . \quad (152)$$

If we integrate this last equation throughout one-quarter period we have already seen that

$$\mathrm{S\,N_1\,B_1} = \int_0^{\frac{\mathrm{T}}{4}} e_1\,dt = m\,.\,e_1\,\frac{\mathrm{T}}{4},$$

but

$$\mathrm{T} = \frac{1}{n}.$$

Hence

$$4\,\mathrm{S\,N_1\,B_1}\,n = m\,.\,e_1. \quad . \quad . \quad . \quad . \quad (153)$$

But if $f$ is the form factor of the curve of primary potential difference, then

$$f = \frac{\sqrt{m\,.\,e_1^{\,2}}}{m\,.\,e_1}. \quad . \quad . \quad . \quad . \quad . \quad (154)$$

Therefore, from equations (153) and (154), we have

$$\sqrt{m\,.\,e_1^{\,2}} = 4\,f\,n\,\mathrm{N_1\,S\,B_1}, \quad . \quad . \quad . \quad (155)$$

which gives us the R.M.S. value of the primary potential difference in terms of the maximum value of the induction density in the core within the primary coil.

If the secondary circuit of the transformer is closed, and a secondary current is being taken from the transformer, then the currents, inductions and potentials are determined by the two equations,

$$e_1 = \mathrm{R_1}\,i_1 + \mathrm{N_1\,S}\,\frac{d\,b_2}{d\,t} \quad . \quad . \quad . \quad . \quad . \quad (156)$$

and

$$0 = \mathrm{R_2}\,i_2 + e_2 + \mathrm{N_2\,S}\,\frac{d\,b_2}{d\,t}. \quad . \quad . \quad . \quad . \quad (157)$$

If the secondary circuit is open, and hence $\mathrm{R_2}\,i_2$ equal to zero, and $\mathrm{R_1}\,i_1$ practically negligible in comparison with $\mathrm{N_2\,S}\,\frac{d\,b_2}{dt}$, we may write (156) and (157)

$$e_1 = \mathrm{N_1\,S}\,\frac{d\,b_1}{d\,t}$$

and

$$e_2 = -\,\mathrm{N_2\,S}\,\frac{d\,b_2}{d\,t},$$

and, therefore, as already shown,

$$m\,.\,e_1 = 4\,n\,\mathrm{N_1\,S\,B_1}, \quad . \quad . \quad . \quad . \quad (158)$$

and

$$m\,.\,e_2 = 4\,n\,\mathrm{N_2\,S\,B_2}. \quad . \quad . \quad . \quad . \quad (159)$$

In the previous section we have shown that the total induction $S B_1$ linked with the $N_1$ primary turns may be expressed as $(Z_1 - Z_2 + Z_2 \beta)$, where $Z_1$ is the maximum value of the total induction due to the primary current alone, and $Z_2$ that due to the secondary current alone.

Hence, as before, writing $Z_1 - Z_2 + Z_2 \beta$ for $S B_1$, and $Z_1 - Z_2 - Z_1 \beta$ for $S B_2$, we have, by substitution in equations (158) and (159), the results

$$m . e_1 = 4 n N_1 (Z_1 - Z_2 + Z_2 \beta),$$

and

$$m . e_2 = 4 n N_2 (Z_1 - Z_2 - Z_1 \beta).$$

It is an experimental fact that the curve of secondary potential is an exact copy of the curve of primary potential at no load, and very nearly also at any load; hence the form factors of the curves of primary and secondary potentials are the same. Writing $f$ for this form factor we have

$$\sqrt{m . e_1^2} = f m . e_1$$

and

$$\sqrt{m . e_2^2} = f m . e_2.$$

Hence the transformation ratio of the transformer T is given by the equation

$$T = \frac{\sqrt{m . e_2^2}}{\sqrt{m . e_1^2}} = \frac{N_2 (Z_1 - Z_2 - Z_1 \beta)}{N_1 (Z_1 - Z_2 + Z_2 \beta)} . \quad . \quad . \quad (160)$$

Accordingly we see that the transformation ratio of the transformer is never exactly equal to the ratio of the turns unless the leakage coefficient $\beta$ is zero, and that the transformation ratio diminishes as $Z_1$ and $Z_2$ increase with load, because their difference $Z_1 - Z_2$ always remains approximately constant at all loads, and is the mean core induction.

The leakage coefficient $\beta$ is a function of the form factor $f$, such that $\beta$ is greater as $f$ is greater. Thus, peaked primary potential curves give greater secondary drop than rounded potential curves, even if they have the same R.M.S. value.

From equation (155) we see that the maximum value of the core induction B, either within the primary or secondary coil, is smaller in proportion as the form factor $f$ is greater, if the R.M.S. value of the primary potential remains constant.

Hence the maximum value B of this core induction is less for pointed or peaked primary electromotive force curves than for rounded or flat curves, the R.M.S. value of the primary terminal potential difference being constant.

This has been experimentally proved by Dr. Roessler in his researches on the influence of the form of the potential and current curves on transformer action.

In Fig. 211 are shown three curves. The curve marked I is the magnetisation curve of a small transformer measured with the ballistic galvanometer in the ordinary way. The curve marked II is the curve of induction as obtained with alternating currents, using a Wechsler alternator, and the curve marked III that obtained in the same way, but by the use of a Ganz alternator. The values of the maximum induction B for the alternating currents are obtained from the primary electromotive force curves, as already described.

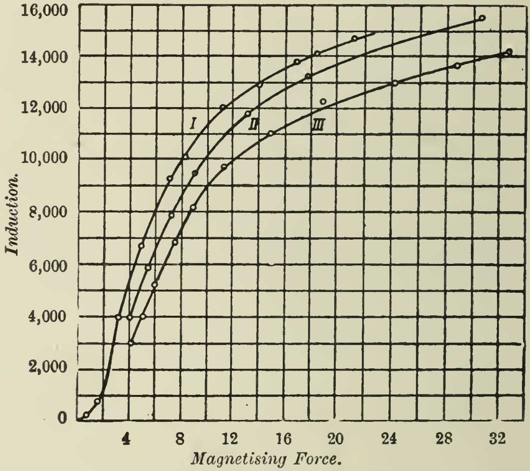

Fig. 211.

It is seen that the curves of induction as obtained by the alternating-current machines lie below that obtained by the continuous currents and ballistic galvanometer.

In other words, for a given induction, the magnetising force required is greater with alternating than with continuous currents. There are two reasons for this: first, the existence of eddy currents in the core, which, acting like smaller closed secondary currents, increase the primary current, and, second, the existence of magnetic leakage when alternating currents are used. The curves show, however, that when the peaked form of primary electromotive force curve given by the Ganz machine is used, the induction corresponding to a given magnetising force is less than when the Wechsler machine with rounded electromotive force curve is employed, the same R.M.S. values of the primary electromotive force being employed.

The root-mean-square values of the primary and secondary currents, viz., $\sqrt{m \cdot i_1^2}$ and $\sqrt{m \cdot i_2^2}$, are connected with the maximum values of these variables by the equations

$$\sqrt{m \cdot i^2_1} = g \ \mathrm{I}_1$$

and

$$\sqrt{m \cdot i_2^2} = g \ \mathrm{I}_2,$$

where $g$ is the quantity already called the *amplitude factor.*

It has been shown that for peaked or pointed curves the amplitude factor is smaller than for flat or rounded curves. Hence for the same root mean-square value of the primary and secondary currents the maximum values of these quantities are greater for peaked current curves than for rounded or flat curves. Hence the inductions created by these currents respectively are greater—that is, $Z_1$ and $Z_2$ will be greater for peaked current and potential curves than for flat or rounded curves.

Accordingly, whilst the respective primary and secondary inductions $Z_1$ and $Z_2$ are greater for electromotive force curves with large form factors, their difference, $Z_1 - Z_2$, which is the resultant core induction B, is less. Hence we see that the secondary drop, or increase of transformation ratio produced by loading up the transformer must be greater when the primary electromotive force curve is peaked than when it is rounded, the same mean-square value of this last being preserved constant.

We have, then, to discuss the form of the curves of primary current under variations of form factor of the electromotive force curve.

If $h_1$ is the instantaneous value of the magnetising force due to the primary current $i_1$, then

$$h_1 = \frac{4\pi}{10} \frac{\mathrm{N}_1 \, i_1}{l},$$

and on open secondary circuit we have

$$e_1 = \mathrm{R}_1\, i_1 + \mathrm{N}_1\, \mathrm{S}\, \frac{d b_1}{d t}.$$

Hence $$e_1 = \mathrm{R}_1\, i_1 + \mathrm{N}_1\, \mathrm{S}_1 \frac{d\, b_1}{d\, h_1} \cdot \frac{d\, h_1}{dt},$$

or $$e_1 = \mathrm{R}_1\, i_1 + \frac{4\pi}{10}\, \frac{\mathrm{N}_1{}^2}{\mathrm{L}}\, \mathrm{S}\, \frac{d\, b_1}{d\, d_1} \cdot \frac{di_1}{d\, t}. \quad . \quad . \quad . \qquad (161)$$

This last equation is true for all forms of primary electromotive force curves.

If the permeability of the iron was constant, the value of $\frac{d\, b_1}{d\, h_1}$ would be constant and equal to $\mu$, but in practice it is not found to be constant. We see, however, that $\frac{d\, b_1}{d h_1}$ is the slope of the geometrical tangent to the hysteresis curve at the instant considered—that is, it is the trigonometrical tangent of the angle which the geometrical tangent to the hysteresis curve makes with the positive direction of the axis of time, and this is not found to be a constant quantity as we travel round the hysteresis curve. The value of $\frac{d\, b_1}{d\, h_1}$, however, is not greatly affected by the form of the curve of $e_1$. Hence, for curves of primary electromotive force which have a peaked form, and therefore a large maximum value, the value of $\frac{d\, i_1}{d\, t}$, or the slope of the current curve, will be greater than for flatter curves of electromotive force. This is seen to be the case by reference to Figs. 203 and 204, which show the no-load primary-current curves and primary electromotive force curves of the same transformer tested by Dr. Roessler on the Ganz and the Wechsler alternator.

The exact predetermination of the form of the primary current at no load from the curve of primary electromotive force is, at any rate as yet, an impossible matter. It would be an easy thing to predetermine if the hysteresis curve always had the same form, but as this last is affected to a considerable extent by variations in the quality of the iron and of the reluctance of the magnetic circuit, it is not of much use to make assumptions which are not justified in practice.

A knowledge of the power factor of transformers of any particular type will always enable us to make an approximation to the value of the magnetising current if the total power taken up in the core is known and the mean-square value of the primary electromotive force. For if X is the total power taken up in the core at no load, and $\sqrt{m\,e_1^2}$, $\sqrt{m\,i_1^2}$ are the R.M.S. values of the primary electromotive force and current, and F is the power factor, then $F = \frac{X}{\sqrt{m\,e_1^2} \cdot \sqrt{m\,i_1^2}}$, from which $\sqrt{m \cdot i_1^2}$ can be obtained.

It is seen, however, that the primary current curve at no load is always a more irregular curve than the curve of primary electromotive force, and that for the same R.M.S. value of the primary electromotive force the R.M.S. value of the primary current at no load (the magnetising current) is less for pointed or peaked potential curves than for flat curves.

§ **12. Iron Core Loss in Transformers and Induction Coils.**—It has already been explained that two distinct causes of energy dissipation exist in the iron cores of transformers and induction coils—viz., the magnetic hysteresis loss and the eddy-current loss. The former of these is not affected or diminished by any amount of lamination of the core, but the latter can be reduced to a very small percentage of the total loss by constructing the core of iron plates of thickness not greater than 0·014 inch, the plates being separated from each other by very thin paper, or a layer of paint or varnish. In the chapter in the Second Volume of this Treatise devoted to the Construction of the Transformer, the various practical details connected with the core construction, and the predetermination of the core loss for plates or wires of given size are considered.

Supposing, however, that the core is properly laminated, and in planes parallel to the lines of induction in the core, there will still be a certain dissipation of energy, by reason of eddy electric currents set up in the iron as the induction changes its direction. If we consider a small circuit described anywhere in the iron plate, in a plane perpendicular to the lines of induction, then, if $e$ is at any instant the electromotive force set up in this circuit by reason of the variation of the induction through it, the mean rate at which energy is being dissipated in this circuit must be equal to some constant, multiplied by the value of the mean of the square of $e$. But we have seen that if B is the maximum value of the induction in the core, then the R.M.S. value of the electromotive force of induction induced in the primary circuit is equal to the value of the expression $4 f \mathrm{N}_1 . n \mathrm{S} \mathrm{B}$, where $f$ is the form factor of the curve of electromotive force.

Hence the mean-square value $(m . e^2)$ of the electromotive force of induction must be numerically proportional to $f^2 n^2 \mathrm{B}^2$; also the same holds good for the eddy-current electromotive force and rate of energy dissipation, and the eddy current loss per unit of volume in the core measured in watts must be proportional to the product of some constant $\xi$ and the quantity $f^2 n^2 \mathrm{B}^2$. In other words, the eddy-current loss will be equal to $\xi f^2 n^2 \mathrm{B}^2$ watts per unit of volume of the core, where $f$ is the form factor of the curve of primary electromotive force, $n$ the frequency, and B the maximum value of the induction.

Mr. Steinmetz has shown that the hysteresis loss in iron cores can be represented by an arbitrary formula, expressing the fact that the hysteresis loss per unit of volume of the core is proportional to the product of a constant, the frequency, and the maximum value of the induction raised to a power very near to 1·6. Hence, if H is the hysteresis loss in the core per unit of volume,

$$\mathrm{H} = \eta n \mathrm{B}^{1 \cdot 6},$$

where $\eta$ is called the hysteretic constant of the iron.

This law, although only an empirical one, deduced entirely from observation, yet appears to be sufficiently exact to guide practice within the limits of the range of induction density employed in transformers.

Hence the total loss T in a transformer core of volume V is given by the expression

$$\mathrm{T} = \mathrm{V} (\eta n \mathrm{B}^{1 \cdot 6} + \xi n^2 f^2 \mathrm{B}^2). \quad . \quad . \quad . \quad (162)$$

The eddy-current loss varies as the square of the maximum value of the induction, and the hysteresis loss as the 1·6th power of the same.

Since the R.M.S. value ($\sqrt{m\,e_1^2}$) of the primary electromotive force has been shown to be related to B by the equation

$$\sqrt{m\,.\,e_1^2} = 4\,f\,N_1\,n\,S\,B,$$

we can substitute for B, in equation (162), its value in terms of $\sqrt{m\,e_1^2}$, and we arrive at the equation

$$T = V\left\{\eta\,n\left(\frac{\sqrt{m\,.\,e_1^2}}{4\,f\,N_1\,n\,S}\right)^{1.6} + \xi\,\frac{m\,.\,e_1^2}{16\,N_1^2\,S^2}\right\}. \quad . \quad (163)$$

This last equation shows us that the eddy-current loss is not affected by the form factor of the curve of primary electromotive force, but that the hysteresis loss is affected by it, because the form factor $f$ appears in the hysteresis term of the expression for T but not in the eddy current term.

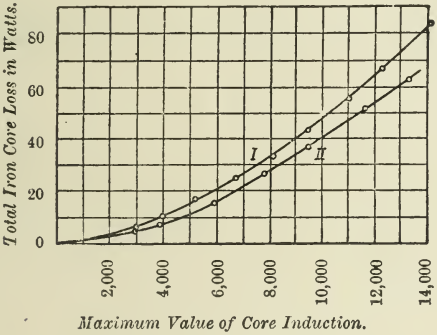

FIG. 212.—Curve I taken with Ganz Machine. Curve II taken with Wechsler Machine.

Hence variation in the form factor of the curve of primary electromotive force will alter the total core loss in the transformer, and make it *less* in proportion as the form factor is greater. This has been pointed out both by the author and by Dr. G. Roessler, and is amply confirmed by experiment.

Hence the total core loss in a transformer is not an absolute and fixed quantity, but depends upon the form of the wave of primary electromotive force to a not inconsiderable degree.

This dependence of the core loss upon the form factor of the curve of electromotive force is shown by the results of Dr. Roessler's experiments embodied in the diagrams in Figs. 206 and 212. From Fig. 212 it will be seen that at a given induction the core loss is greater when the transformer is worked off the Ganz alternator than off the Wechsler alternator; and from Fig. 206 it is shown that for the same R.M.S. value of the primary potential difference the core loss is greater with the Wechsler than with the Ganz alternator.

Broadly speaking, pointed or peaked potential curves give rise to greater core loss than rounded or flat primary potential curves.

In the Second Volume of this Treatise, we return to the discussion of these matters, and enter into more details as to the practical considerations to which they lead in the Construction of the Induction Coil and Transformer.

END OF VOLUME I.

# APPENDIX.

## NOTE A. *(See page 19.)*

**The Magnetic Force at any Point in the Plane of a Circular Current.**—The magnetic force at any point in the plane of a circular conductor conveying a current may be found by an elegant geometrical method due to Mr. A. Russell (see *The Electrician*, Vol. XXXI., p. 284).

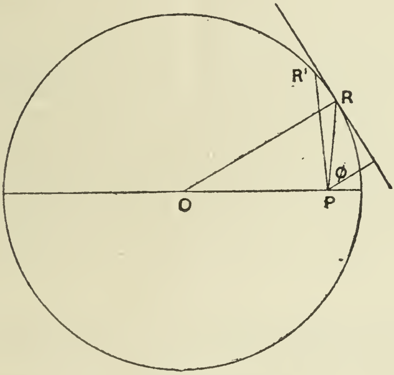

FIG. 1.

Let P be any point in the plane of a circular current, and let F be the magnetic force at P due to a current of strength $i$ in the wire. Let $d\,s$ be an element of length of the circle, and let O R be the radius of the circle. Take any point R on the circumference of the circle (*see* Fig. 1), draw the diameter through O P, and at R draw a tangent to the circle. Then draw the radius O R, and through P draw P N perpendicular to the tangent. Join P R. Let $OP = a$, $OR = R$, $PR = r$, $PN = p$, and the angle $RPN = \phi$.

Then, by Ampère's law, the magnetic force F at P due to the element $d\,s$ of the circuit at R is equal to $\frac{i\,d\,s}{r^2}\,\frac{p}{r}$. Hence

$$\mathrm{F} = \int \frac{i\,d\,s}{r^2}\,\frac{p}{r}.$$

But $$p\,d\,s = r^2\,d\,\phi,$$

hence $$\mathrm{F} = \int_0^{2\pi} \frac{i\,d\,\phi}{r};$$

and, since $$r = \sqrt{\mathrm{R}^2 - a^2 \sin^2 \phi} - a \cos \phi,$$

therefore, $$\frac{1}{r} = \frac{\sqrt{\mathrm{R}^2 - a^2 \sin^2 \phi} + a \cos \phi}{\mathrm{R}^2 - a^2}.$$

$$\text{Hence, } \mathrm{F} = \frac{i}{\mathrm{R}^2 - a^2} \int_0^{2\pi} \sqrt{\mathrm{R}^2 - a^2 \sin^2 \phi}\; d\,\phi + \frac{i\,a}{\mathrm{R}^2 - a^2} \int_0^{2\pi} \cos \phi\, d\,\phi.$$

The second integral, taken between the assigned limits, is zero. The first integral, $\int_0^{2\pi} \sqrt{\mathrm{R}^2 - a^2 \sin^2 \phi}\; d\,\phi$, is called an elliptic integral of the second order, and represents the length of the circumference of an ellipse which has O for its centre, P for one of its foci and 2 R for its major axis.

Hence, if we describe an ellipse on the diameter of the circle, with O as its centre and P as its focus, the magnetic force F at the point P due to a current $i$ in the circle is equal to the value of $\frac{i}{\mathrm{R}^2 - a^2}$ × the length of the circumference of this ellipse.

In practice we can easily describe this ellipse by means of two pins and a thread, and then measure the length of its circumference with a measuring-wheel, such as is used for measuring distances upon a map, or by laying a thread round the ellipse and measuring its length.

In this way a practical measurement can be made of the magnetic force due to a circular current at any point in its plane.

Up to the limit of $a = 0{\cdot}8\,\mathrm{R}$, the length $l$ of the circumference of the ellipse can be calculated approximately from the formula—

$$l = \frac{\pi}{2}\left\{\mathrm{R} + \sqrt{\mathrm{R}^2 - a^2} + \sqrt{4\,\mathrm{R}^2 - 2\,a^2}\right\}.$$

## NOTE B. *(See page 35.)*

**The Total Induction in a Circular Solenoid.**—If we consider a circular-sectioned ring to be closely wound over with turns of wire, we obtain a circular solenoid.

Let $a$ be the mean radius of the circular cross-section of the solenoid, and let R be the radius of the circular axis of the solenoid. The whole solenoid may be considered to be resolved into elementary solenoids. Let $d\,S$ be the cross-section of one of these, and let $x$ be the radius of the circular axis of the circular elementary solenoid.

Let N be the total number of turns of wire on the ring. Then, for the elementary solenoid, there are $\frac{N}{2\pi x}$ turns per unit of length.

The magnetic force in the interior of the elementary solenoid is H, and

$$H = \frac{4\pi N I}{2\pi x} = \frac{2 N I}{x},$$

where I is the current in the solenoid.

Hence, if the medium is non-magnetic, the surface integral of magnetic force is the measure of the induction. Hence the magnetic induction in the interior of the elementary solenoid is $\frac{2 N I}{x} d\,S$. Hence the whole induction Q through the whole solenoid is obtained by integrating this last expression over the area of the solenoid, viz., $\pi a^2$. Therefore between proper limits

$$Q = 2 N I \int \frac{d\,S}{x}.$$

Now the integral $\int \frac{d\,S}{x}$ taken over the circular cross-section can be shown to be equal to $2\pi\{R - \sqrt{R^2 - a^2}\}$,

and hence $$Q = 4\pi N I\{R - \sqrt{R^2 - a^2}\}.$$

This, then, is the expression which should be employed for the total induction over the circular cross-section of the ring, if the mean radius of cross-section $a$ is not very small compared with the radius of the circular axis R.

If $\frac{a}{R}$ is a small quantity, then, since

$$\{R - \sqrt{R^2 - a^2}\} = \frac{1}{2} R \left(\frac{a^2}{R^2} + \frac{1}{4}\frac{a^4}{R^4} + \frac{1}{8}\frac{a^6}{R^6} + \&c.\right),$$

we have $R - \sqrt{R^2 - a^2} = \frac{a^2}{2R}$ nearly when $\frac{a}{R}$ is small, and hence

$$Q = 4\pi N I \frac{a^2}{2R} = 4\pi \frac{N I}{2\pi R} (\pi a^2),$$

or $4\pi$ times the current turns per unit of length of solenoid multiplied by the area of cross-section of the solenoid; and for a very large thin circular solenoid the magnetic force in the centre is $4\pi$ times the current turns per unit of length.

### NOTE C. *(See page 52.)*

**The Calibration of the Ballistic Galvanometer.**—A galvanometer consists generally of two parts—a magnet and a coil of wire. The magnet may be fixed and the coil suspended and movable; or the magnet suspended and movable and the coil fixed. If the arrangements are such that the movable system when disturbed is brought to rest without vibration, or with very few vibrations, the galvanometer is called a *damped* or *aperiodic* galvanometer. If, however, the resistance to motion is so small that the movable part, when disturbed or made to oscillate, continues for a long time to oscillate with very slowly decreasing amplitude of vibration, the galvanometer is called a *ballistic* galvanometer.

If a body is suspended so as to vibrate about a vertical axis, and if when given an angular displacement about that axis it returns when released to its original position, and oscillates about it, the body will in general execute vibrations of decreasing amplitude. If I is the moment of inertia of the body about that axis, and if $\mu$ is the torque or couple which tends to restore the body to its original position if displaced round the axis by a unit angle, then, if at any instant the angular velocity of the oscillating body is $\omega$, the equation of motion, neglecting for the moment resistance to motion, is

$$I \frac{d\omega}{dt} = \mu \theta,$$

where $\theta$ is the angle of displacement at the instant when the angular velocity is $\omega$.

Since $\omega = \frac{d\theta}{dt}$, we have as the equation of motion,

$$\frac{d^2\theta}{dt^2} = \frac{\mu}{I}\theta,$$

and the time $t$ of one complete vibration of the body will be

$$t = 2\pi\sqrt{\frac{I}{\mu}}.$$

This gives us the periodic time of oscillation of the body, assuming that the resistance to motion due to friction is nothing.

If there is any small resistance to the motion, such as air resistance or other causes of energy dissipation, the amplitude of the swings of the vibrating body will gradually decrease. If the experiment is tried of setting the needle or coil of a ballistic galvanometer in motion, and observing the amplitudes $x_1$, $x_2$, $x_3$, &c., of successive swings to the right and left, it will be found that $x_1$, $x_2$, $x_3$, &c., form a descending geometrical progression, and that the values of $\log x_1$, $\log x_2$, $\log x_3$, &c., form a descending arithmetic progression.

If the difference of the logarithms of the amplitudes of two successive swings, one to the right and one to the left, is taken, this difference, denoted by $\lambda$, is called the *logarithmic decrement* of the galvanometer, and it will be found that this difference is approximately constant for any two successive swings right and left during the progress of the decay of the excursions. That is, $\log x_1 - \log x = \lambda$, also $\log x_2 - \log x_3 = \lambda$, &c. Hence, if the logarithm of the amplitude falls off or decreases by an amount $\lambda$ in the course of the passage of the needle or coil of the galvanometer from the extremity of one swing to the right to the extremity of one swing to the left, it may fairly be assumed that, if the coil or needle is started from rest by a sudden blow, the excursion $x_0$ it would make if there were no resistance is related to the excursion $x_1$ it does actually make with the actual resistance existing by the relation

$$\log x_0 - \log x_1 = \frac{\lambda}{2},$$

or

$$\log_{10}\frac{x_0}{x_1} = \frac{\lambda}{2},$$

or

$$\frac{x_0}{x_1} = 10^{\frac{\lambda}{2}}.$$

If the base of the logarithms is 10, or the logarithms are ordinary ones, then by the exponential theorem we have

$$10^{\frac{\lambda}{2}} = 1 + M\frac{\lambda}{2} + \frac{M^2}{4}\,\frac{\lambda^2}{1.2.3} + \&c.,$$

where M is the *modulus* 2·303, or the multiplier for converting ordinary logarithms into Napierian logarithms. M is the logarithm of 10 to the base $e$, the base of Napierian logarithms.

If $\frac{\lambda}{2}$ is small, we may neglect $\frac{\lambda^2}{4}$ and higher powers in comparison with $\frac{\lambda}{2}$, and write

$$10^{\frac{\lambda}{2}} = 1 + \mathrm{M}\frac{\lambda}{2}, \text{ nearly.}$$

Hence $$\frac{x_0}{x_1} = 1 + \mathrm{M}\frac{\lambda}{2}, \text{ nearly,}$$

or $$x_0 = x_1\left(1 + \mathrm{M}\frac{\lambda}{2}\right), \text{ nearly.}$$

In other words, we can find what the excursion $x_0$ would be if there were no air resistance by multiplying the actual first excursion $x_1$ by a factor $\left(1 + \mathrm{M}\frac{\lambda}{2}\right)$ called the correcting factor.

Hence, if the movable system of a ballistic galvanometer receives a sudden blow or impulse when at rest, and if it then makes an excursion the angular magnitude of which is $x_1$, and if such excursion is slightly resisted by air friction, we can eliminate the results of this and correct for the frictional resistance by multiplying $x_1$ by the correcting factor $\left(1 + \mathrm{M}\frac{\lambda}{2}\right)$.

If the logarithmic decrement is measured directly in Napierian logarithms, then the correcting factor is simply

$$\left(1 + \frac{\lambda}{2}\right)$$

where $\lambda$ equals the Napierian logarithmic decrement, or the logarithm of the ratio of one swing to the next.

Returning to the equation of motion of the needle or coil, since we have at any moment

$$\mathrm{I}\frac{d\,\omega}{d\,t} = \mu\,\theta,$$

let us consider a small magnetic needle of magnetic length $l$ hanging in the centre of a coil so wound that when a current flows through the coil there is a uniform magnetic field due to the current in all the region within which the needle lies. Let the normal position of the needle when at rest be such that the magnetic axis of the needle is at right angles to the direction of the field due to the current in the coil.

Then let M be the magnetic moment of the needle, H the strength of the controlling field, the direction of H being at right angles to the field due to the current in the coil.

Let the needle be set oscillating by a sudden impulsive couple acting upon it when at rest. Let $\theta$ be the angular displacement of the magnetic axis of the needle at any instant $t$.

The restoring couple acting on this needle at that instant is $\mathrm{M\,H} \sin\theta$, and, by the equation of motion,

$$\mathrm{I}\frac{d\omega}{dt} = \mathrm{M\,H}\sin\theta.$$

Hence,

$$\mathrm{I}\,d\omega\frac{d\theta}{dt} = \mathrm{M\,H}\sin\theta\,d\theta;$$

but

$$\frac{d\theta}{dt} = \omega.$$

Hence

$$\mathrm{I}\,\omega\,d\omega = \mathrm{M\,H}\sin\theta\,d\theta.$$

Integrate the above equation from the instant when the needle leaves its position of rest with a finite angular velocity $\Omega$ until it reaches a displacement $\theta$ and has a zero angular velocity. We have

$$\mathrm{I}\int_{\Omega}^{0}\omega\,d\omega = \mathrm{M\,H}\int_{0}^{\theta}\sin\theta\,d\theta,$$

or

$$\tfrac{1}{2}\,\mathrm{I}\,\Omega^2 = \mathrm{M\,H}\,(1-\cos\theta),$$

$$= 2\,\mathrm{M\,H}\sin^2\frac{\theta}{2};$$

therefore

$$\Omega = 2\sqrt{\frac{\mathrm{M\,H}}{\mathrm{I}}}\sin\frac{\theta}{2}. \quad . \quad . \quad . \quad (1)$$

If the impulse which starts the needle from rest with a finite velocity is that due to the flow of a quantity of electricity through the coil surrounding the needle, which flow is all over before the needle has had time to move sensibly from rest, we can obtain a relation between the quantity of electricity so sent through and the excursion of the needle.

For let $i$ be the current in the coil at any instant, and let G be a constant depending upon the form of the galvanometer coils, such that $G\,i$ is the magnetic field due to the current. Then, since M is the moment of the needle, $M\,G\,i$ is the whole couple acting on needle, and the impulse of this couple is $M\,G\,i\,d\,t$ in a time $d\,t$. If the whole impulse is over before the needle has had time to move, then the whole impulse of the couple must be equal to the total gain of angular momentum $I\,d\,\omega$ taking place in the time $d\,t$.

Hence $$M\,G\,i\,d\,t = I\,d\,\omega\,;$$

but if $d\,q$ is the quantity of electricity which has flowed through the galvanometer coils in the time $d\,t$, we have

$$i = \frac{d\,q}{d\,t}$$

and $$M\,G\,i\,d\,t = M\,G\,d\,q = I\,d\,\omega.$$

But since the whole impulse is over before the needle has time to leave its position of rest, we have, by integrating from $\omega = 0$ to $\omega = \Omega$, the equation

$$M\,G\,Q = I\,\Omega,$$

where $\Omega$ is the angular velocity with which the needle leaves its position of rest. But by equation (1),

$$\Omega = 2\sqrt{\frac{M\,H}{I}}\sin\frac{\theta}{2}\,;$$

therefore $$Q = \sqrt{\frac{4\,I\,H}{M\,G^2}}\sin\frac{\theta}{2},$$

or $$Q = k\sin\frac{\theta}{2}.$$

Hence we see that, when a quantity Q of electricity is discharged suddenly through the coils of a ballistic galvanometer, the whole discharge being over in a very short time compared with the period of free vibration of the needle, the quantity of electricity is proportional to a constant, $k$, called the *ballistic constant*, multiplied by the sine of half the angle of excursion of the needle.

If the galvanometer has a logarithmic decrement $\lambda$, which is moderately small, say not more than 10 per cent., then to a close approximation

$$Q = k\sin\frac{\theta}{2}\left(1 + \frac{\lambda}{2}\right).$$

To determine $k$ for any galvanometer, the easiest way is to charge a condenser having a capacity of $c$ microfarads to a potential of $v$ volts by placing it in contact with a battery of known voltage, and then discharging this quantity $c\,v$ microcoulombs through the ballistic galvanometer. If this quantity $c\,v$ gives a "throw" $\theta$, we have

$$k = \frac{c\,v}{\sin\frac{\theta}{2}\left(1 + \frac{\lambda}{2}\right)}.$$

Hence $k$ is determined.

Such a calibrated ballistic galvanometer can immediately be employed to measure a magnetic field. For if a loop of wire having N turns is placed in a field of induction so that the induction is linked with the circuit, and if the loop is connected with the ballistic galvanometer, then on suddenly withdrawing the loop from the field we shall get a "throw" of the galvanometer which can be interpreted to mean so much quantity in microcoulombs passing through the galvanometer. Then, by the principles explained on page 31, the total change in induction linked with the circuit is equal to the product of the resistance of the circuit and the quantity set flowing through it, or

$$\text{Induction} \times \text{linkages} = \text{resistance} \times \text{quantity}.$$

If induction is measured in microwebers—one weber being $10^8$ C.G.S. lines or units of induction, and one microweber therefore 100 C.G.S. units—we have the rule—

$$\text{Microwebers} \times \text{linkages} = \text{microcoulombs} \times \text{ohms};$$

and if one microweber of induction passing through the loop linked once with this circuit, is suppressed, it will cause one microcoulomb of electric quantity to flow through the circuit if its resistance is one ohm.

In employing the ballistic galvanometer to measure magnetic induction, it must be observed that the condition of application of the above principles is that the whole change of induction must be completed in a very short time compared with the periodic time of the needle of the galvanometer.

To suggest, as is sometimes done, that the induction in the field magnets of a dynamo can be measured by surrounding them with a loop of wire connected to a ballistic galvanometer, and then short-circuiting or breaking the field circuit, is to presuppose a most unlikely event, viz., that the time during which such change of induction takes place is small compared with the periodic time of an ordinary ballistic galvanometer.

If the galvanometer is one with a movable coil of the d'Arsonval type, then it is better to standardize the galvanometer by means of a coil of known length, turns and resistance, placed in the axis of a long cylindrical coil, the turns per unit of length of which are known. The reason for this is that if the galvanometer circuit is always closed the movement of the coil in the strong magnetic field induces currents in the galvanometer which resist its motion. Hence a powerful damping action comes into play from this cause alone. To ascertain what induction change caused a given galvanometer throw we proceed thus:—The standardizing secondary coil should be joined up in series with the galvanometer coil and with the secondary or exploring coil, and it should be placed in the axis of a long coil, of which the field can be calculated. If, then, under the influence of an unknown induction linked or unlinked with the exploring coil we obtain a galvanometer throw $\theta$, we can find out what was the induction causing this throw by interrupting or reversing a known current in the long standard field coil, and thus linking or unlinking a known induction with the standard secondary coil. If the current through the standard field coil is altered until it gives a similar throw $\theta$ on being interrupted or reversed, we know at once that the value of the unknown induction linked or unlinked with the secondary or exploring coil which gives an equal throw, must be the same as that calculable induction linked with the standardizing secondary coil.

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